Universal Asymptotic Distribution Functions mod 1

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I. INTRODUCTION.

The question considered is roughly this: what happens to the asymptotic distribution mod 1 (defined below) of a sequence when one applies high iterates of a transformation T?

Let $\{x_k\}$ denote the fractional part of the real number x_k, i.e. $\{x_k\} := x_k \bmod 1$ We say that g is an a.d.f. (asymptotic distribution function mod 1) for the sequece of real numbers $\langle x_k \rangle$ iff

$$g(x) := \lim_{N\to\infty} \frac{1}{N}\#\Big\{k \leq N : \{x_k\} \in [0,x)\Big\}$$

for each $x \in [0,1]$. Now, given a sequence $\langle x_k \rangle_{k=1}^{\infty}$ and a map $T : [0,1] \to [0,1]$, define $\langle x_k^{(n)} \rangle_{k=1}^{\infty}$ recursively by

$$x_k^{(0)} := \{x_k\} = x_k \bmod 1$$

$$x_k^{(n)} := T(x_k^{(n-1)})$$

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Finally g_n will denote an a.d.f. for $\langle x_k^{(n)} \rangle_{k=1}^{\infty}$. We shall always assume that g_0 exists and and that g_0' is bounded away from zero.

II. TENT FUNCTIONS

We consider first the case where T is one of the family of "tent functions" T defined here. Define h by

$$h(x) := \begin{cases} 2x, & 0 \le x \le 1/2; \\ 2 - 2x, & 1/2 \le x \le 1 \end{cases}$$

Our main result is:

Theorem 1 *Let $k \in C^2([0,1])$ with $k(0) = 0, k(1) = 1$, and $k'(x) > \delta > 0 \ \forall x$. Let $T = k^{-1} \circ h \circ k$. Then $g_n(x) = k(x) + O(e^{-\lambda\sqrt{n}})$*

Remark: Both λ and the constant implicit in the big-Oh notaion depend on k and g_0. As usual, we assume that g_0 is C^2, and that $g_0'(x)$ is bounded away from zero.

The proof contains many technical details, so we begin with a brief outline. Readers familiar with Khintchine(1974) will recognize the influence of Kuzmin. We shall find an increasing sequence $\langle A_r \rangle$, and a decreasing sequence $\langle B_r \rangle$, such that

$$(*) \qquad A_r k'(x) < g_{rn}'(x) < B_r k'(x)$$

and

$$1 - c_1 e^{-c_2 n} < A_r < B_r < 1 + c_1 e^{-c_2 n}$$

(for all r, where c_1 and c_2 are constants depending only on g_0 and k.) This implies that

$$\left| g_{n^2}'(x) - k'(x) \right| = O(e^{-c_2 n})$$

The theorem then follows, by an easy argument, from the lemma 2:

Lemma 2 *If α and β are constants,and*

$$\alpha k'(x) < g_m'(x) < \beta k'(x) \ \ \forall x$$

then

$$\alpha k'(x) < g_{m+1}'(x) < \beta k'(x) \ \ \forall x$$

The proofs of $(*)$ and lemma 2 depend heavily on the fact that $d\mu := k'(x)dx$ is invariant under T. We therefore begin by verifying this fact. Let L and R denote restrictions of T to $[0, k^{-1}(1/2)]$ and $[k^{-1}(1/2), 1]$ respectively. Then

Lemma 1 *$d\mu := k'(x)dx$ is $T-$invariant.*

Pf. Let P_T be the Frobenius-Perron operator:

$$P_T f(x) := \frac{d}{dx} \int_{T^{-1}([0,x])} f(s) ds$$

It suffices to show that $P_T k' = k'$. But

$$P_T k'(x) = \frac{d}{dx}\left(\int_0^{L^{-1}(x)} k'(s)ds + \int_{R^{-1}(x)}^{1} k'(s)ds \right)$$

$$= \frac{d}{dx} \int_{k(0)}^{k\circ L^{-1}(x)} dt + \frac{d}{dx} \int_{k\circ R^{-1}(x)}^{1} dt$$

$$= \frac{1}{2}k'(x) + \frac{1}{2}k'(x) = k'(x)$$

Corollary

$$k'(x) = \frac{k'(L^{-1}(x))}{|L'(L^{-1}(x))|} + \frac{k'(R^{-1}(x))}{|R'(R^{-1}(x))|}$$

One can verify that g_m satisfies an identity similar to that in the preceeding corollary:

$$(**) \qquad g'_{m+1}(x) = \frac{g'_m(L^{-1}(x))}{|L(L^{-1}(x))|} + \frac{g'_m(R^{-1}(x))}{|R(R^{-1}(x))|}$$

With these two identities, it is easy to prove lemma 2. Repeated application of (**) also yields:

Lemma 3

$$g'_n(x) = \sum_{\langle f_i \rangle} g'_0(f_1^{-1} \circ f_2^{-1} \circ \ldots \circ f_n^{-1}) \left| \frac{d}{dx}\left(f_1^{-1} \circ f_2^{-1} \circ \ldots \circ f_n^{-1}\right) \right|$$

where the sum is over all 2^n *sequences* $\langle f_i \rangle_{i=1}^{n}$ *with* $f_i = L$ or R.

We shall need a few more technical lemmas. Their proofs are omitted because they are tedious and straight forward.

Lemma 4 *Let* $\Pi := \left\{ f_1^{-1} \circ f_2^{-1} \circ \ldots \circ f_n^{-1}(x) \ : f_i = L \text{ or } R \right\}$. *For* $f \in \Pi$, *let* $I_f = f([0,1])$ *be the range of* f. *Then*

(a) *Each* $f \in \Pi$ *is monotonic on* $[0,1]$

(b) $\bigcup_{f \in \Pi} I_f = [0,1]$

(c) *For* $f \neq g$, $I_f \cap I_g$ *is empty or a single point.*

Observation: The I_f's form a partition of the unit interval into 2^n subintervals.

Now for any g, let $M(g) := \max_{x\in[0,1]}(g(x))$, and let $m(g) := \min_{x\in[0,1]}(g(x))$. Then we have

Lemma 5 *If $f = f_1 \circ f_2^{-1} \circ \ldots f_n^{-1} \in \Pi$, then*

(a) $\frac{m(k')}{M(k')}2^{-n} \le |f'(x)| \le \frac{M(k')}{m(k')}2^{-n}$

(b) $|f''(x)| \le 2^{-n}\left(\frac{M(k'')}{m(k')} + \frac{M^2(k')M(k'')}{m^3(k')}\right)$

(c) *Let $|I_f| = |f(1) - f(0)|$ be the length of the subinterval correspo*

Then

$$|I_f| \le 2^{-n}\left(\frac{M(k')}{m(k')}\right)$$

Finally we have:

Lemma 6

$$|g_n''(x)| \le 2^{-n}\frac{M(g_0')M^2(k')}{m^2(k')} + M(g_0)\left(\frac{M(k'')}{m(k')} + \frac{M^2(k')M(k'')}{M^3(k')}\right)$$

Now at last we can carry out the proof of theorem 1. Choose constants A_0 and B_0 that

$$A_0 k'(x) < g_0'(x) < B_0 k'(x) \quad \forall x$$

Let $\Phi_n(x) := g_n'(x) - A_0 k'(x)$. As in lemma 3, we have

$$\Phi_n(x) := \sum_{f \in \Pi} \Phi_0(f(x))|f'(x)|$$

It follows from lemma 5 that

$$\Phi_n(x) > \frac{m^2(k')}{M^2(k')}\sum_f \Phi_0(f(x))|I_f|$$

On the other hand, the mean value theorem implies that

$$\int_0^1 \Phi_0(\xi)d\xi = \sum_f \Phi_0(c_f)|I_f|$$

where $c_f \in I_f$. Thus

$$\Phi_n(x) > \frac{m^2(k')}{M^2(k')}\int_0^1 \Phi_0(\xi)d\xi + \frac{m^2(k')}{M^2(k')}\sum_f(\Phi_0(f(x)) - \Phi_0(c_f))|I_f|$$

Since $|\Phi_0(f(x)) - \Phi_0(c_f)| < (M(g_0'') + A_0 M(k''))|I_f|$, it follows that

$$g_n'(x) > A_0 k'(x) + \frac{m^2(k')}{M^2(k')}\int_0^1 \Phi_0(\xi)d\xi - \frac{m^2(k')}{M^2(k')}\left(M(g_0'') + A_0 M(k'')\right)\sum_f |I_f|^2$$

But by lemma 5, $|I_f| < \frac{M(k')}{m(k')}2^{-n}$, so we have

$$g_n'(x) > A_0k'(x) + \frac{m^2(k')}{M^2(k')}\int_0^1 \Phi_0(\xi)d\xi - 2^{-n}\frac{m(k')}{M(k')}\Big(M(g_0'') + A_0M(k'')\Big)\sum_f |I_f|$$

If we let $c := \frac{m^2(k')}{M^3(k')}\int_0^1 \Phi_0(\xi)d\xi$, let $b := \frac{1}{M^2(k')}(M(g_0') + A_0M(k''))$, and set $A_1 = A_0 + c - b2^{-n}$, then we have already proved that

$$g_n'(x) > A_1k'(x)$$

In the same way, one can define $\Psi_n(x) = B_0k'(x) - g_n'(x)$. One then finds that

$$g_n'(x) < (B_0 - c' + b2^{-n})k'(x) := B_1k'(x)$$

where $c' = \frac{m^2(k')}{M^3(k')}\int_0^1 \Psi_0(t)dt$. Furthermore, for n sufficiently large, we have $A_0 < A_1 < B_1 < B_0$, and $B_1 - A_1 = B_0 - A_0 - (c' + c) + 2b2^{-n}$, where $(c + c') = \frac{m^2(k')}{M^3(k')}\int_0^1(\Phi_0(t) + \Psi_0(t))dt = \frac{m^2(k')}{M^3(k')}(B_0 - A_0)$.

Now begin again with $\langle x_k^{(n)}\rangle$ as the initial sequence, and repeat all the same arguments. Thus we begin with g_n instead of g_0, and by similar reasoning arrive at the inequality $A_2k'(x) < g_{2n}'(x) < B_2k'(x)$ where $A_1 < A_2 < B_2 < B_1$ and $B_2 - A_2 \le \delta(B_1 - A_1) + 2b_12^{-n}$. Here $b_1 := \frac{M(g_n') + A_1M(k')}{M(k')}$ and $\delta := 1 - \frac{m^2(k')}{M^3(k')}$ is a fixed constant in $(0, 1)$.

Continuing in the same way, we obtain recursively

$$A_rk'(x) < g_{2r}'(x) < B_rk'(x)$$

$$A_{r-1} < A_r < B_r < B_{r-1}$$

and

$$B_r - A_r \le \delta(B_{r-1} - A_{r-1}) + 2b_{r-1}2^{-n}$$

where

$$b_{r-1} := \frac{M(g_{n(r-1)}') + A_{r-1}M(k')}{M(k')}$$

Furthermore, lemma 6 implies that b_r is bounded above by a fixed (independent of r) constant T. Then

$$B_n - A_n \le (B - A)\delta^n + 2T2^{-n}(\delta^{n-1} + \delta^{n-2} + \dots\delta + 1)$$

Thus

$$0 \le B_n - A_n \ll e^{-\lambda n}$$

where λ is a constant depending only on g_0 and k.

The limit $\lim_{r\to\infty} A_r = \lim_{r\to\infty} B_r = L$ exists. We claim that $L = 1$:

$$L = L\int_0^1 k'(x) = \int_0^1 g'_{n^2}(x)dx + O(e^{-\lambda n})$$

$$= 1 + O(e^{-\lambda n})$$

Thus

$$|g_N - k'(x)| \ll e^{-\lambda\sqrt{N}}$$

as $N = n^2 \to \infty$. The theorem then follows, by an easy argument, from lemma 1.∎

III. OTHER TRANSFORMATIONS

One important feature of theorem 1 is that the limit distribution g is essentially independent of the initial distribution g_0. This is not a special property of tent functions; it seems to be true for many different choices of T. To emphasize this point, we consider two more examples. The first is the map $T : x \mapsto \{\frac{1}{\{x\}}\}$ Assume that $x_n \notin Z$ and that g_0 is C^2 It is well known (see Kuipers and Niederreiter(1974)) that

$$g_1(x) = \sum_{k=1}^{\infty}\left(g_0(\frac{1}{k}) - g_0(\frac{1}{k+x})\right)$$

Then

$$g'_1(x) = \sum_{k=1}^{\infty} g'_0(\frac{1}{k+x})\frac{1}{(k+x)^2}$$

Furthermore g_1 is C^2, so we can repeat. In this way one obtains the identity

$$g'_{n+1}(x) = \sum_{k=1}^{\infty} g'_n(\frac{1}{k+x})\frac{1}{(k+x)^2}$$

This functional equation was considered by Kuzmin in his treatment of Gauss' problem in the metric theory of continued fractions (see Khintchine(1974)). His results imply the following

Theorem 2 $\qquad g_n(x) = \frac{\log(1+x)}{\log(2)} + O(e^{-c\sqrt{n}})$

uniformly for $x \in [0,1]$ (here c is an absolute constant)

Finally we consider the classical Bernoulli shift $x \mapsto \{2x\}$. We show that $g_n(x) = x + O(2^{-n})$. More precisely, we prove

Theorem 2 *Let $M := |g''_0|_\infty$ Then for all n and x*

$$|g_n(x) - x| \le \frac{M}{2^n}$$

Proof. We begin by establishing the identity

$$g_{n+1}(x) = g_n(\frac{x}{2}) + g_n(\frac{x+1}{2}) - g_n(\frac{1}{2}) \qquad *(1)*$$

For $n = 0$ we have

$$g_1(x) := \lim_{N\to\infty} \frac{1}{N}\#\Big\{n \le N : x_n^1 \in [0,x)\Big\}$$

$$= \lim_{N\to\infty} \frac{1}{N}\Big(\#\Big\{n \le N : x_n^0 \in [0,\frac{x}{2})\Big\} + \#\Big\{n \le N : x_n^0 \in [\frac{1}{2},\frac{x+1}{2})\Big\}\Big)$$

$$= g_0(\frac{x}{2}) - g_0(0) + g_0(\frac{x+1}{2}) - g_0(\frac{1}{2})$$

$$= g_0(\frac{x}{2}) + g_0(\frac{x+1}{2}) - g_0(\frac{1}{2})$$

This identity, and the fact that g_0 is C^2, together imply that g_1 is C^2. Hence we can repeat the argument for $n = 1$, and recursively for $n > 1$. In this way one obtains the identity *(1)*.

Next let $\Delta_n(x) := g_n(x) - x$. We shall prove that Δ_n is small. By equation *(1)* we have

$$\Delta_{n+1}(x) = \Delta_n(\frac{x}{2}) + \Delta_n(\frac{x+1}{2}) - \Delta_n(\frac{1}{2})$$

Differentiating yields

$$\Delta_n'(x) = \frac{1}{2}\Big(\Delta_n'(\frac{x}{2}) + \Delta_n'(\frac{x+1}{2})\Big)$$

Using this identity and mathematical induction, one obtains (for $n \ge 1$):

$$\Delta_n'(x) = \sum_{\epsilon_i = 0 \text{ or } 1} \frac{1}{2^n}\Delta_0'\Big(\frac{x + \epsilon_0 2^0 + \epsilon_1 2^1 + \ldots + \epsilon_{n-1}2^{n-1}}{2^n}\Big)$$

On the other hand, by the mean value theorem, one has

$$\int_0^1 \Delta_0'(t)dt = \sum_{\vec{\epsilon}} \frac{1}{2^n}\Delta_n'\big(\xi_{\vec{\epsilon}}\big)$$

where $\xi_{\vec{\epsilon}}$ is some point in the interval of length 2^{-n} whose left endpoint is $\frac{\sum \epsilon_i 2^i}{2^n}$ Thus we have

$$|\Delta_n'(x) - \int_0^1 \Delta_0'(t)dt| =$$

$$\Big|\sum_{\vec{\epsilon}} \frac{1}{2^n}\Delta_0'(\frac{x + \epsilon_0 2^0 + \epsilon_1 2^1 + \ldots + \epsilon_{n-1}2^{n-1}}{2^n}) - \sum_{\vec{\epsilon}} \frac{1}{2^n}\Delta_n'(\xi_{\vec{\epsilon}})\Big|$$

$$\le \frac{1}{2^n}\sum_{\vec{\epsilon}}\Big|\Delta_0'(\frac{x + \epsilon_0 2^0 + \epsilon_1 2^1 + \ldots + \epsilon_{n-1}2^{n-1}}{2^n}) - \Delta_n'(\xi_{\vec{\epsilon}})\Big|$$

Hence we have

$$\left|\Delta_n'(x) - \int_0^1 \Delta_0'(t)dt\right| \leq \frac{M}{2^n}$$

where $M = |g_0''|_\infty$ Observe that $\int_0^1 \Delta_0'(t)dt = \Delta(1) - \Delta(0) = 0$. Thus

$$|\Delta_n'(x)| \leq \frac{M}{2^n}$$

and consequently

$$\left|g_n(x) - x\right| = |\Delta_n(x)| \leq \frac{M}{2^n}.$$

■

CONCLUSION

For several choices of T, we proved that $g_n \to g$ uniformly, and that g is un the sense that it is essentially independent of the initial distribution g_0. To the knowledge, no one has studied this universal feature of the invariant measure with a transformation T.

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