Math 235, Fall 2012: Solutions to homework 1

Ex 1 What is the maximum size of a subset of $\{3, 11, 19, 27, \dots, 139, 147, 155\}$ for which no two elements add up to 158?

→ Consider the 10 subsets of $\{3, 11, 19, \dots, 139, 147, 155\}$ defined as

$$S_1 = \{3, 155\},\ S_2 = \{11, 147\}, \dots,\ S_{10} = \{75, 83\}.$$

Note that two elements add up to 158 if and only if they belong to the same S_i. By the pigeonhole principle, if 11 elements are considered, two of them must belong to a common S_i, so their sum is 158. Thus, a subset of $\{3, 11, 19, \dots, 139, 147, 155\}$ for which no two elements add up to 158 must have size ≤ 10.

On the other hand, $\{3, 11, \dots, 75\}$ is a subset of size $= 10$ for which no two elements add up to 158. Therefore,

<u>desired maximum size = 10</u>.

Ex 2 Prove that at any party there are two guests with the same number of friends present.

→ Let n denote the number of guest, and let m_i denote the number of friends of the i-th guest, $i \in [1:n]$. A priori, the m_i all belong to $[0:n-1]$, but in fact they all belong to $[0:n-2]$ or $[1:n-1]$ — this translates the fact that if there is a guest with no friends, then no guest can be friend with everybody, and if there is a guest who is friend with everybody, then there is no guest with no friends. In any case, the n integers $m_1, \dots, m_n$ all belong to a set of size $n-1$, so by the pigeonhole principle, two of them must be equal. This is what needed to be shown.

<u>Ex3</u> How many numbers between 1 and 2012 (inclusive) are divisible neither by 2, 3, 5, nor 7?

- Let us introduce the sets

$S_2 := \{ n \in [1:2012] : n \text{ is divisible by } 2\}$,

$S_3 := \{ n \in [1:2012] : n \text{ is divisible by } 3\}$,

$S_5 := \{ \text{———————————} 5\}$,

$S_7 := \{ \text{———————————} 7\}$.

We want to know the size of $[1:2012] \setminus (S_2 \cup S_3 \cup S_5 \cup S_7)$. By the inclusion-exclusion principle, we have:

$$
\begin{aligned}
|S_2 \cup S_3 \cup S_5 \cup S_7| = {} & |S_2| + |S_3| + |S_5| + |S_7| \\
& - |S_2 \cap S_3| - |S_2 \cap S_5| - |S_2 \cap S_7| - |S_3 \cap S_5| - |S_3 \cap S_7| - |S_5 \cap S_7| \\
& + |S_3 \cap S_5 \cap S_7| + |S_2 \cap S_5 \cap S_7| + |S_2 \cap S_3 \cap S_7| + |S_2 \cap S_3 \cap S_5| \\
& - |S_2 \cap S_3 \cap S_5 \cap S_7| .
\end{aligned}
$$

We observe that : $S_2 = \{2, 4, 6, \dots, 2012\}$, $|S_2| = 1006$,

$S_3 = \{3, 6, 9, \dots, 2010 = 3 \times 670\}$, $|S_3| = 670$,

$S_5 = \{5, 10, 15, \dots, 2010 = 5 \times 402\}$, $|S_5| = 402$,

$S_7 = \{7, 14, 21, \dots, 2009 = 7 \times 287\}$, $|S_7| = 287$.

Moreover, we have: $S_2 \cap S_3 = \{ n \in [1:2012] : n \text{ is divisible by } 2 \times 3 = 6\}$, so

$S_2 \cap S_3 = \{6, 12, 18, \dots, 2010 = 6 \times 335\}$, $|S_2 \cap S_3| = 335$,

likewise: $S_2 \cap S_5 = \{10, 20, 30, \dots, 2010 = 10 \times 201\}$, $|S_2 \cap S_5| = 201$,

$S_2 \cap S_7 = \{14, 28, 42, \dots, 2002 = 14 \times 143\}$, $|S_2 \cap S_7| = 143$,

$S_3 \cap S_5 = \{15, 30, 45, \dots, 2010 = 15 \times 134\}$, $|S_3 \cap S_5| = 134$,

$S_3 \cap S_7 = \{21, 42, 63, \dots, 1995 = 21 \times 95\}$, $|S_3 \cap S_7| = 95$,

$S_5 \cap S_7 = \{35, 70, 105, \dots, 1995 = 35 \times 57\}$, $|S_5 \cap S_7| = 57$.

Continuing in this way, we get $S_3 \cap S_5 \cap S_7 = \{ n \in [1:2012 : n \text{ divisible by } 3 \times 5 \times 7 = 105\}$,

$S_3 \cap S_5 \cap S_7 = \{105, 210, \dots, 1995 = 105 \times 19\}$, $|S_3 \cap S_5 \cap S_7| = 19$,

likewise: $S_2 \cap S_5 \cap S_7 = \{70, 140, \dots, 1960 = 70 \times 28\}$, $|S_2 \cap S_5 \cap S_7| = 28$,

$S_2 \cap S_3 \cap S_7 = \{42, 84, \dots, 1974 = 42 \times 47\}$, $|S_2 \cap S_3 \cap S_7| = 47$,

$S_2 \cap S_3 \cap S_5 = \{30, 60, \dots, 2010 = 30 \times 67\}$, $|S_2 \cap S_3 \cap S_5| = 67$.

Finally, $S_2 \cap S_3 \cap S_5 \cap S_7 = \{210, 420, \dots, 1890 = 210 \times 9\}$, $|S_2 \cap S_3 \cap S_5 \cap S_7| = 9$.

Putting everything together, we conclude that

$$|S_2 \cup S_3 \cup S_5 \cup S_7| = (1006 + 670 + 402 + 287)$$
$$- (335 + 201 + 143 + 134 + 95 + 57) + (19 + 28 + 47 + 67) - 9$$
$$= 2365 - 965 + 161 - 9 = 1552,$$

and the solution to the problem is $2012 - 1552 = \boxed{460}$

<u>Ex4</u> Let S be a subset of $[1:100]$ of size 10. Show that there are two subsets of S for which the sum of the elements are the same.

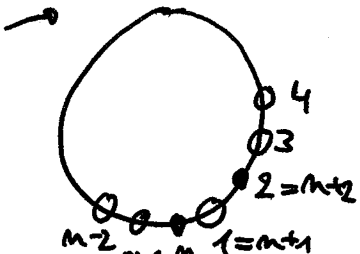
→ The largest possible value for the sum of the elements of a subset of S is: $100 + 99 + \dots + 91 = (100+91) + (99+92) + \dots + (96+95) = 5 \times 191 = 955$, so there are at most 955 possible values that can be taken by the sums. However, there are $2^{10} - 1 = 1023$ subsets of S. By the pigeonhole principle, there must be two different subsets yielding the same value for the sum of their elements.

<u>Ex5</u> Calculate the generating function of the sequence $\frac{D_n}{n!}$.

→ Using the expression for D_n given in the notes, we obtain

$$\sum_{n \geq 0} \frac{D_n}{n!} z^n = \sum_{n \geq 0} \sum_{r=0}^{n} \frac{(-1)^r}{r!} z^n = \sum_{r=0}^{\infty} \sum_{n \geq r} \frac{(-1)^r}{r!} z^n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} z^r \sum_{n \geq r} z^{n-r}$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} z^r \times \frac{1}{1-z} = \boxed{\frac{e^{-z}}{1-z}}.$$

Ex6 Show that if a coller made of n pearls has more than $(k-1)n/k$ white pearls, then there is a string of k consecutive white pearls.



We count the number of white pearls by counting their number in each of the strings $[1:k], [2:k+1], \dots, [n : k+n-1 \equiv k-1]$. Note that in this way, each pearl is counted k times. Thus,

$$\sum_{i=1}^{n} \text{number of white pearls in } [i : i+k-1] = k \times \text{number of white pearls} > (k-1)n.$$

Therefore, there is an $i \in [1:n]$ such that the number of white pearls in $[i : i+k-1]$ is $> k-1$, i.e., it is $= k$. The string $[i : i+k-1]$ is made of k consecutive white pearls.

Ex7 Given real numbers $x_1, x_2, \dots, x_n$, prove that

$$\max\{x_1, x_2, \dots, x_n\} = \sum_{r=1}^{n} (-1)^{r-1} \sum_{1 \le i_1 < \dots < i_r \le n} \min\{x_{i_1}, \dots, x_{i_r}\}.$$

Let m be a number such that $m \le \min\{x_1, \dots, x_n\}$. Let us apply the inclusion-exclusion principle to the set $S_i := [m, x_i]$, $i \in [1:n]$:

$$|S_1 \cup S_2 \cup \dots \cup S_n| = \sum_{r=1}^{n} (-1)^{r-1} \sum_{1 \le i_1 < \dots < i_r \le n} |S_{i_1} \cap \dots \cap S_{i_r}|.$$

We note that $S_1 \cup S_2 \cup \dots \cup S_n = [m, \max\{x_1, x_2, \dots, x_n\}]$, so that $|S_1 \cup S_2 \cup \dots \cup S_n| = \max\{x_1, x_2, \dots, x_n\} - m$, and $\{S_{i_1} \cap \dots \cap S_{i_r}\} = [m, \min\{x_{i_1}, \dots, x_{i_r}\}]$ so that $|S_{i_1} \cap \dots \cap S_{i_r}| = \min\{x_{i_1}, \dots, x_{i_r}\} - m$. Thus,

$$\circledast \quad \max\{x_1, x_2, \dots, x_n\} - m = \sum_{r=1}^{n} (-1)^{r-1} \sum_{1 \le i_1 < \dots < i_r \le n} \left(\min\{x_{i_1}, \dots, x_{i_r}\} - m\right)$$

$$= \sum_{i=1}^{n} (-1)^{r-1} \sum_{1 \le i_1 < \dots < i_r \le n} \min\{x_{i_1}, \dots, x_{i_r}\} - m \sum_{r=1}^{n} (-1)^{r-1} \binom{n}{r}.$$

It now suffices to remark that $\sum_{r=1}^{n} (-1)^{r-1} \binom{n}{r} = -\sum_{r=0}^{n} (-1)^{r} \binom{n}{r} + 1 = -(1-1)^n + 1 = 1$ and to add m to both sides of $\circledast$.