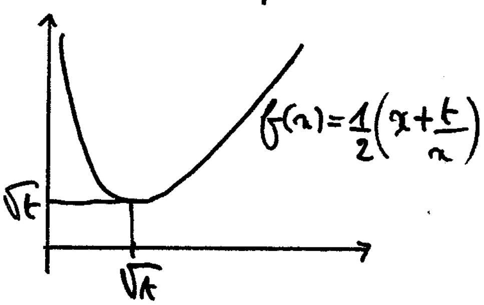
Ex1 For $t>0$, define (x_n) by $x_0>0$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{t}{x_n}\right)$, $n \geq 0$.

Does (x_n) converge, and if so what is its limit?

→ If the limit ℓ exists, we must have $\ell = \frac{1}{2}\left(\ell + \frac{t}{\ell}\right)$, i.e., $\ell^2 = t$,

and $\ell \geq 0$ (because each x_n is positive), so $\underline{\ell = \sqrt{t}}$

Let us prove that the sequence (x_n) actually does converge to $\sqrt{\ell}$.



Note that $x_{n+1} = f(x_n) \geq \sqrt{t}$ for any $n \geq 0$.

Note also that

$$x_{n+1} - x_n = \frac{1}{2}\Big(\underbrace{\frac{t}{x_n} - x_n}_{\text{decreasing in } x_n}\Big) \leq \frac{1}{2}\left(\frac{t}{\sqrt{t}} - \sqrt{t}\right) = 0 \text{ for any } n \geq 1,$$

so that the sequence $(x_n)_{n\geq 1}$ is decreasing. It is bounded from below (by $\sqrt{t}$), hence <u>it does converge.</u>

Ex2 Let (u_n) be a sequence converging to u, prove that the Cesaro mean $\frac{1}{n}\sum_{k=1}^{n} u_k$ also converge to u.

→ This is an ε-argument. We need to show:

$\forall \varepsilon > 0$, $\exists N \geq 0$ such that: $n \geq N \Longrightarrow \left| \frac{1}{n}\sum_{k=1}^{n} u_k - u \right| < \varepsilon$.

~~Since~~ Consider $\varepsilon > 0$, since $u_n \to u$, we know that:

$\exists N_0 \geq 0$ such that: $n \geq N_0 \Longrightarrow |u_n - u| < \varepsilon/2$.

Thus, for $n \geq N_0$:

$$\left| \frac{1}{n}\sum_{k=1}^{n} u_k - u \right| = \frac{1}{n}\left|\sum_{k=1}^{n}(u_k - u)\right| \leq \frac{1}{n}\left(\sum_{k=1}^{N_0}|u_k - u| + \sum_{k=N_0+1}^{n}|u_k - u|\right)$$

$$\leq \frac{1}{n}\sum_{k=1}^{N_0}|u_k - u| + \frac{n - N_0}{n}\,\varepsilon/2$$

Since $\frac{1}{n}\sum_{k=1}^{N_0}|u_k - u| \xrightarrow[n\to\infty]{} 0$, there exists $N_1 \geq 0$ such that: $n \geq N_1 \Rightarrow \frac{1}{n}\sum_{k=1}^{N_0}|u_k - u| < \varepsilon/2$.

Now, for ~~$n\geq$~~ $n \geq N := \max(N_0, N_1)$, we have

$$\underline{\left| \frac{1}{n}\sum_{k=1}^{n} u_k - u \right| \leq \varepsilon/2 + \underbrace{\frac{n-N_0}{n}}_{<1}\,\varepsilon/2 < \varepsilon},$$

as desired.

Ex3 Prove that $\sum_n \frac{1}{n\ln^a(n)}$ converges iff $a>1$.

→ If $a \le 0$, $\frac{1}{n\ln^a(n)} \geq \frac{1}{n}$ for $n \geq 3$, so $\sum \frac{1}{n\ln^a(n)}$ diverges.

Assume now that $a>0$, and use that $\frac{1}{x\ln^a(x)}$ is decreasing to write

$$\frac{1}{(n+1)\ln^a(n+1)} \leq \int_n^{n+1} \frac{1}{x\ln^a(x)}\,dx \leq \frac{1}{n\ln^a(n)}$$

Now notice that

$$\int_X^\infty \frac{1}{x\ln^a(x)}\,dx \overset{a\neq 1}{=} \left[\frac{\ln^{1-a}(x)}{1-a}\right]_X^\infty = \begin{cases} +\infty & \text{if } a<1 \\ <+\infty & \text{if } a>1 \end{cases}$$

$$\overset{a=1}{=} \Big[\ln(\ln(x))\Big]_X^\infty = +\infty$$

Thus, for $\underline{a \leq 1}$, $\sum_{n\geq 3} \frac{1}{n\ln^a(n)} \geq \sum_{n\geq 3}\int_n^{n+1}\frac{1}{x\ln^a(x)}\,dx = \int_3^\infty \frac{1}{x\ln^a(x)}\,dx$ $\underline{\text{diverges}}$

and for $\underline{a>1}$, $\sum_{n\geq 3}\frac{1}{n\ln^a(n)} = \sum_{n\geq 2}\frac{1}{(n+1)\ln^a(n+1)} \leq \sum_{n\geq 2}\int_n^{n+1}\frac{1}{x\ln^a(x)}\,dx = \int_2^\infty \frac{dx}{x\ln^a(x)}$ $\underline{\text{converges.}}$

Ex4 Given $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$, find $\sum_{n=0}^\infty \frac{1}{(2n+1)^2}$.

→ Note that $\sum_{n=1}^\infty \frac{1}{n^2} = \sum_{k=1}^\infty \frac{1}{(2k)^2} + \sum_{k=0}^\infty \frac{1}{(2k+1)^2} = \frac{1}{4}\sum_{k=1}^\infty \frac{1}{k^2} + \sum_{k=0}^\infty \frac{1}{(2k+1)^2}$

hence $\sum_{k=0}^\infty \frac{1}{(2k+1)^2} = \frac{3}{4}\sum_{n=1}^\infty \frac{1}{n^2} = \frac{3}{4}\frac{\pi^2}{6}$, i.e. $\underline{\sum_{k=0}^\infty \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}}$.

Ex6 Prove that $\sum_{k=1}^\infty \frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}$ converges and find its value

→ Since $0 \leq \frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}} \leq \frac{1}{2k^{3/2}}$, and since $\sum_{k=1}^\infty \frac{1}{k^{3/2}}$ converges,

the series $\underline{\sum_{k=1}^\infty \frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}}$ converges, too.

To find its value, notice that

$$\frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}} = \frac{1}{\sqrt{k(k+1)}\left(\sqrt{k+1}-\sqrt{k}\right)} = \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}} = \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}},$$

so that $\underline{\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}} = \left(\frac{1}{\sqrt{1}} - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right) + \cancel{\dots} \quad \underline{= 1.}$

<u>Ex7</u> Evaluate $\sum_{k=0}^{m} \cos\left(\frac{k\pi}{m}\right)$ and $\sum_{k=0}^{m} \sin\left(\frac{k\pi}{m}\right)$.

→ Observe that $\sum_{k=0}^{m} \cos\left(\frac{k\pi}{m}\right) = \sum_{k=0}^{m} \mathrm{Re}\left(e^{i\frac{k\pi}{m}}\right) = \mathrm{Re}\left(\sum_{k=0}^{m} e^{i\frac{k\pi}{m}}\right)$

$$\sum_{k=0}^{m} \sin\left(\frac{k\pi}{m}\right) = \sum_{k=0}^{m} \mathrm{Im}\left(e^{i\frac{k\pi}{m}}\right) = \mathrm{Im}\left(\sum_{k=0}^{m} e^{i\frac{k\pi}{m}}\right).$$

We calculate

$$\sum_{k=0}^{m} e^{i\frac{k\pi}{m}} = \frac{1-e^{i\frac{\pi}{m}(m+1)}}{1-e^{i\frac{\pi}{m}}} = \frac{1+e^{i\frac{\pi}{m}}}{1-e^{i\frac{\pi}{m}}}$$

$$= \frac{e^{\frac{i\pi}{2m}}}{e^{\frac{i\pi}{2m}}} \frac{e^{-\frac{i\pi}{2m}}+e^{\frac{i\pi}{2m}}}{e^{-\frac{i\pi}{2m}}-e^{\frac{i\pi}{2m}}} = \frac{2\cos(\pi/2m)}{-2i\sin(\pi/2m)} = i\cot(\pi/2m).$$

Identifying real and imaginary parts, we conclude:

$$\underline{\sum_{k=0}^{m} \cos\left(\frac{k\pi}{m}\right) = 0} \quad , \quad \underline{\sum_{k=0}^{m} \sin\left(\frac{k\pi}{m}\right) = \cot\left(\frac{\pi}{2m}\right).}$$

<u>Ex8</u> Consider (u_k) defined by $u_{k+1} = r u_k + d$, $k \geq 0$. Find an explicit form for the general term.

→ Rewrite the recurrence relation as $u_{k+1} + c = r(u_k + c) + d - rc + c$ and choose c to make the additive term disappear, i.e. $c = \frac{d}{r-1}$.

Thus, $\left(u_{k+1} + \frac{d}{r-1}\right) = r\left(u_k + \frac{d}{r-1}\right)$ shows that $u_k + \frac{d}{r-1}$ is a geometric progression. Therefore, $u_k + \frac{d}{r-1} = r^k\left(u_0 + \frac{d}{r-1}\right)$, or: $\underline{u_k = r^k u_0 - \frac{d}{r-1}\left(1 - r^k\right).}$

Ex 9 For the power series expansion $\frac{1}{1-2z-z^2} = \sum_{n=0}^{\infty} a_n z^n$,

prove that, for all $n \geq 0$, $a_n^2 + a_{n+1}^2 = a_m$ for some $m \geq 0$.

→ We shall determine the a_n's exactly. First, remark that

$$1-2z-z^2 = (\sqrt{2}+1-z)(\sqrt{2}-1+z) = \underbrace{(\sqrt{2}+1)(\sqrt{2}-1)}_{=1}\left(1-\frac{z}{\sqrt{2}+1}\right)\left(1+\frac{z}{\sqrt{2}-1}\right)$$

$$= (1-(\sqrt{2}-1)z)(1+(\sqrt{2}+1)z).$$

We deduce the partial fraction decomposition

$$\frac{1}{1-2z-z^2} = \frac{(2-\sqrt{2})/4}{1-(\sqrt{2}-1)z} + \frac{(2+\sqrt{2})/4}{1+(\sqrt{2}+1)z},$$

so that

$$\frac{1}{1-2z+z^2} = \frac{2-\sqrt{2}}{4}\sum_{n=0}^{\infty}(\sqrt{2}-1)^n z^n + \frac{2+\sqrt{2}}{4}(\sqrt{2}+1)^n(-1)^n z^n$$

$$= \sum_{n=0}^{\infty} \frac{(2-\sqrt{2})(\sqrt{2}-1)^n + (2+\sqrt{2})(-1)^n(\sqrt{2}+1)^n}{4} z^n$$

$$= \sum_{n=0}^{\infty} \frac{\sqrt{2}\left[(\sqrt{2}-1)^{n+1} - (-1)^{n+1}(\sqrt{2}+1)^{n+1}\right]}{4} z^n$$

We obtain

$$a_n = \frac{(\sqrt{2}-1)^{n+1} - (-1)^{n+1}(\sqrt{2}+1)^{n+1}}{2\sqrt{2}}$$

Thus

$$a_n^2 = \frac{1}{8}\left((\sqrt{2}-1)^{2n+2} + (\sqrt{2}+1)^{2n+2} - 2(-1)^{n+1}\underbrace{\left((\sqrt{2}-1)(\sqrt{2}+1)\right)^{n+1}}_{=1}\right)$$

and

$$a_{n+1}^2 = \frac{1}{8}\left((\sqrt{2}-1)^{2n+4} + (\sqrt{2}+1)^{2n+4} - 2(-1)^{n+1}\right).$$

It follows that

$$a_n^2 + a_{n+1}^2 = \frac{1}{8}\left((\sqrt{2}-1)^{2n+2} + (\sqrt{2}-1)^{2n+4} + (\sqrt{2}+1)^{2n+2} + (\sqrt{2}+1)^{2n+4}\right)$$

$$= \frac{1}{8}\left((\sqrt{2}-1)^{2n+3}\underbrace{\left(\frac{1}{\underbrace{\sqrt{2}-1}_{=\sqrt{2}-1}} + \sqrt{2}-1\right)}_{=2\sqrt{2}} + (\sqrt{2}+1)^{2n+3}\underbrace{\left(\frac{1}{\underbrace{\sqrt{2}+1}_{=\sqrt{2}-1}} + \sqrt{2}+1\right)}_{=2\sqrt{2}}\right)$$

$$= \frac{1}{2\sqrt{2}}\left((\sqrt{2}-1)^{2n+3} + (\sqrt{2}+1)^{2n+3}\right).$$

Finally: $\boxed{a_n^2 + a_{n+1}^2 = a_{2n+2}}$

Ex 10 Partition $\mathbb{N}$ into the sets $\{1\}, \{2,3\}, \{4,5,6\}, \{7,8,9,10\}, \dots$

Find a simple expression for the sum of the integers in the n-th set.

→ The first element in the n-th set is $1+2+\dots+(n-1)+1 = \frac{(n-1)n}{2}+1$,

the second element is $\frac{(n-1)n}{2}+2$, until the last element which is $\frac{(n-1)n}{2}+n$.

Thus, the sum of integers in the n-th set is:

$$\sum_{k=1}^{n}\left(\frac{(n-1)n}{2}+k\right) = \frac{(n-1)n^2}{2} + \frac{n(n+1)}{2} = \frac{n}{2}\left(n^2-n+n+1\right) = \boxed{\frac{n(n^2+1)}{2}}$$

Ex 11 If $\sum a_n$ is convergent with $a_n \geq 0$, prove that $\sum \sqrt[n]{a_1 a_2 \cdots a_n}$ is also convergent.

→ We will use the inequality between geometric and arithmetic means

$$\sqrt[n]{u_1 u_2 \cdots u_n} \leq \frac{u_1+u_2+\dots+u_n}{n} \quad \text{for } u_1, u_2, \dots, u_n \geq 0,$$

not applied to $u_k = a_k$ but rather to $u_k = k\,a_k$. Thus,

$$\sqrt[n]{a_1 a_2 \cdots a_n} = \sqrt[n]{\frac{a_1}{1} \times \frac{2 \times a_2}{2} \times \dots \times \frac{n \times a_n}{n}} \leq \frac{1}{\sqrt[n]{n!}} \quad \frac{a_1+2a_2+\dots+na_n}{n}$$

The Stirling approximation $n! \underset{n\to\infty}{\sim} \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \geq \left(\frac{n}{e}\right)^n$ gives

$n! \geq c\left(\frac{n}{e}\right)^n$ for some constant $c>0$ and all $n \geq 1$. Then

$\sqrt[n]{n!} \geq c^{1/n}\,\frac{n}{e} \geq d\,\frac{n}{e}$ for some constant $d>0$, since $c^{1/n} \xrightarrow[n\to\infty]{} 1$.

It follows that:

$$\sum_{n=1}^{N}\sqrt[n]{a_1 a_2 \cdots a_n} \leq \sum_{n=1}^{N} \frac{1}{d\frac{n}{e}} \sum_{k=1}^{n} \frac{k a_k}{n} = \frac{e}{d}\sum_{k=1}^{N} k\, a_k \sum_{n=k}^{N}\frac{1}{n^2}.$$

Note that $\sum_{n=k}^{N}\frac{1}{n^2} \leq \sum_{n=k}^{N}\int_{n-1}^{n}\frac{1}{x^2}\,dx = \int_{k-1}^{N}\frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{k-1}^{N} = \frac{1}{k-1}-\frac{1}{N} \leq \frac{1}{k-1}$, so

$$\underline{\sum_{n=1}^{N}\sqrt[n]{a_1 a_2 \cdots a_n}} \leq \frac{e}{d}\sum_{k=1}^{N}\frac{k}{k-1}a_k \leq \frac{2e}{d}\sum_{k=1}^{N} a_k \underline{< +\infty}$$

separate the case $k=1$