# Math 235, Fall 2012: Solutions to homework 3

**Ex1** Prove that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

→ By induction on $n \geq 1$. For $n=1$, LHS $= 1^2 = 1$ and RHS $= \frac{1\times 2\times 3}{6} = 1$, so the base case holds. Suppose the result true up to an integer $n-1$, $n\geq 2$, and let us prove that it is true for $n$, too. We have

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2 = \frac{(n-1)(n)(2n-1)}{6} + n^2 = \frac{n}{6}\left((n-1)(2n-1)+6n\right) = \frac{n}{6}\left(n^2-3n+1+6n\right)$$

$$= \frac{n}{6}\left(n^2+3n+1\right) = \frac{n}{6}(n+1)(2n+1),$$ i.e., the result is true for $n$.

This completes the proof.

**Ex2** Prove that $n(n-1)(n+1)(3n+2)$ is divisible by 24 for all $n\geq 1$.

→ By induction on $n \geq 1$. For $n=1$, the quantity of interest is 0, which is divisible by 24. Suppose now that $q_k := k(k-1)(k+1)(3k+2)$ is divisible by 24 for any $k \leq n-1$, and let us prove that $q_n$ is also divisible by 24. We have

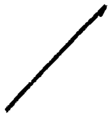
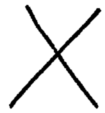
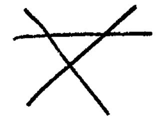
$$q_n - q_{n-1} = n(n-1)(n+1)(3n+2) - (n-1)(n-2)n(3n-1) = (n-1)n\left[(n+1)(3n+2)-(n-2)(3n-1)\right]$$

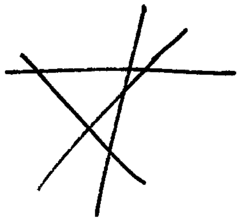
$$= (n-1)n\left[3n^2+5n+2-3n^2+7n-2\right] = (n-1)n\left[12n\right].$$

Since $(n-1)n$ is always divisible by 2 (one of $n-1$ or $n$ is even), we see that $q_n - q_{n-1}$ is divisible by 24, and since $q_{n-1}$ is divisible by 24, we deduce that $q_n$ is divisible by 24. Thus the inductive hypothesis is proved for $n$, and the proof is complete.

**Ex3** Find the number $R(n)$ of regions in which the plane can be divided by $n$ straight lines.

→ We start by experimenting on the small values of $n$:

 $R(1)=2$  $R(2)=4$  $R(3)=7$

 $R(4)=11$

This "suggests" the general expression $\boxed{R(n)=\frac{n^2+n+1}{2}}$ (if an expression of the form $R(n)=an^2+bn+c$ is suspected, then $R(1)$, $R(2)$, and $R(3)$ allow one to determine $a$, $b$, and $c$, and the resulting form can be checked on $R(4)$).

Let us verify this by induction on $n \geq 1$. The base cases have already been verified. We now assume the induction hypothesis true up to $n$, and we aim to prove that it is true for $n+1$, too. By adding an $(n+1)$st line to $n$ lines already considered, this line will intersect (in general) the other ones in $n$ different points, hence creating $n+1$ segments. Each of these segments divides a previous region in two, hence creating $n+1$ new regions. Thus,

$$R(n+1) = R(n) + n+1 = \frac{n^2+n+1}{2} + \frac{2n+2}{2} = \frac{(n+1)^2+(n+1)+1}{2}.$$

This shows that the induction hypothesis is true for $n+1$, and completes the proof.

<u>Ex4</u> Find a closed-form formula for the Fibonacci numbers defined by $F_0=0$, $F_1=1$ and $F_{n+2}=F_{n+1}+F_n$ for $n \geq 2$.

- The roots of $z^2-z-1=0$ are

$$\phi=\frac{1+\sqrt{5}}{2} \quad \text{and} \quad -1/\phi=\frac{1-\sqrt{5}}{2}.$$

Therefore, the Fibonacci numbers can be expressed as

$$F_n = \alpha\,\phi^n + \beta\,(-1/\phi)^n.$$

The values $F_0=0$ and $F_1=1$ impose $\alpha+\beta=0$ and $\alpha\phi-\beta/\phi=1$, i.e.,

$\alpha(\underbrace{\phi + \frac{1}{\phi}}_{=\sqrt{5}}) = 1$. We obtain $\alpha = \frac{1}{\sqrt{5}}$ and $\beta = -\frac{1}{\sqrt{5}}$. Finally,

$$\boxed{F_n = \frac{\phi^n - (1/\phi)^n}{\sqrt{5}}}$$

**Ex 5** Find a formula for $\sum_{k=1}^{n} k^4$, and prove it rigorously.

→ Let us look at the first values of $u_n := \sum_{k=1}^{n} k^4$. We have.

$u_1 = 1$, $u_2 = 17$, $u_3 = 98$.

We are told that $u_n$ has the form $u_n = (an^2 + bn + c)\, n(n+1)(2n+1)$, so

$u_1: \quad 1 = (a+b+c) \times 6 \quad , \; a+b+c = 1/6 \quad , \; a+b+c = 1/6 \; , \; a+b+c = 1/6$

$u_2: \quad 17 = (4a+2b+c) \times 30 \quad , \; 4a+2b+c = \frac{17/5}{6} \; , \; 3a+b = 2/5 \; , \; 3a+b = 2/5$

$u_3: \quad 98 = (9a+3b+c) \times 84 \quad \; 9a+3b+c = \frac{7}{6} \; , \; 5a+b = 3/5 \; , \; 2a = 1/5$

This gives $a = 1/10$, $b = 1/10$, and $c = -1/30$. So let us now prove by induction on $n \geq 1$ that $\boxed{u_n = \frac{(3n^2+3n-1)n(n+1)(2n+1)}{30}}$ (*)

The base cases have just been treated. Let us now assume that (*) holds up to $n-1$, $n \geq 2$, and let us prove that it holds for $n$, too. We have

$$u_n - \frac{(3n^2+3n-1)n(n+1)(2n+1)}{30} = u_{n-1} + n^4 - \frac{(3n^2+3n-1)n(n+1)(2n+1)}{30}$$

$$= \frac{(3(n-1)^2 + 3(n-1) - 1)(n-1)n(2n-1)}{30} + \frac{30n^4}{30} - \frac{(3n^2+3n-1)n(n+1)(2n+1)}{30}$$

$$= \frac{n}{30}\left[(3n^2-3n-1)(n-1)(2n-1) + 30n^3 - (3n^2+3n-1)(n+1)(2n+1)\right]$$

To prove that $[\ ]=0$, instead of expanding everything, we can show that it equals $0$ at $4$ different points (since it is a polynomial in $n$ of degree $3$ — not that the term in $n^4$ is $=0$. We have

$$[\ ]\big|_{n=1} = 0+30-5\times 2\times 3 = 0\,,\quad [\ ]\big|_{n=-1} = 5\times(-2)\times(-3)-30-0 = 0\,,$$

$$[\ ]\big|_{n=1/2} = 0+\frac{30}{8}-\frac{5}{4}\times\frac{3}{2}\times 2 = 0\,,\quad [\ ]\big|_{n=-1/2} = \frac{5}{4}\times\left(\frac{-3}{2}\right)\times(-2)-\frac{30}{8}-0 = 0\,.$$

This implies that $u_n = \dfrac{(3n^2+3n-1)\,n(n+1)(2n+1)}{30}$, and concludes the induction.

Ex6 Consider the recurrence relation $u_{n+2} = 2r\,u_{n+1} - r^2 u_n$ ($r\neq 0$). Prove that the sequence $u_n = \alpha r^n + \beta n r^n$, $n\geq 1$, where $\alpha,\beta$ are determined by the values of $u_1$ and $u_2$

→ The base cases impose $\boxtimes\begin{cases}\alpha r+\beta r = u_1\\ \alpha r^2+2\beta r^2 = u_2\end{cases}$, which completely determines the values of $\alpha$ and $\beta$, since $\det\begin{pmatrix} r & r\\ r^2 & 2r^2\end{pmatrix} = r^3 \neq 0$. Now assume that $\alpha$ and $\beta$ are determined by $\boxtimes$. We prove by induction on $n\geq 1$ that $u_n = \alpha r^n + \beta n r^n$. The base cases $n=1$ and $n=2$ hold because of $\boxtimes$. Now suppose the induction hypothesis true up to $n-1$, $n\geq 3$. We get

$$u_n = 2r\,u_{n-1} - r^2 u_{n-2} = 2r\left(\alpha r^{n-1}+\beta(n-1)r^{n-1}\right) - r^2\left(\alpha r^{n-2}+\beta(n-2)r^{n-2}\right)$$
$$= \alpha\left(2r^n - r^n\right) + \beta\left((2n-2)r^n - (n-2)r^n\right)$$
$$= \alpha r^n + \beta n r^n,$$

hence the induction hypothesis holds for $n$. This concludes the inductive proof.

**Ex 8** For $n \geq 2$, prove that any $2n$ points joined by at least $n^2+1$ segments contains a triangle. Show that it is not so if there are only $n^2$ segments

→ We proceed by induction on $n \geq 2$. For $n = 2$, the situation is as follows:

and if all points have $\leq 2$ segments attached to them, then the total number of segments is $\leq \frac{4 \times 2}{2} = 4 < 2^2+1$,

So a point must have at least 3 segment attached to it.

At this point, any other segment produces a triangle, as expected.

Let us now assume that the induction hypothesis holds up to $n-1$, $n \geq 3$, and let us prove that it holds for $n$. Out of the $(2n+2)$ points joined by at least $(n+1)^2+1 = n^2+2n+2$, let us isolate two points connected by one line segment:

$2n$ points

We can assume that any of the $2n$ points on the right is connected to only one of the two isolated points, otherwise we already have a triangle.

Thus, the number of segments among the $2n$ points is

$$\geq n^2+2n+2 - 2n - 1 = n^2+1.$$

By the induction hypothesis, there is a triangle among the $2n$ points on the right. This shows that the induction hypothesis holds for $n+1$.

The proof is now complete.

**Ex 9** For integers $n, d \geq 0$, prove the relation

$$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+d}{n} = \binom{n+d+1}{n+1}$$

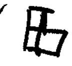
note the typo in the original statement

→ By induction on $d \geq 0$. The result is clear for $d = 0$. Then, if it is true for $d-1$, $d \geq 1$, we get

$$\binom{m}{m} + \binom{m+1}{m} + \dots + \binom{m+d-1}{m} + \binom{m+d}{m} = \binom{m+d}{m+1} + \binom{m+d}{m} = \frac{(m+d)!}{(m+1)!\,(d-1)!} + \frac{(m+d)!}{m!\,d!}$$

$$= \frac{(m+d)!}{(m+1)!\,d!}\left( d + m+1 \right) = \frac{(m+d+1)!}{(m+1)!\,d!} = \binom{m+d+1}{m+1},$$

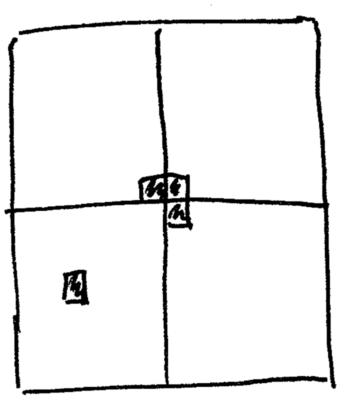
i.e., the result is true for $d$. This concludes the inductive proof.

<u>Ex 10</u> Consider a $2^n \times 2^n$ checkerboard from which an arbitrary square has been removed. Can it be paved by polyminos? (  )

→ The answer is yes. It is worth noticing that a surface paved by polyminos has a number of squares which is a multiple of 3, and so has the checkerboard with one square removed since

$$4^n - 1 = (4-1)\left(4^{n-1} + 4^{n-2} + \dots + 4 + 1\right).$$

Now let us prove the claim by induction on $n \geq 1$. For $n = 1$, we have the following situation , and the resulting surface can be paved by one polymino. Next, we suppose that the claim holds up to $n \geq 1$, and we prove that it holds for $n+1$. We decompose the $2^{n+1} \times 2^{n+1}$ checkerboard as four $2^n \times 2^n$ checkerboards, as follows



Suppose for instance that the removed square is in the lower-left checkerboard. Then we place a polymino covering one square in each of the three other checkerboards, as indicated.
By the induction hypothesis, each of the four

checkerboards can be covered by polyminos. Thus, the whole checkerboard is also pavable by polyminos. This concludes the induction.

<u>Ex 11</u> Prove the expression of the Vandermonde determinant:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_p \\ \vdots & & & \vdots \\ x_1^{p-1} & x_2^{p-1} & \cdots & x_p^{p-1} \end{vmatrix} = \prod_{1 \le i < j \le p} (x_j - x_i)$$

→ The determinant on the LHS, when considered as a function of $x_1$, is a polynomial of degree $\le p-1$ that vanishes at $x_2, x_3, \ldots, x_p$. Therefore, we must have

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_p \\ \vdots & & & \vdots \\ x_1^{p-1} & x_2^{p-1} & \cdots & x_p^{p-1} \end{vmatrix} = C \, (x_p - x_1) \cdots (x_3 - x_1)(x_2 - x_1) .$$

Note that $C$ is the coefficient of $(-x_1)^{p-1}$, which in the determinant appears to be $\begin{vmatrix} 1 & \cdots & 1 \\ x_2 & \cdots & x_p \\ x_2^{p-2} & \cdots & x_p^{p-2} \end{vmatrix}$ (by expanding along the first column).

This has the form $\prod_{2 \le i < j \le p} (x_j - x_i)$ (a proper inductive proof would be appropriate).

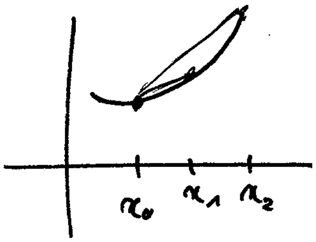
Altogether, we have

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_p \\ \vdots & & & \\ x_1^{p-1} & x_2^{p-1} & \cdots & x_p^{p-1} \end{vmatrix} = \prod_{2 \le i < j \le p} (x_j - x_i) \times (x_p - x_1) \cdots \times (x_2 - x_1) = \prod_{1 \le i < j \le p} (x_j - x_i),$$

as expected.

**Ex 12** Given a convex function $\varphi$ on an interval $I$ and given $x_0 \in I$, prove that the slope $\dfrac{f(x)-f(x_0)}{x-x_0}$ nondecreasing with $x \in I$, $x \neq x_0$.

→ We are going to prove that the slope is an nondecreasing function "to the right" of $x_0$. The case "to the left" is similar.



We want to prove that, if $x_2 > x_1 (> x_0)$,

$$\frac{f(x_2)-f(x_0)}{x_2-x_0} \overset{?}{\geq} \frac{f(x_1)-f(x_0)}{x_1-x_0}.$$

Since $x_1 \in (x_0, x_2)$, we can write $x_1 = (1-t)x_0 + t x_2$ for some $t \in (0,1)$. Hence the RHS takes the form

$$\frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f((1-t)x_0+tx_2)-f(x_0)}{t(x_2-x_0)}.$$

Rearranging the prospective inequality, we need to prove

$$t\left(f(x_2)-f(x_0)\right) \overset{?}{\geq} f((1-t)x_0+tx_2) - f(x_0),$$

or:

$$f((1-t)x_0+tx_2) \overset{?}{\leq} (1-t)f(x_0) + t f(x_2).$$

This is indeed true — it follows from the convexity of $f$.