Math 235, Fall 2012 : Solutions to homework 4

Ex1 Find the sequence $(a_n)_{n\geq 0}$ given by $a_0 = 1$ and

$$a_n = \frac{1}{2}\left(1 - \sum_{k=1}^{n-1} a_k a_{n-k}\right) \quad \text{for } n \geq 1$$

→ Rewrite the definition of (a_n) as: $a_0 = 1$, $\sum_{k=0}^{n} a_k a_{n-k} = 1$, $n \geq 1$.

Then $\frac{1}{1-z} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k a_{n-k}\right) z^n = \left(\sum_{n=0}^{\infty} a_n z^n\right)^2$, so that

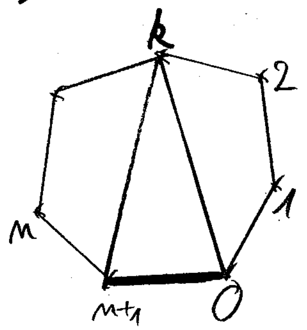
$\sum_{n=0}^{\infty} a_n z^n = (1-z)^{-1/2}$ — we reject $-(1-z)^{-1/2}$ because of the value at $z=0$.

We derive : $a_n = \frac{(-1/2)(-3/2)\cdots(-\frac{1}{2}-n+1)}{n!}(-1)^n = \frac{1\times 3\times\cdots\times(2n-1)}{2^n n!} = \frac{2n!}{4^n n! n!}$

$$\boxed{a_n = \frac{1}{4^n}\binom{2n}{n}}$$

Ex2 Find the number of different ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.

→ Let u_n be the required number. The segment $[0, n+1]$ (see figure)



can be part of n triangles ($(0,k,n+1)$ for $k \in [1:n]$).

There are u_{k-1} ways to triangulate the part on the right, and u_{n-k} to triangulate the part on the left.

Altogether, we derive $u_n = \sum_{k=1}^{n} u_{k-1} u_{n-k} = \sum_{i=0}^{n-1} u_i u_{n-1-i}$.

This is the recurrence relation for the Catalan numbers.

We deduce (in view of $u_1 = 1 = C_1$), that $u_n = C_n = \frac{1}{n+1}\binom{2n}{n}$.

Ex3 Prove that the number of partitions of an integer into odd positive integers equals the number of its partition into distinct integers.

→ The generating function for the numbers of partitions of n into odd integers is:

$$(1+z+z^2+\dots)(1+z^3+z^6+\dots)(1+z^5+z^{10}+\dots) = \frac{1}{1-z}\,\frac{1}{1-z^3}\,\frac{1}{1-z^5}\cdots \qquad (1)$$

The generating function for the numbers of partitions of n into distinct integers is:

$$(1+z)(1+z^2)(1+z^3)\cdots \qquad (2)$$

The problem now consists in proving that (1) and (2) are the same, i.e., that $P(z) := (1+z)(1+z^2)(1+z^3)(1+z^4)(1+z^5)\cdots(1-z)(1-z^3)(1-z^5)\cdots \overset{?}{=} 1.$

Note that

$$P(z) = (1+z)(1-z)(1+z^3)(1-z^3)(1+z^5)(1-z^5)\cdots(1+z^2)(1+z^4)\cdots$$

$$= (1-z^2)(1-z^6)(1-z^{10})\cdots(1+z^2)(1+z^4)\cdots \quad = P(z^2)$$

Repeating this, we obtain, for $|z|<1$:

$$P(z) = P(z^{2^n}) \xrightarrow[n\to\infty]{} P(0) = 1 \quad , \quad \text{i.e., } P(z) = 1, \text{ as expected.}$$

Ex4 Recover $\sum_{k=0}^{n} \left[{n \atop k}\right] = n!$ from the expression $\sum_{n=0}^{\infty} \left[{n \atop k}\right] \frac{z^n}{n!} = \frac{1}{k!}\left(\ln\left(\frac{1}{1-z}\right)\right)^k$ of the exponential generating function.

→ Notice that $\displaystyle\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \left[{n \atop k}\right]\frac{z^n}{n!} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\ln\left(\frac{1}{1-z}\right)\right)^k = \exp\left(\ln\left(\frac{1}{1-z}\right)\right) = \frac{1}{1-z}$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left[{n \atop k}\right]\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left[{n \atop k}\right]\right)\frac{z^n}{n!} \, ,$$

and identify the term in z^n ~~with the~~ in this expansion and the one of $\frac{1}{1-z}$.

Ex5 The positive differences of the four numbers $0,2,5,6$ are $1,2,3,4,5,6$, each taken exactly once. Prove that this cannot occur with more than four numbers.

→ Let $a_1, a_2, \dots, a_n$ be the numbers in question. There are $\frac{n(n-1)}{2}$ positive differences, and we want to prove that they cannot be $1,2,3,\dots,\frac{n(n-1)}{2}$.

Form the generating function $z^{a_1}+z^{a_2}+\dots+z^{a_n}$, and notice that the product $(z^{a_1}+z^{a_2}+\dots+z^{a_n})(z^{-a_1}+z^{-a_2}+\dots+z^{-a_n})$ will feature power of z with exponent the positive differences, their opposite, and 0 (n times). Thus, we want to prove that

$$(z^{a_1}+z^{a_2}+\dots+z^{a_n})(z^{-a_1}+z^{-a_2}+\dots+z^{-a_n}) = \underbrace{z^{-\frac{n(n-1)}{2}}+\dots+z^{-1}+n+z+\dots+z^{\frac{n(n-1)}{2}}}$$

$$= n-1+\frac{z^{-\frac{n(n-1)}{2}}-z^{\frac{n(n-1)}{2}+1}}{1-z}$$

is impossible.

Let us set $z=e^{i\theta}$ and notice that $\dfrac{z^{-\frac{n(n-1)}{2}}-z^{\frac{n(n-1)}{2}+1}}{1-z} = \dfrac{\sin((n^2-n+1)\theta/2)}{\sin(\theta/2)}$.

Thus, we want to prove that

$$\left|e^{ia_1\theta}+e^{ia_2\theta}+\dots+e^{ia_n\theta}\right|^2 = n-1+\frac{\sin((n^2-n+1)\theta/2)}{\sin(\theta/2)}$$

is impossible for some choice of θ when $n \geqslant 4$. We choose $\theta = \frac{3\pi}{n^2-n+1}$, so that $\sin((n^2-n+1)\theta/2) = -1$ and $\sin(\theta/2) < \frac{3\pi}{2(n^2-n+1)}$, hence $n-1+\dfrac{\sin((n^2-n+1)\theta/2)}{\sin(\theta/2)} < n-1-\dfrac{2(n^2-n+1)}{3\pi} < (n-1)\left(1-\dfrac{2n}{3\pi}\right) < 0$ if $n \geqslant 5$.

Therefore, this expression cannot be a modulus squared, and we have our contradiction.