Pigeonhole and Inclusion-Exclusion Principles

1 Pigeonhole principle

This is the observation that, if n objects have to be placed in less than n sets, at least one of the sets will contain two objects or more. More generally, if n objects have to be placed in k sets, at least one of the sets will contain $\lceil n/k \rceil$ objects or more. Here are some famous theorems where this principle is applied.

Theorem 1 (Dirichlet). If x is an irrational number, then there are infinitely many pairs (p, q) of integers such that $|x - p/q| < 1/q^2$.

Proof. Suppose on the contrary that the set S of $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that $|x - p/q| < 1/q^2$ is finite. Define

$$\varepsilon := \min_{(p,q) \in \mathbb{Z} \times \mathbb{N}} |x - p/q| > 0.$$

Now consider an integer $Q \geq 1/\varepsilon$. The $Q+1$ fractional parts $\{qx\}$, $q \in [0 : Q]$, belong to $[0, 1)$, which is the union of the Q intervals $[\ell/Q, (\ell+1)/Q)$, $\ell \in [0, Q-1]$. Therefore, there exist $q', q'' \in [0 : Q]$ with $q' > q''$, say, such that $\{q'x\}$ and $\{q''x\}$ belong to the same interval, hence $|\{q'x\} - \{q''x\}| < 1/Q$. This reads

$$\big|q'x - \lfloor q'x \rfloor - (q''x - \lfloor q''x \rfloor)\big| = \big|(q' - q'')x - (\lfloor q'x \rfloor - \lfloor q''x \rfloor)\big| < \frac{1}{Q}.$$

Setting $q := q' - q'' \in [1 : Q]$ and $p := \lfloor q'x \rfloor - \lfloor q''x \rfloor \in \mathbb{Z}$, we derive

$$\left| x - \frac{p}{q} \right| < \frac{1}{qQ} \leq \begin{cases} 1/q^2, \\ 1/Q \leq \varepsilon. \end{cases}$$

This shows that $(p, q) \in S$ and that $|x - p/q| < \varepsilon$, which is a contradiction. □

Theorem 2 (Erdős–Szekeres). Every sequence of $(m-1)(n-1)+1$ distinct real numbers admits either an increasing subsequence of length m or a decreasing subsequence of length n.

Proof. Suppose that the sequence — denote it by $(u_i)_{i \in [1:(m-1)(n-1)+1]}$ — has no increasing subsequence of length m. This means that

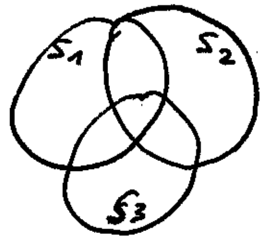
$$S_k := \{i \in [1 : (m-1)(n-1)+1] : \text{ the largest increasing subsequence starting at } i \text{ has length } k\}$$

is nonempty only for $k = 1, 2, \ldots, m-1$. Since every $i \in [1 : (m-1)(n-1)+1]$ belongs to one of $S_1, S_2, \ldots, S_{m-1}$, there is an S_k with size $\geq n$. Consider $i_1 < i_2 < \cdots < i_n$ in S_k. For $j \in [1 : n]$, notice that an increasing subsequence of length k starts at i_{j+1} and that no increasing sequence of length $k+1$ starts at i_j, therefore we must have $u_{i_j} > u_{i_{j+1}}$. This gives a decreasing subsequence of length n, namely $(u_{i_j})_{j \in [1:n]}$. □

2 Inclusion-exclusion principle

The inclusion-exclusion principle generalizes the formulas

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|,$$
$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|.$$



Theorem 3. Given finite sets $S_1, S_2, \ldots, S_n$,

$$|S_1 \cup \cdots \cup S_n| = \sum_{r=1}^{n} (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le n} |S_{i_1} \cap \cdots \cap S_{i_r}|.$$

Remark. Finite sets can be replaced by measurable sets, say, with $|\cdot|$ meaning measure.

Proof. We could proceed by induction on n. Here is another instructive argument based on characteristic functions. Recall first that, for two sets A and B, one has $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$ and $\chi_{A\cap B} = \chi_A \chi_B$. It follows that $1 - \chi_{A\cap B} = (1-\chi_A)(1-\chi_B)$. We then derive

$$\begin{aligned} 1 - \chi_{S_1 \cup S_2 \cup \cdots \cup S_n}(x) &= (1 - \chi_{S_1}(x))(1 - \chi_{S_2}(x)) \cdots (1 - \chi_{S_n}(x)) \\ &= 1 + \sum_{r=1}^{n} (-1)^r \sum_{1 \le i_1 < \cdots < i_r \le n} \chi_{S_{i_1}}(x) \chi_{S_{i_2}}(x) \cdots \chi_{S_{i_r}}(x) \\ &= 1 + \sum_{r=1}^{n} (-1)^r \sum_{1 \le i_1 < \cdots < i_r \le n} \chi_{S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_r}}(x). \end{aligned}$$

It now suffices to sum over all possible x's. □

As an application, we count the derangements of $[1:n]$, i.e., the permutations of $[1:n]$ with no fixed points.

Theorem 4. The number of derangements of $[1:n]$ is

$$D_n = n! \sum_{r=0}^{n} \frac{(-1)^r}{r!}.$$

Remark. The proportion of derangements among all permutations approaches $1/e \approx 0.3679$ when n grows, since $D_n/n! \to \sum_{r=0}^{\infty} (-1)^r/r! = e^{-1}$.

Proof. Let $\mathcal{P}$ be the set of permutations of $[1:n]$ and $\mathcal{D}$ the set of derangements of $[1:n]$. For each $i \in [1:n]$, we consider the set $\mathcal{P}_i$ of $\sigma \in \mathcal{P}$ such that $\sigma(i) = i$. Since a permutation is not a derangement iff it belongs to one of the $\mathcal{P}_i$, we have $\mathcal{P} \setminus \mathcal{D} = \bigcup_{i=1}^n \mathcal{P}_i$. Therefore,

$$|\mathcal{P} \setminus \mathcal{D}| = \sum_{r=1}^{n} (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le n} |\mathcal{P}_{i_1} \cap \cdots \cap \mathcal{P}_{i_r}|.$$

Note that $|\mathcal{P} \setminus \mathcal{D}| = |\mathcal{P}| - |\mathcal{D}| = n! - D_n$, that there are $\binom{n}{r}$ ways of choosing $(i_1 < \cdots < i_r)$ in $[1:r]$, and that $|\mathcal{P}_{i_1} \cap \cdots \cap \mathcal{P}_{i_r}| = (n-r)!$ since a permutation of $[1:n]$ that fixes $i_1, \ldots, i_r$ is equivalent to a permutation of $[1:n] \setminus \{i_1, \ldots, i_r\}$. This yields

$$n! - D_n = \sum_{r=1}^{n} (-1)^{r-1} \binom{n}{r} (n-r)! = \sum_{r=1}^{n} (-1)^{r-1} \frac{n!}{r!}.$$

It suffices to rearrange the latter to obtain the announced formula. □

3 Exercises

Ex.1: What is the maximum size of a subset of $\{3, 11, 19, 27, \ldots, 139, 147, 155\}$ for which no two elements add up to 158?

Ex.2: Prove that at any party there are two guests with the same number of friends present.

Ex.3: How many numbers between 1 and 2012 (inclusive) are not divisible by 2, 3, 5, and 7?

Ex.4: Let S be a subset of $[1:100]$ of size 10. Show that there are two subsets of S for which the sums of the elements are the same.

Ex.5: Calculate the generating function of the sequence $(D_n/n!)$, i.e., the formal power series

$$\sum_{n \ge 0} \frac{D_n}{n!} z^n.$$

Ex.6: Show that, if a collar made of n pearls has more than $(k-1)n/k$ white pearls, then there is a string of k consecutive white pearls.

Ex.7: Given real numbers $x_1, x_2, \ldots, x_n$, prove that

$$\max\{x_1, \ldots, x_n\} = \sum_{r=1}^{n} (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le n} \min\{x_{i_1}, \ldots, x_{i_r}\}.$$

Ex.8: Prove that, in a party with n guests, one can find two guests such that at least half of the remaining guests know either both or neither of them.

Ex.9: How many matrices in $\{0,1\}^{m \times n}$ have no row and no column consisting only of zeros?

Ex.10: Show that the number of permutations σ of $[1:n]$ with $\sigma(i+1) \neq \sigma(i)$ for all $i \in [1:n]$ equals $D_n + D_{n-1}$.