Linear Algebra

1 Range and null space

For $A \in \mathcal{M}_{m\times n}(\mathbb{C})$, define its range

$$\operatorname{ran} A := \{Ax, x \in \mathbb{C}^m\},$$

and its null space

$$\ker A := \{x \in \mathbb{C}^n : Ax = 0\}.$$

These are linear subspaces of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. The rank and the nullity of A are defined by

$$\operatorname{rk} A := \dim(\operatorname{ran} A), \qquad \operatorname{nul} A := \dim(\ker A).$$

They are deduced form one another by the rank-nullity theorem

$$\operatorname{rk} A + \operatorname{nul} A = n.$$

Recall that $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ is injective if $\ker A = \{0\}$, and surjective if $\operatorname{ran} A = \mathbb{C}^m$. Note that a *square* matrix A is injective (or surjective) iff it is both injective and surjective, i.e., iff it is bijective. Bijective matrices are also called invertible matrices, because they are characterized by the existence of a unique square matrix B (the inverse of A, denoted by A^{-1}) such that $AB = BA = I$.

2 Trace and determinant

The trace and determinants are functions taking *square* matrices and returning scalars. The trace of $A \in \mathcal{M}_n(\mathbb{C})$ is the sum of its diagonal elements, i.e.,

$$\operatorname{tr} A := \sum_{i=1}^{n} a_{i,i} \quad \text{where} \quad A = [a_{i,j}]_{i,j=1}^{n}.$$

Notice that the trace is linear (i.e., $\operatorname{tr}(\lambda A + \mu B) = \lambda \operatorname{tr}(A) + \mu \operatorname{tr}(B)$) and that

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) \qquad \text{whenever } A \in \mathcal{M}_{m\times n}(\mathbb{C}) \text{ and } B \in \mathcal{M}_{n\times m}(\mathbb{C}).$$

As for the determinant, it can be defined in several equivalent ways:

1. As a function of the columns of a marix, it is the only function $f : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ that is linear with respect to each columns ($f(\ldots, \lambda x + \mu y, \ldots) = \lambda f(\ldots, x, \ldots) + \mu f(\ldots, y, \ldots)$), alternating ($f(\ldots, x, \ldots, y, \ldots) = -f(\ldots, y, \ldots, x, \ldots)$), and unit-normalized ($f(I) = 1$). This can be used to derive the identity

$$\det(AB) = \det(A)\det(B) \quad \text{for all } A, B \in \mathcal{M}_n(\mathbb{C}).$$

2. $\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$,
   where S_n is the set of $n!$ permutations of $[1:n]$ and $\operatorname{sgn}(\sigma) = (-1)^s$, $s =$ number of pairwise interchanges composing σ (hence the computation rules for 2×2 and 3×3 determinants). This can be used to prove that

$$\det A^\top = \det A \qquad \text{for all } A \in \mathcal{M}_n(\mathbb{C}).$$

3. Laplace expansion with respect to a row or a column, e.g. with respect to the ith row

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det A_{i,j},$$

   where $A_{i,j}$ is the submatrix of A obtained by deleting the ith row and the jth column. The matrix $B \in \mathcal{M}_n(\mathbb{C})$ with entries $b_{i,j} := (-1)^{i+j} \det A_{i,j}$ is called the comatrix of A — note that $B^\top$ is also called the adjoint of A (*classical* adjoint, not to be confused with *hermitian* adjoint). Laplace expansion can be used to prove that $AB^\top = (\det A)I$. In turn, it is deduced that $A \in \mathcal{M}_n(\mathbb{C})$ is invertible iff $\det A \neq 0$, in which case $\mathbf{A}^{-1} = (1/\det(A))B^\top$.

3 Eigenvalues and eigenvectors

Given a *square* matrix $A \in \mathcal{M}_n(\mathbb{C})$, if there exist $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n$, $x \neq 0$, such that

$$Ax = \lambda x,$$

then λ is called an eigenvalue of A and x is called an eigenvector corresponding to the eigenvalue λ. The set of all eigenvectors corresponding to an eigenvalue λ is called the eigenspace corresponding to the eigenvalue λ — it is indeed a linear space. Note that λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$, i.e., iff λ is a zero of the characteristic polynomial of A defined by

$$p_A(x) := \det(A - xI).$$

Observe that p_A is a polynomial of the form

$$p_A(x) = (-1)^n x^n + (-1)^{n-1} \operatorname{tr}(A)\, x^{n-1} + \cdots + \det(A).$$

Since this polynomial can also be written in factorized form as $(\lambda_1 - x) \cdots (\lambda_n - x)$, where $\{\lambda_1, \ldots, \lambda_n\}$ is the set of eigenvalues of A (complex and possibly repeated), we have

$$\operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n, \qquad \det(A) = \lambda_1 \cdots \lambda_n.$$

The existence of n linearly independent eigenvectors $v_1, \ldots, v_n \in \mathbb{C}^n$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A \in \mathcal{M}_n$ (which occurs in particular if A has n distinct eigenvalues) is equivalent to the existence of a invertible matrix $V \in \mathcal{M}_n$ and of a diagonal matrix $D \in \mathcal{M}_n$ such that

$$A = VDV^{-1}.$$

The columns of V are the $v_i's$ and the diagonal entries of D are the λ_i's. In this case, we say that the matrix A is diagonalizable. More generally, two matrices A and B are called similar if there exists an invertible matrix V such that $A = VBV^{-1}$. Note that two similar matrices have the same characteristic polynomial, hence the same eigenvalues (counting multiplicities), and in particular the same trace and determinant.

4 Exercises

Ex.1: We recall that $\mathrm{rk}\, A^* = \mathrm{rk}\, A$, where $A^* \in \mathcal{M}_{n\times m}(\mathbb{C})$ denotes the conjugate transpose of a matrix $A \in \mathcal{M}_{m\times n}$. In general, is it true that $\mathrm{nul}\, A^* = \mathrm{nul}\, A$? Establish that $\ker A = \ker A^*A$, deduce that $\mathrm{nul}\, A = \mathrm{nul}\, A^*A$ and that $\mathrm{rk}\, A = \mathrm{rk}\, A^*A = \mathrm{rk}\, A^* = \mathrm{rk}\, AA^*$, and finally conclude that $\mathrm{ran}\, A = \mathrm{ran}\, AA^*$.

Ex.2: Calculate $\mathrm{tr}(A^*A)$ and observe that $A = 0$ iff $\mathrm{tr}(A^*A) = 0$.

Ex.3: For $A, B \in \mathcal{M}_n(\mathbb{C})$, prove that $AB = I$ implies $BA = I$. Is this true if A and B are not square?

Ex.4: Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & t & \cdots & t \\ t & 1 & t & \vdots \\ t & \cdots & \ddots & t \\ t & \cdots & t & 1 \end{bmatrix},$$

and diagonalize it.

Ex.5: For $A \in \mathcal{M}_n(\mathbb{Z})$, suppose that there exists a prime number p dividing $\sum_{j=1}^n a_{i,j}$ for all $i \in [1:n]$. Prove that p divides $\det(A)$.

Ex.6: Determine if the following statement is true or false: there exists $A \in \mathcal{M}_n(\mathbb{R})$ such that $A^2 + 2A + 5I = 0$ if and only if n is even.