Sequences and Series

1 Some convergence criteria

For real-valued sequences,

- If a nondecreasing sequence is bounded above, then it is convergent;
- A sequence $(u_n)_{n\geq 0}$ is convergent if and only if it is a Cauchy sequence, i.e.,

$$\forall \varepsilon, \exists n_0 \geq 0 : \forall n \geq n_0, \forall p \geq 0, |u_{n+p} - u_n| < \varepsilon.$$

The counterparts for real-valued series are:

- If $u_k \geq 0$ for all k and if $\sum_{k=0}^{n} u_k \leq U$ for some $U > 0$ and all n, then $\sum_{k=0}^{\infty} u_k$ converges; [Consequence: if $0 \leq u_k \leq v_k$ and $\sum_k v_k$ converges, then $\sum_k u_k$ converges, too.]
- A series $\sum_{k=0}^{\infty} u_k$ converges if and only if

$$\forall \varepsilon > 0, \exists n_0 \geq 0 : \forall n \geq n_0, \forall p \geq 0, \left|\sum_{k=n}^{n+p} u_k\right| < \varepsilon.$$

[Consequence: If $\sum_k |u_k|$ converges (absolute convergence), then $\sum_k u_k$ converges, too.]

The latter comparison criterion is useful when classical series are at our disposal, such as:

$$\sum_n x^n \text{ converges iff } |x| < 1, \qquad \sum_n \frac{1}{n^a} \text{ converges iff } a > 1,$$

$$\sum_n \frac{1}{n \ln^a(n)} \text{ converges iff } a > 1. \tag{1}$$

From there, we can deduce the ratio test: if a positive sequence satisfies $u_{n+1}/u_n \underset{n\to\infty}{\longrightarrow} \rho$ with $0 \leq \rho < 1$, then $\sum_k u_k$ converges. Once can also prove, for instance, that the series $\sum 1/(n(n+1))$ converges by remarking that $0 \leq 1/(n(n+1)) \leq 1/n^2$ and that $\sum 1/n^2$ converges, which implies that $\sum 1/(n(n+1))$ converges, too. To find the exact value of the sum, we interpret it as a telescoping sum, i.e.,

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1.$$

Another useful technique is the summation by parts, consisting of the following manipulation: given $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$, define $A_0 = 0$, $A_n = \sum_{k=1}^n a_k$ for $n \geq 1$, and write

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n}(A_k - A_{k-1})b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_{k-1} b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=0}^{n-1} A_k b_{k+1} \\ &= A_n b_n + \sum_{k=1}^{n-1} A_k (b_k - b_{k+1}).\end{aligned}$$

2 Power series

A power series is a series of the type $\sum u_n z^n$. Here are a few classical ones, together with their regions of absolute convergence in the complex plane:

$$(2) \qquad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 \cdots, \qquad |z| < 1$$

$$(3) \qquad \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n = 1 + 2z + 3z^2 + 4z^3 \cdots, \qquad |z| < 1$$

$$(4) \qquad \ln(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} \cdots, \qquad |z| < 1$$

$$(5) \qquad (1+z)^a = \sum_{n=0}^{\infty} \frac{a(a-1)\cdots(a-n+1)}{n!} z^n = 1 + az + \frac{a(a-1)}{2} z^2 + \frac{a(a-1)(a-2)}{6} z^3 \cdots, \qquad |z| < 1$$

$$(6) \qquad \exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{1}{2} z^2 + \frac{1}{6} z^3 \cdots, \qquad \text{all } z$$

$$(7) \qquad \cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = 1 + \frac{1}{2} z^2 + \frac{1}{24} z^4 \cdots \qquad \text{all } z$$

$$(8) \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = z + \frac{1}{6} z^3 + \frac{1}{120} z^5 \cdots \qquad \text{all } z$$

$$(9) \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{1}{2} z^2 + \frac{1}{24} z^4 \cdots \qquad \text{all } z$$

$$(10) \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{1}{6} z^3 + \frac{1}{120} z^5 \cdots \qquad \text{all } z$$

Some of these equalities may remain valid at particular points on the boundary of the region of absolute convergence. For instance, (4) holds for $z = -1$, and it then reads

$$\ln(2) = \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots.$$

3 Progression and sums

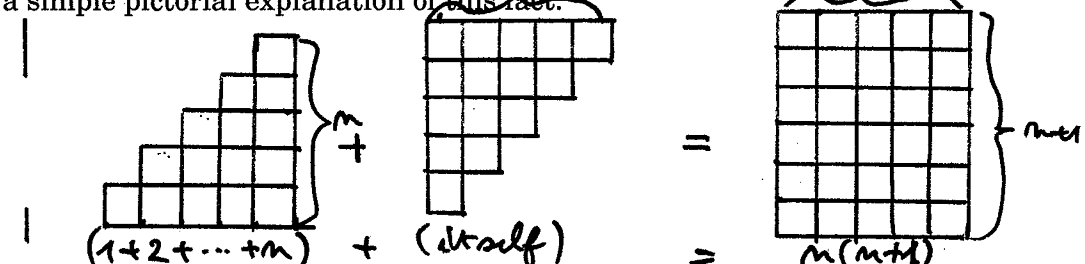
The fundamental identity (2) is a consequence of the general expression for the sum of a geometric progression. A geometric progression is a sequence $(u_k)_{k\geq 0}$ obeying the recurrence relation $u_{k+1} = ru_k$ for some $r \in \mathbb{C}$ and all $k \geq 0$. Alternatively, it can be viewed as the sequence defined by $u_k = r^k u_0$ for all $k \geq 0$. The sum of its first $n+1$ terms is derived from the identity

$$\sum_{k=0}^{n} r^k = \begin{cases} \dfrac{1-r^{n+1}}{1-r} & \text{if } r \neq 1, \\ n+1 & \text{if } r = 1. \end{cases}$$

An arithmetic progression is a sequence $(u_k)_{k\geq 0}$ obeying the recurrence relation $u_{k+1} = u_k + d$ for some $d \in \mathbb{C}$ and all $k \geq 0$. Alternatively, it can be viewed as the sequence defined by $u_k = u_0 + kd$ for all $k \geq 0$. The sum of its first $n+1$ terms is derived from the identity

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

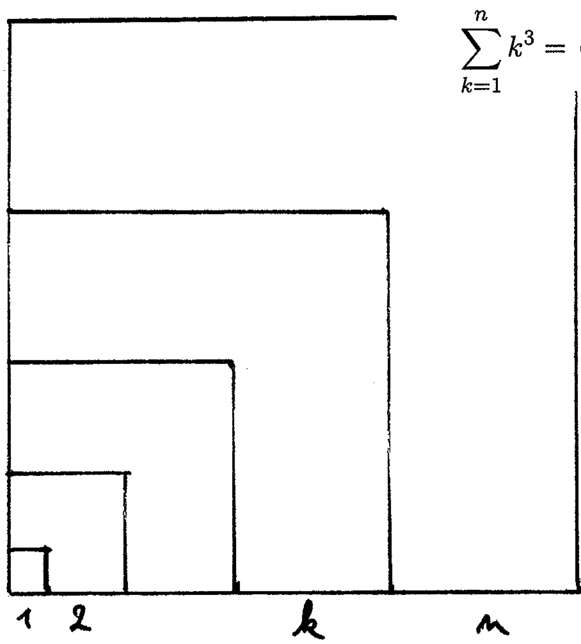
Here is a simple pictorial explanation of this fact.



Let us also point out the values of the sums of the first squares and of the first cubes:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$



total area $= (1+2+\cdots+n)^2 = \left(\frac{n(n+1)}{2}\right)^2$

$= \sum_{k=1}^{n}$ kth subarea

note that the kth subarea is

$2 \times k \times (1+2+\cdots+k) - k^2$

$= k\,k(k+1) - k^2 = k^3$.

4 Exercises

Ex.1: Given $t > 0$, consider the sequence defined recursively by $x_0 > 0$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{t}{x_n}\right), \qquad n \geq 0.$$

Does the sequence $(x_n)_{n\geq 0}$ converge, and if so what is the value of its limit?

Ex.2: Let $(u_n)_{n\geq 1}$ be a convergent sequence. Prove that the Cesaro mean $\frac{1}{n}\sum_{k=0}^{n} u_k$ converges to the same limit as the original sequence when $n \to \infty$.

Ex.3: Prove the assertion (1).

Ex.4: Given the value $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find the values of $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

Ex.5: State the power series expansions of $1/(1-z)$, $1/(1+z)^2$, $\ln(1+z)$, and $\sqrt{1+z}$. Derive (3) and (4) from (2) by formal differentiation and integration. Show that (5) reduces to the binomial theorem when a is an integer. Derive (7), (8), (9), and (10) from (6) using the expressions of the (hyperbolic) trigonometric functions in terms of the exponential function.

Ex.6: Prove that the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}(k+1) + k\sqrt{k+1}}$ converges and find its value.

Ex.7: Evaluate the sums $\sum_{k=0}^{n} \cos\left(\frac{k\pi}{n}\right)$ and $\sum_{k=0}^{n} \sin\left(\frac{k\pi}{n}\right)$.

Ex.8: Consider a sequence $(u_k)_{k\geq 0}$ obeying the recurrence relation $u_{k+1} = ru_k + d$ for some $r, d \in \mathbb{C}$ and all $k \geq 0$. Find the explicit form for the general term u_k and deduce an expression for the sum $\sum_{k=0}^{n} u_k$.

Ex.9: Consider the power series expansion

$$\frac{1}{1-2z-z^2} = \sum_{n=0}^{\infty} a_n z^n.$$

Prove that, for all $n \geq 0$, there exists $m \geq 0$ such that $a_n^2 + a_{n+1}^2 = a_m$.

Ex.10: Consider partitioning the natural numbers into the sets $\{1\}$, $\{2,3\}$, $\{4,5,6\}$, $\{7,8,9,10\}$, etc. Find a simple expression for the sum of the integers in the nth set.

Ex.11: If $\sum a_n$ is a convergent series of nonnegative terms, prove that $\sum \sqrt[n]{a_1 a_2 \cdots a_n}$ is also convergent.