# Induction and Recurrence

## 1 Mathematical induction

The principle of mathematical induction states that if an assertion $\mathcal{P}_n$ is true for an integer $n = n_0$ (the base case) and if $\mathcal{P}_{n+1}$ is true as soon as $\mathcal{P}_n$ is true (the inductive step), then the assertion $\mathcal{P}_n$ is true for all integers $n \geq n_0$. Equivalently, if an assertion $\mathcal{P}_n$ is true for an integer $n = n_0$ and if $\mathcal{P}_{n+1}$ is true as soon as $\mathcal{P}_n, \mathcal{P}_{n-1}, \ldots, \mathcal{P}_{n_0}$ are true, then the assertion $\mathcal{P}_n$ is true for all integers $n \geq n_0$. To see how the first version implies the second one, apply it to $\mathcal{P}'_n := (\mathcal{P}_{n_0}$ and $\mathcal{P}_{n_0+1}$ and $\ldots$ and $\mathcal{P}_n)$. It is advisable to systematically work with the second version. Sometimes, several base cases may need to be verified (see Section 2, for instance).

The validity of the principle of mathematical induction is a consequence of the well-ordering principle: any nonempty set of nonnegative integers has a minimal element. Indeed, consider the set $\mathcal{S} := \{n \geq n_0 : \mathcal{P}_n \text{ is wrong}\}$. If $\mathcal{S}$ was nonempty, it would have a minimal element $n_1$, and necessarily $n_1 > n_0$ (by the base case). The minimality implies that $n_1 - 1 \notin \mathcal{S}$, i.e., $\mathcal{P}_{n_1-1}$ is true. But then $\mathcal{P}_{n_1}$ is also true (by the inductive step), meaning that $n_1 \notin \mathcal{S}$. This is a contradiction. Hence $\mathcal{S}$ is empty, or in other words $\mathcal{P}_n$ is true for all $n \geq n_0$.

## 2 Recurrence relation

Mathematical induction is often used to establish rigorously a statement guessed from the first few cases (or given by the question). For instance, consider a sequence $(u_n)_{n\geq 1}$ given by the values $u_1, u_2, \ldots, u_p$ and the $p$-term recurrence relation $u_{n+p} = f(u_{n+p-1}, \ldots, u_{n+1}, u_n)$ for $n \geq 1$. One can compute in turn $u_{p+1}$, next $u_{p+2}$, then $u_{p+3}$, etc. If we see a pattern emerging for a closed-form formula, we can justify it using mathematical induction. We have already seen arithmetic and geometric progressions as examples of sequences defined by one-term recurrence relations. We now consider the particular case of a linear function $f$, i.e.,

$$u_{n+p} = c_{p-1}u_{n+p-1} + \cdots + c_1 u_{n+1} + c_0 u_n, \qquad c_0 \neq 0.$$

If the polynomial $p(z) := z^p - c_{p-1}z^{p-1} - \cdots - c_1 z - c_0$ has distinct roots $r_1, r_2, \ldots, r_n$ (the case of repeated roots can be treated, too), then it is proved below that the general term is given by

$$(1) \qquad u_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_p r_p^n \qquad \text{for all } n \geq 1,$$

where the $p$ coefficients $\alpha_1, \alpha_2, \ldots, \alpha_p$ are uniquely determined by the $p$ values of $u_1, u_2, \ldots, u_p$. We proceed by induction on $n$. The $p$ base cases hold since $\alpha_1, \alpha_2, \ldots, \alpha_p$ are determined precisely for this purpose. Assuming that (1) holds up to $n \geq 1$, let us now prove that it also holds for $n+1$. Using the recurrence relation, the induction hypothesis for $n-1, n-2, \ldots, n-p+1$, and the fact that $r_j^p = c_{p-1}r_j^{p-1} + \cdots + c_1 r_j + c_0$ for all $j \in [1:p]$, we derive

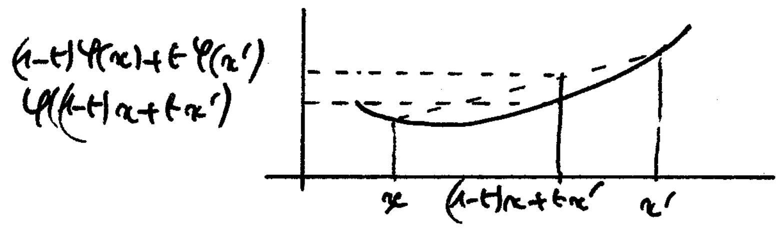
$$u_{n+1} = \sum_{k=1}^{p} c_{k-1}u_{n-p+k} = \sum_{k=1}^{p} c_{k-1}\sum_{j=1}^{p}\alpha_j r_j^{n-p+k} = \sum_{j=1}^{p}\alpha_j r_j^{n-p+1}\sum_{k=1}^{p} c_{k-1}r_j^{k-1} = \sum_{j=1}^{p}\alpha_j r_j^{n-p+1} r_j^p$$
$$= \sum_{j=1}^{p}\alpha_j r_j^{n+1}.$$

This shows that (1) holds for $n+1$. The principle of mathematical induction allows us to conclude that (1) holds for all $n \geq 1$.

## 3 An application: Jensen's inequality

A function $\varphi$ defined on an interval $I$ is called convex if

$$\varphi((1-t)x + tx') \leq (1-t)\varphi(x) + t\varphi(x') \qquad \text{for all } x, x' \in I \text{ and all } t \in [0,1].$$




Given a convex function $\varphi$ on an interval $I$, Jensen's inequality says that the image of a convex combination is smaller than or equal to the convex combination of the images. Precisely, if $x_1, \ldots, x_n \in I$ and if $t_1, \ldots, t_n \geq 0$ satisfy $t_1 + \cdots + t_n$, then

$$\varphi\Big(\sum_{j=1}^{n} t_j x_j\Big) \leq \sum_{j=1}^{n} t_j \varphi(x_j). \tag{2}$$

This can be proved by induction on $n$. Indeed, in the base case $n = 1$, (2) holds with equality. Let us now assume that (2) holds up to an integer $n-1$, $n \geq 2$, and let us prove that it also holds for the integer $n$. To this end, consider $x_1, \ldots, x_n \in I$ and $t_1, \ldots, t_n \geq 0$ with $t_1 + \cdots + t_n = 1$. If $t_n = 1$, then all other $t_j$ are zero, and (2) holds with equality. So we may assume that $t_n < 1$, and we set $t'_j := t_j/(1-t_n) \geq 0$ for $j \in [1:n-1]$. Notice that $\sum_{j=1}^{n-1} t'_j = (\sum_{j=1}^{n} t_j)/(1-t_n) = 1$. Applying the defining property of a convex function and then the induction hypothesis, we get

$$\varphi\Big(\sum_{j=1}^{n} t_j x_j\Big) = \varphi\Big(\sum_{j=1}^{n-1} t_j x_j + t_n x_n\Big) = \varphi\Big((1-t_n)\sum_{j=1}^{n-1} t'_j x_j + t_n x_n\Big)$$
$$\leq (1-t_n)\varphi\Big(\sum_{j=1}^{n-1} t'_j x_j\Big) + t_n\varphi(x_n) \leq (1-t_n)\sum_{j=1}^{n-1} t'_j\varphi(x_j) + t_n\varphi(x_n) = \sum_{j=1}^{n} t_j\varphi(x_j).$$

This shows that (2) holds for $n$. The principle of mathematical induction allows us to conclude that (1) holds for all $n \geq 1$.

## 4 Exercises

Ex.1: Prove that, for any integer $n \geq 1$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Ex.2: For $n \geq 1$, prove that $n(n-1)(n+1)(3n+2)$ is divisible by 24.

Ex.3: Find the number $R(n)$ of regions in which the plane can be divided by $n$ straight lines.

Ex.4: The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Find a closed-form formula for $F_n$ involving the golden ration $\phi := (1+\sqrt{5})/2$.

Ex.5: Guess a formula for the sum $1^4 + 2^4 + \cdots + n^4$, and then provide a rigorous proof (Hint: the answer is a polynomial in $n$, it has degree 5, and it is divisible by $n(n+1)(2n+1)$).

Ex.6: Given $r \in \mathbb{R}$, consider the two-term recurrence relation $u_{n+2} = 2ru_{n+1} - r^2 u_n$ (note that the polynomial $p(z) = z^2 - 2rz + r^2$ has a double root at $r$). Prove that $u_n = \alpha r^n + \beta n r^n$ for all $n \geq 1$, where the coefficients $\alpha, \beta$ are uniquely determined by the values of $u_1, u_2$.

Ex.7: Find a formula for the general term of the sequence defined by $u_1 = 3$ and the recurrence relation $u_{n+1} = u_n(u_n + 2)$ for $n \geq 1$.

Ex.8: For $n \geq 2$, prove that any $2n$ points joined by at least $n^2 + 1$ segments contain at least one triangle. Show that this is not true if the number of segments is $n^2$.

Ex.9: For integers $n, d \geq 0$, prove the relation

$$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+d}{n} = \binom{n+d+1}{n}.$$

Ex.10: Consider a $2^n \times 2^n$ checkerboard from which an arbitrary square has been removed. Can it be paved with polyominos, that is L-shaped tiles covering three squares each?

Ex.11: In Section 2, we have used the fact that if $r_1, \ldots, r_p$ are distinct nonzero numbers, then the system of $p$ linear equations $\alpha_1^n r_1^n + \cdots + \alpha_p^n r_p^n = u_n$, $n \in [1:p]$, in the $p$ unknowns $\alpha_1, \ldots, \alpha_p$ has a unique solution. In linear algebra terms, this condition is equivalent to the invertibility of the matrix whose $(i,j)$th entry is $r_j^i$. Establish this invertibility by proving the formula for the Vandermonde determinant, i.e.,

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_p \\ \vdots & \cdots & \ddots & \vdots \\ r_1^{p-1} & r_2^{p-1} & \cdots & r_p^{p-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (r_j - r_i).$$

Ex.12: Given a convex function $\varphi$ on an interval $I$ and given $x_0 \in I$, prove that the slope $x \in I \mapsto (f(x) - f(x_0))/(x - x_0)$ is an increasing function of $x \in I$.