Generating Functions

1 Definition and first examples

Generating functions offer a convenient way to carry the totality of the information about a sequence in a condensed form. Precisely, the (ordinary) generating function of the sequence $(a_n)_{n\geq 0}$ is defined as the formal power series

$$\sum_{n=0}^{\infty} a_n z^n.$$

For instance, the power series of the constant sequence $(1)_{n\geq 0}$ is $\sum_{n=0}^{\infty} z^n = 1/(1-z)$. From there, k successive differentiations lead to the generating of the sequence $\left(\binom{n+k}{k}\right)_{n\geq 0}$:

$$\sum_{n=0}^{\infty} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}}.$$

A more striking illustration concerns the number p_n of partitions of an integer n, i.e., the number of ways to write it as the sum of an nondecreasing sequence. For instance, $p_4 = 5$, since 4 can be written as $1+1+1+1 = 1+1+2 = 1+3 = 2+2 = 4$. Although there is no simple form for the sequence (p_n), its generating function admits a nice expression (uncovered by Euler), namely

$$\sum_{n=0}^{\infty} p_n z^n = \prod_{k=1}^{\infty} \frac{1}{1-z^k}.$$

This can be understood by looking at the coefficient of z^n in the right-hand side expressed as

$$(1+z+z^2+\cdots)(1+z^2+z^4+\cdots)(1+z^3+z^6+\cdots)\cdots.$$

Indeed, the coefficient of z^n is the number of ways to write

$$n = n_1 + 2n_2 + 3n_3 + \cdots = (1+\cdots+1) + (2+\cdots+2) + (3+\cdots+3) + \cdots,$$

which is precisely p_n.

2 Two classics: Fibonacci and Catalan

Sometimes, the cumbersome determination of the general term of a sequence can be shortcut by an argument exploiting generating functions. As a first example, consider the Fibonacci numbers defined by $F_0 = 1$, $F_1 = 1$, and

$$F_{n+2} = F_{n+1} + F_n \qquad \text{for } n \geq 0. \tag{1}$$

Let $f(z) := \sum_{n=0}^{\infty} F_n z^n$ denote the generating function of $(F_n)_{n\geq 0}$. Multiplying (1) by z^{n+2} and summing over all $n \geq 0$, we obtain

$$f(z) - F_0 - F_1 z = z f(z) - F_0 z + z^2 f(z), \qquad \text{i.e.,} \qquad f(z) = \frac{z}{1-z-z^2}.$$

Since $1 - z - z^2 = (1 - \phi z)(1 + z/\phi)$, where $\phi = (1 + \sqrt{5})/2$, we derive the partial fraction decomposition (remember to multiply through by $1 - \phi z$ and to take the value $z = 1/\phi$, next to multiply through by $1 + z/\phi$ and to take the value $z = -\phi$)

$$\frac{z}{1 - z - z^2} = \frac{1/\sqrt{5}}{1 - \phi z} - \frac{1/\sqrt{5}}{1 + z/\phi}.$$

Calling upon known power series expansions, we deduce

$$f(z) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} (\phi z)^n - \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} (-z/\phi)^n = \sum_{n=0}^{\infty} \frac{\phi^n - (-1/\phi)^n}{\sqrt{5}} z^n.$$

By identifying the coefficients of z^n, we conclude that

$$F_n = \frac{\phi^n - (-1/\phi)^n}{\sqrt{5}}.$$

As a second example, consider the Catalan numbers C_n defined (among many alternative definitions) as the numbers of binary trees that possess n branching nodes (hence $n+1$ leaves). Starting from $C_0 = 1$, they obey the recurrence relation

$$C_{n+1} = \sum_{i+j=n} C_i C_j, \qquad n \geq 0. \tag{2}$$

This translates the fact that a binary tree with $n + 1$ branching nodes is decomposed, when the root is removed, as two binary trees with i and j branching nodes satisfying $i + j = n$. Let $f(z) := \sum_{n=0}^{\infty} C_n z^n$ be the generating function of the Catalan numbers. Multiplying (2) by z^{n+1} and summing over all $n \geq 0$ leads to

$$f(z) - 1 = \sum_{n=0}^{\infty} \left(\sum_{i+j=n} C_i C_j \right) z^{n+1} = z \left(\sum_{i=0}^{\infty} C_i z^i \right) \left(\sum_{j=0}^{\infty} C_j z^j \right) = z f(z)^2.$$

Solving this quadratic equation in $f(z)$ gives (note that the second solution is rejected in view of its value at $z = 0$)

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

Calling upon known power series expansions, we deduce

$$\begin{aligned} f(z) &= \frac{1}{2z} \left(- \sum_{n=1}^{\infty} \frac{(1/2)(-1/2)(-3/2)\cdots(1/2 - n + 1)}{n!} (-4z)^n \right) = \sum_{n=1}^{\infty} 2^{n-1} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-3)}{n!} z^{n-1} \\ &= \sum_{n=0}^{\infty} 2^n \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{(n+1)\, n!} z^n = \sum_{n=0}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n-1) \cdot (2n)}{(n+1)\, n!\, n!} z^n = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} z^n. \end{aligned}$$

By identifying the coefficients of z^n, we conclude that

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

3 Stirling numbers

The Stirling numbers of the second kind, denoted $\left\{ n \atop k \right\}$, count the number of ways to partition $[1:n]$ into k nonempty blocks. For instance, $\left\{ 4 \atop 3 \right\} = 6$, since $\{1,2,3,4\}$ can be partitioned as $\{1\} \cup \{2\} \cup \{3,4\}$, $\{1\} \cup \{3\} \cup \{2,4\}$, $\{1\} \cup \{4\} \cup \{2,3\}$, $\{2\} \cup \{3\} \cup \{1,4\}$, $\{2\} \cup \{4\} \cup \{1,3\}$, and $\{3\} \cup \{4\} \cup \{1,2\}$. Note that $\left\{ n \atop k \right\} = 0$ for $k > n$ and for $k \le 0$ (unless $n = 0$, in which case the convention $\left\{ 0 \atop 0 \right\} = 1$ is used). The Stirling numbers of the second kind obey the recurrence relation

$$\left\{ n+1 \atop k \right\} = \left\{ n \atop k-1 \right\} + k \left\{ n \atop k \right\}. \tag{3}$$

This translates the fact that, when partitioning $[1:n+1]$ into k blocks, the element $n+1$ either forms a block on its own, leading to $k-1$ blocks that partition $[1:n]$, or it joins one of k blocks that partition $[1:n]$. For $k \ge 0$, consider the generating function $f_k(z) := \sum_{n=0}^{\infty} \left\{ n \atop k \right\} z^n$. With $k \ge 1$, multiplying (3) by z^{n+1} and summing over all $n \ge 0$ leads to

$$f_k(z) = z f_{k-1}(z) + kz f_k(z), \qquad \text{i.e.,} \qquad f_k(z) = \frac{z}{1-kz} f_{k-1}(z).$$

In view of $f_0(z) = 1$, we obtain by immediate induction

$$f_k(z) = \frac{z^k}{(1-z)(1-2z)\cdots(1-kz)}.$$

The partial fraction decomposition of the latter is

$$f_k(z) = c_0 + \frac{c_1}{1-z} + \cdots + \frac{c_k}{1-kz}.$$

We have $c_0 = \lim_{n\to\infty} f_k(z) = (-1)^k/k!$ and, for $j \in [1:k]$,

$$c_j = [f_k(z)(1-jz)]_{|z=1/j} = \frac{(1/j)^k}{(1-1/j)\cdots(1-(j-1)/j)(1-(j+1)/j)\cdots(1-k/j)} = \frac{(-1)^{k-j}}{j!(k-j)!}.$$

In conjunction with

$$f_k(z) = \sum_{j=0}^{k} \frac{c_j}{1-jz} = \sum_{j=0}^{k} c_j \sum_{n=0}^{\infty} (jz)^n = \sum_{n=0}^{\infty} \sum_{j=0}^{k} c_j j^n z^n,$$

we conclude that

$$\left\{ n \atop k \right\} = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!(k-j)!} j^n = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j} j^n.$$

The Stirling numbers of the first kind, denoted $\left[n \atop k \right]$, count the number of partitions of $[1:n]$ with k cycles. For instance, $\left[3 \atop 2 \right] = 3$, since $(1)(32)$, $(2)(13)$, and $(3)(12)$ are the three permutations of $\{1,2,3\}$ with two cycles. Note that $\left[n \atop k \right] = 0$ for $k > n$ and for $k \le 0$ (unless $n = 0$, in which case the convention $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$ is used). The Stirling numbers of the first kind obey the recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + n \begin{bmatrix} n \\ k \end{bmatrix}. \tag{4}$$

This translates the fact that, when considering a partition of $[1 : n+1]$ with k cycles, the element $n+1$ either forms a cycle on its own, leading to a permutation of $[1 : n]$ with $k-1$ cycles, or it incorporates (at one of n possible positions) one of k cycles making a partition of $[1 : n]$. For $k \geq 0$, consider now the exponential generating function of $\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{n \geq 0}$ given by

$$f_k(z) := \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{z^n}{n!}.$$

Multiplying (4) by $z^n/n!$ and summing over all $n \geq 0$ leads to

$$f_k'(z) = f_{k-1}(z) + z f_k'(z), \qquad \text{i.e.,} \qquad f_k'(z) = \frac{f_{k-1}(z)}{1-z}.$$

In view of $f_0(z) = 1$, we obtain by immediate induction

$$f_k(z) = \frac{1}{k!} \ln^k \left(\frac{1}{1-z} \right). \tag{5}$$

4 Exercises

Ex.1: Find the sequence $(a_n)_{n \geq 0}$ given by $a_0 = 1$ and $a_n = \dfrac{1 - \sum_{k=1}^{n-1} a_k a_{n-k}}{2}$ for $n \geq 1$.

Ex.2: Find the number of different ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.

Ex.3: Prove that the number of partitions of an integer into odd positive integers equals the number of its partitions into distinct positive integers.

Ex.4: It follows from the definition of the Stirling numbers of the first kind that $\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!$. Recover this fact from the expression (5) of the exponential generating function.

Ex.5: The positive differences of the four numbers $0, 2, 5, 6$ are the numbers $1, 2, 3, 4, 5, 6$, each taken exactly once. Prove that this phenomenon cannot occur if there are more than four numbers.