Classical Inequalities

Arithmetic-geometric means: The arithmetic mean $(a+b)/2$ of two nonnegative numbers a and b is always larger than or equal to its geometric mean $\sqrt{ab}$, with equality if and only if $a = b$. This can be seen from $a + b - 2\sqrt{ab} = (\sqrt{a} - \sqrt{b})^2 \geq 0$. The inequality generalizes to more than two numbers: for all $a_1, a_2, \ldots, a_n \geq 0$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. A weighted version involves weights $w_1, w_2, \ldots, w_n$ not all equal to $1/n$. Namely, given $w_1, w_2, \ldots, w_n > 0$ with $w_1 + w_2 + \cdots + w_n = 1$, for all $a_1, a_2, \ldots, a_n \geq 0$,

$$\sum_{i=1}^{n} w_i a_i \geq \prod_{i=1}^{n} a_i^{w_i}, \tag{1}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. This can be proved as follows.
Set $G := a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}$ and $A := w_1 a_1 + w_2 a_2 + \cdots + w_n a_n$. Assume without loss of generality that $a_1 \leq a_2 \leq \cdots \leq a_n$. Since $a_1 \leq G \leq a_n$, we consider the integer $k \in [1 : n-1]$ such that $a_k \leq G \leq a_{k+1}$. Then one can write

$$\sum_{i=1}^{k} w_i \int_{a_i}^{G} \left(\frac{1}{x} - \frac{1}{G} \right) dx + \sum_{i=k+1}^{n} w_i \int_{G}^{a_i} \left(\frac{1}{G} - \frac{1}{x} \right) dx \geq 0. \tag{2}$$

It follows that

$$\sum_{i=1}^{n} w_i \int_{G}^{a_i} \frac{dx}{G} \geq \sum_{i=1}^{n} w_i \int_{G}^{a_i} \frac{dx}{x}, \qquad \text{i.e.,} \quad \frac{A}{G} - 1 \geq \sum_{i=1}^{n} w_i \ln \frac{a_i}{G} = 0,$$

as desired. Equality throughout means equality in (2), i.e., $a_1 = \cdots = a_k = a_{k+1} = \cdots = a_n = G$.

Cauchy–Schwarz inequality: For all real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$,

$$\Big(\sum_{j=1}^{n} a_j b_j \Big)^2 \leq \Big(\sum_{j=1}^{n} a_j^2 \Big) \Big(\sum_{j=1}^{n} b_j^2 \Big),$$

with equality if and only if $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$. Cauchy–Schwarz inequality extends to other situations, for instance we can replace sums by integrals and obtain, for all real-valued functions f, g that are continuous on $[a, b]$,

$$\Big(\int_a^b f(x) g(x) dx \Big)^2 \leq \Big(\int_a^b f(x)^2 dx \Big) \Big(\int_a^b g(x)^2 dx \Big),$$

with equality if and only if $f = g$.

Hölder inequality: This is a generalization of Cauchy–Schwarz inequality to all $p, q > 1$ satisfying $1/p + 1/q = 1$ rather than $p = q = 2$. It reads, for all real numbers $a_1, \dots, a_n, b_1, \dots, b_n$,

$$\sum_{j=1}^{n} a_j b_j \le \Big(\sum_{j=1}^{n} |a_j|^p\Big)^{1/p} \Big(\sum_{j=1}^{n} |b_j|^q\Big)^{1/q},$$

with equality if and only if $a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$. The integral version reads, for all real-valued functions f, g that are continuous on $[a, b]$,

$$\int_a^b f(x)g(x)dx \le \Big(\int_a^b |f(x)|^p dx\Big)^{1/p} \Big(\int_a^b |g(x)|^q dx\Big)^{1/q},$$

with equality if and only if $f = g$. For the proof, set $u_j = |a_j|/A$ where $A := \big(\sum_{j=1}^n |a_j|^p\big)^{1/p}$ and $v_j = |b_j|/B$ where $B := \big(\sum_{j=1}^n |b_j|^q\big)^{1/q}$. Notice that is is enough to prove that $\sum_{j=1}^n u_j v_j \le 1$, knowing that $u_1, \dots, u_n, v_1, \dots, v_n \ge 0$, $\sum_{j=1}^n u_j^p = 1$, and $\sum_{j=1}^n v_j^q = 1$. In turn, it is enough to prove that $uv \le u^p/p + v^q/q$ for all $u, v \ge 0$ — this is known as Young's inequality. To justify the latter, rewrite it as $1 \le u^{p-1}v^{-1}/p + (p-1)u^{-1}v^{1/(p-1)}/p$, or, with $t := u^{-1}v^{1/(p-1)}$, as $t^{-(p-1)} + (p-1)t - 1 \ge 0$. This can now be seen by studying the variations of the function $f(x) := x^{-(p-1)} + (p-1)x - 1$ on $[0, \infty)$.

Jensen inequality: Let φ be a convex function on an interval I — if φ is twice differentiable, this means that $\varphi''(x) \ge 0$ for all $x \in I$. We have seen in 'Induction and Recurrence' that, if $x_1, \dots, x_n \in I$ and if $t_1, \dots, t_n \ge 0$ satisfy $t_1 + \cdots + t_n = 1$, then

$$\varphi\Big(\sum_{j=1}^{n} t_j x_j\Big) \le \sum_{j=1}^{n} t_j \varphi(x_j). \tag{3}$$

The integral version of Jensen inequality reads

$$\varphi\Big(\frac{1}{b-a}\int_a^b f(x)dx\Big) \le \frac{1}{b-a}\int_a^b \varphi(f(x))dx \tag{4}$$

for any continuous fonction f on $[a, b]$.

Chebyshev inequality: If $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ or $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, then

$$\frac{1}{n}\sum_{j=1}^{n} a_j b_j \ge \Big(\frac{1}{n}\sum_{j=1}^{n} a_j\Big)\Big(\frac{1}{n}\sum_{j=1}^{n} b_j\Big). \tag{5}$$

An easy argument consists in rearranging the inequality $\sum_{i,j=1}^n (a_i - a_j)(b_i - b_j) \ge 0$. An integral version of Chebyshev inequality reads, for functions f, g both nondecreasing on $[a, b]$ or both nonincreasing on $[a, b]$,

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \ge \Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big) \tag{6}$$

Rearrangement inequality: If $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ and if σ is a permutation of $[1:n]$, then

$$\sum_{j=1}^{n} a_j b_{n+1-j} \le \sum_{j=1}^{n} a_j b_{\sigma(j)} \le \sum_{j=1}^{n} a_j b_j. \tag{7}$$

One can use the technique of summation by parts for the proof of the rightmost inequality in (7), say. Setting $B_0 = 0$, $B'_0 = 0$, and

$$B_j = \sum_{i=1}^{j} b_{\sigma(i)}, \qquad B'_j = \sum_{i=1}^{n} b_i, \qquad j \in [1:n],$$

we have $B'_j \le B_j$ for $j \in [1:n-1]$ and $B'_n = B_n$. It follows that

$$\sum_{j=1}^{n} a_j b_{\sigma(j)} = \sum_{j=1}^{n} a_j B_j - \sum_{j=1}^{n} a_j B_{j-1} = a_n B_n + \sum_{j=1}^{n-1} \underbrace{(a_j - a_{j+1})}_{\le 0} \underbrace{B_j}_{\ge B'_j}$$

$$\le a_n B'_n + \sum_{j=1}^{n-1} (a_j - a_{j+1}) B'_j = \sum_{j=1}^{n} a_j b_j,$$

where the last equality is just the reversal of the summation by parts process.

1 Exercises

Ex.1: Prove the inequality between the geometric and harmonic means, namely

$$\frac{n}{1/a_1 + 1/a_2 + \cdots + 1/a_n} \le \sqrt[n]{a_1 a_2 \ldots a_n},$$

for all $a_1, a_2, \ldots, a_n > 0$.

Ex.2: For a continuous convex function φ on $[a,b]$, deduce (4) from (3).

Ex.3: For $a, b, c, d, \ldots \ge 0$, prove that

$$\sqrt{a+b+c+d+\cdots} + \sqrt{b+c+d+\cdots} + \sqrt{c+d+\cdots} + \cdots \ge \sqrt{a+4b+9c+16d+\cdots}.$$

Ex.4: Prove the leftmost inequality of (7).

Ex.5: Deduce (1) from Jensen inequality.

Ex.6: Prove Chebyshev inequality (5) using summation by parts.

Ex.7: Let $P(x)$ be a polynomial with positive coefficients. Prove that $P(1/x) \ge 1/P(x)$ for all $x > 0$, provided $P(1) \ge 1$.

Ex.8: If f is a continuous real-valued function on $[0,1]^2$, prove that

$$\int_0^1 \Big(\int_0^1 f(x,y)dx \Big)^2 dy + \int_0^1 \Big(\int_0^1 f(x,y)dy \Big)^2 dx \le \Big(\int_0^1 \int_0^1 f(x,y)dxdy \Big)^2 + \int_0^1 \int_0^1 f(x,y)^2 dxdy.$$