Lecture 4: Jordan Canonical Forms

This lecture introduces the Jordan canonical form of a matrix — we prove that every square matrix is equivalent to a (essentially) unique Jordan matrix and we give a method to derive the latter. We also introduce the notion of minimal polynomial and we point out how to obtain it from the Jordan canonical form. Finally, we make an encounter with companion matrices.

1 Jordan form and an application

Definition 1. A Jordan block is a matrix of the form

\[ J_1(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix} \]

when \( k = 1 \) and

\[ J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix} \]

when \( k \geq 2 \).

Theorem 2. For any \( A \in M_n \), there is an invertible matrix \( S \in M_n \) such that

\[ A = S \begin{bmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{n_2}(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & J_{n_k}(\lambda_k) \end{bmatrix} S^{-1} =: SJS^{-1}, \]

where \( n_1 + n_2 + \cdots + n_k = n \). The numbers \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are the (not necessarily distinct) eigenvalues of \( A \). The Jordan matrix \( J \) is unique up to permutation of the blocks.

Observation: two matrices close to one another can have Jordan forms far apart, for instance

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \]

Thus, any algorithm determining the Jordan canonical form is inevitably unstable! Try typing help jordan in MATLAB...

The Jordan form can be useful when solving a system of ordinary differential equations in the form \([x' = Ax, x(0) = x_0]\). If \( A = SJS^{-1} \) is the Jordan canonical form, then the change of unknown functions \( \tilde{x} = S^{-1}x \) transforms the original system to \([\tilde{x}' = J\tilde{x}, \tilde{x}(0) = \tilde{x}_0]\). Writing

\[ J = \begin{bmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{n_2}(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & J_{n_k}(\lambda_k) \end{bmatrix} \quad \text{and} \quad \tilde{x} = \begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(k) \end{bmatrix}, \]
the new system decouples as \( u'_{(\ell)} = J_n(\lambda_\ell)u_{(\ell)}, \, \ell \in [1 : k] \). Each of these \( k \) systems has the form

\[
\begin{bmatrix}
    u'_1 \\
    u'_2 \\
    \vdots \\
    u'_m
\end{bmatrix} =
\begin{bmatrix}
    \lambda & 1 & \cdots & 0 \\
    0 & \lambda & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & \lambda
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2 \\
    \vdots \\
    u_m
\end{bmatrix}.
\]

This can be solved by backward substitution: first, use \( u_m(t) = \lambda u_m(t) \) to derive

\[
u_m(t) = u_m(0)e^{\lambda t}.
\]

Then, \( u'_{m-1}(t) = \lambda u_{m-1}(t) + u_m(t) \) reads \( \frac{d}{dt}[u_{m-1}(t)e^{-\lambda t}] = [u'_{m-1}(t) - \lambda u_{m-1}(t)]e^{-\lambda t} = u_m(0) \), so that

\[
u_{m-1}(t) = [u_{m-1}(0) + u_m(0)t]e^{\lambda t}.
\]

Next, \( u'_{m-2}(t) = \lambda u_{m-2}(t)+u_{m-1}(t) \) reads \( \frac{d}{dt}[u_{m-2}(t)e^{-\lambda t}] = [u'_{m-2}(t) - \lambda u_{m-2}(t)]e^{-\lambda t} = u_{m-1}(0) + u_m(0)t \), so that

\[
u_{m-2}(t) = [u_{m-2}(0) + u_{m-1}(0)t + u_m(0)t^2]e^{\lambda t},
\]

etc. The whole vector \( [u_1, \ldots, u_m]^{\top} \) can be determined in this fashion.

## 2 Proof of Theorem 2

**Uniqueness:** The argument is based on the observation that, for

\[
N_k := \left\{ \begin{bmatrix}
    0 & 1 & 0 & \cdots & 0 \\
    0 & 0 & 1 & \cdots & \vdots \\
    0 & \ddots & \ddots & \ddots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & 1 & 0
\end{bmatrix} \right\}_{k},
\]

we have \( N_k^\ell = 0 \) if \( \ell \geq k \) and

\[
N_k^\ell = \begin{bmatrix}
    \leftarrow \ell \rightarrow \\
    0 & \cdots & 1 & \cdots & 0 \\
    0 & 0 & \ddots & 1 & \vdots \\
    0 & \ddots & \ddots & 1 & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\]

has rank \( k - \ell \) if \( \ell < k \).
Let \( \mu_1 < \ldots < \mu_\ell \) be the distinct values of \( \lambda_1, \ldots, \lambda_k \). After permutation,

\[
A \sim \begin{bmatrix}
J_{n_1,1}(\mu_1) & \cdots & \cdots & J_{n_\ell,1}(\mu_\ell) \\
& J_{n_1,h_1}(\mu_1) & \cdots & J_{n_\ell,h_\ell}(\mu_\ell) \\
& & \cdots & \\
& & & \end{bmatrix}.
\]

We are going to prove that for each \( \mu_i \) (say, for \( \mu_1 \)) and for each \( m \), the number of blocks \( J_m(\mu_i) \) of size \( m \) is uniquely determined by \( A \). From

\[
A - \mu I \sim \begin{bmatrix}
N_{n_1,1} & \cdots & \cdots & \\
& N_{n_1,h_1} & \cdots & \\
& & \cdots & \\
& & & \end{bmatrix},
\]

we obtain, for each \( m \geq 0 \),

\[
(A - \mu I)^m \sim \begin{bmatrix}
N_{n_1,1}^m & \cdots & \cdots & \\
& N_{n_1,h_1}^m & \cdots & \\
& & \cdots & \\
& & & \end{bmatrix},
\]

Therefore, we derive, for each \( m \geq 1 \),

\[
\text{rank} ((A - \mu I)^{m-1}) - \text{rank} ((A - \mu I)^m) = \begin{pmatrix} 1 & \text{if } m \leq n_{1,1} \\ 0 & \text{otherwise} \end{pmatrix} + \cdots + \begin{pmatrix} 1 & \text{if } m \leq n_{1,h_1} \\ 0 & \text{otherwise} \end{pmatrix} = \text{[number of Jordan blocks of size } \geq m] =: j_{\geq m}.
\]

Since the numbers \( j_{\geq m} \) are completely determined by \( A \), so are the numbers \( j_m = j_{\geq m} - j_{\geq m+1} \), which represent the numbers of Jordan blocks of size \( = m \).

Let us emphasize that the previous argument is also the basis of a method to find the Jordan canonical form. We illustrate the method on the example

\[
A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & 4 & -1 \end{bmatrix}.
\]
The eigenvalues of $A$ are the roots of $\det(xI - A) = (1 - x)^3$ — calculation left to the reader — hence $1$ is the unique eigenvalue of $A$. Therefore, there are three possibilities for the Jordan canonical form $J$ of $A$:

$$J_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

The observation that $\operatorname{rank}(J - I) = \operatorname{rank}(A - I) = 1$ — calculation left to the reader — shows that $J = J_2$ (since $\operatorname{rank}(J_1 - I) = 0$, $\operatorname{rank}(J_2 - I) = 1$, $\operatorname{rank}(J_3 - I) = 2$).

**Existence:** Let $\mu_1 < \cdots < \mu_\ell$ be the distinct eigenvalues of $A$. We know (see Lecture 1) that

$$A \sim \begin{bmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & T_\ell \end{bmatrix}, \quad \text{where} \quad T_i = \begin{bmatrix} \mu_i & x & \cdots & x \\ 0 & \mu_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \mu_i \end{bmatrix},$$

It is now enough to show that each matrix $T_i$ is equivalent to a Jordan matrix, i.e., that

$$T = \begin{bmatrix} \mu & x & \cdots & x \\ 0 & \mu & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \mu \end{bmatrix} \in \mathcal{M}_m \implies T \sim J = \begin{bmatrix} J_{m_1}(\mu) & 0 & \cdots & 0 \\ 0 & J_{m_2}(\mu) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{m_\ell}(\mu) \end{bmatrix},$$

with $m_1 + m_2 + \cdots + m_\ell = m$, or equivalently that

$$T = \begin{bmatrix} 0 & x & \cdots & x \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathcal{M}_m \implies T \sim M = \begin{bmatrix} N_{m_1} & 0 & \cdots & 0 \\ 0 & N_{m_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & N_{m_\ell} \end{bmatrix},$$

In other words, we want to find vectors

$$x_1, T(x_1), \ldots, T^{m_1-1}(x_1), \ldots, x_k, T(x_k), \ldots, T^{m_\ell-1}(x_k)$$

forming a basis of $\mathbb{C}^m$ and with $T^{m_1}(x_1) = 0, \ldots, T^{m_\ell}(x_k) = 0$. This is guaranteed by the following lemma, stated at the level of linear transformations.

**Lemma 3.** If $V$ is a vector space of dimension $m$ and if $T : V \to V$ is a linear map with $T^p = 0$ for some integer $p \geq 1$, then there exist integers $m_1, \ldots, m_\ell \geq 1$ with $m_1 + \cdots + m_\ell = m$ and vectors $v_1, \ldots, v_k \in V$ such that $(v_1, T(v_1), \ldots, T^{m_1-1}(v_1), \ldots, v_k, T(v_k), \ldots, T^{m_\ell-1}(v_k))$ forms a basis for $V$ and that $T^{m_1}(v_1) = 0, \ldots, T^{m_\ell}(v_k) = 0$. 

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Proof. We proceed by induction on $n \geq 1$. For $n = 1$, there is nothing to do. Let us now suppose the result true up to $m - 1$, $m \geq 2$, and let us prove that it holds for the integer $m$, too. Note that $U := \text{ran} \ T$ is a vector space of dimension $n < m$ (otherwise $T$ would be surjective, hence bijective, and $T^p = 0$ would be impossible). Consider the restriction of $T$ to $U$, i.e., the linear map $\tilde{T} : x \in U \mapsto Tx \in U$, and note that $\tilde{T}^p = 0$. Applying the induction hypothesis, there exist integers $n_1, \ldots, n_\ell \geq 1$ with $n_1 + \cdots + n_\ell = n$ and vectors $u_1, \ldots, u_\ell \in U$ such that $(u_1, \tilde{T}(u_1), \ldots, \tilde{T}^{n_1-1}(u_1), \ldots, u_\ell, \tilde{T}(u_\ell), \ldots, \tilde{T}^{n_\ell-1}(u_\ell))$ forms a basis for $U$ and that $\tilde{T}^{n_1}(u_1) = 0, \ldots, \tilde{T}^{n_\ell}(u_\ell) = 0$. Since $u_1, \ldots, u_\ell \in U = \text{ran} \ T$, there exist $v_1, \ldots, v_\ell \in V$ such that $u_i = Tv_i$. Note that $T^{n_1+1}(v_1) = 0$, so that $(T^{n_1}(v_1), \ldots, T^{n_\ell}(v_\ell))$ is a linearly independent system of $\ell$ vectors in $\ker \ T$, which has dimension $m - n$. Complete this system with vectors $v_{\ell+1}, \ldots, v_{m-n} \in \ker \ T$ to form a basis for $\ker \ T$. Now consider the system $(v_1, T(v_1), \ldots, T^{n_1}(v_1), \ldots, v_\ell, T(v_\ell), \ldots, T^{n_\ell}(v_\ell), v_{\ell+1}, \ldots, v_{m-n})$. Observe that this is a linearly independent system. Observe also that the number of vectors in this system is $n_1 + 1 + \cdots + n_\ell + 1 + m - n - \ell = n_1 + \cdots + n_\ell + m - n = m$, so that the system is in fact a basis for $V$. Finally, observe that $T^{n_1+1}(v_1) = 0, \ldots, T^{n_\ell+1}(v_\ell) = 0, T(v_{\ell+1}) = 0, \ldots, T(v_{m-n}) = 0$ to conclude that the induction hypothesis is true for the integer $m$. This finishes the proof. \qed

3 Minimal polynomial

Let $A \in \mathcal{M}_n$ be given. Consider the set of monic (i.e., having a leading coefficient equal to 1) polynomials $p$ that annihilates $A$ (i.e., such that $p(A) = 0$). According Cayley–Hamilton’s theorem, this set contains at least one polynomial, namely the characteristic polynomial $p_A$ (possibly multiplied by $(-1)^n$, depending on the convention). In this set, we can consider a polynomial of minimal degree. It turns out that there is only one such polynomial.

Theorem 4. Given $A \in \mathcal{M}_n$, there exists a unique monic polynomial of minimal degree that annihilates $A$. This polynomial, called the minimal polynomial of $A$ and denoted $q_A$, divides all polynomials that annihilate $A$.

Proof. Let $m$ be a monic polynomial of minimal degree that annihilates $A$, and let $p$ be an arbitrary polynomial that annihilates $A$. The Euclidean division of $p$ by $m$ reads

\[ p(x) = q(x)m(x) + r(x), \quad \text{deg}(r) < \text{deg}(m). \]

Note that $p(A) = q(A)m(A) + r(A) = 0$ and $m(A) = 0$ imply $r(A) = 0$. But since $\text{deg}(r) < \text{deg}(m)$, we must have $r = 0$. This means that $m$ divides $p$.

Now let $\tilde{m}$ be another monic polynomial of minimal degree that annihilates $A$. By the previous argument, $m$ divides $\tilde{m}$ and $\tilde{m}$ divides $m$, so that $m$ and $\tilde{m}$ are constant multiples of each other. Since they are both monic, we deduce that $m = \tilde{m}$. This means that a monic polynomial of minimal degree that annihilates $A$ is unique. \qed
Observation: Since the minimal polynomial $q_A$ divides the characteristic polynomial $p_A$, if $p_A(x) = (x - \mu_1)^{m_1} \cdots (x - \mu_k)^{m_k}$ with $n_1, \ldots, n_k \geq 1$, then $q_A(x) = (x - \mu_1)^{m_1} \cdots (x - \mu_k)^{m_k}$ with $1 \leq m_1 \leq n_1, \ldots, 1 \leq m_k \leq n_k$. (Note that $m_i \geq 1$ holds because, if $x_i \neq 0$ is an eigenvector corresponding to $\mu_i$, then $0 = q_A(A)(x_i) = q_A(\mu_i)x_i$, hence $q_A(\mu_i) = 0$).

Observation: One can read the minimal polynomial out of the Jordan canonical form (since equivalent matrices have the same minimal polynomial): the $m_i$ are the order of the largest Jordan block of $A$ corresponding to the eigenvalue $\mu_i$.

**Theorem 5.** For $a_0, a_1, \ldots, a_{n-1}$, the matrix

$$
A = \begin{bmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & \ddots & \vdots & -a_1 \\
0 & 1 & \ddots & 0 & \vdots \\
\vdots & \ddots & \ddots & 0 & -a_{n-2} \\
0 & \cdots & 0 & 1 & -a_{n-1}
\end{bmatrix}
$$

has characteristic polynomial and minimal polynomial given by

$$
p_A(x) = q_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.
$$

The matrix $A$ is called the companion matrix of the polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$.

**Proof.** We first determine the characteristic polynomial as

$$
p_A(x) = \det(xI - A) = \begin{vmatrix}
x & 0 & \cdots & 0 & a_0 \\
-1 & x & \ddots & \vdots & a_1 \\
0 & -1 & \ddots & 0 & \vdots \\
\vdots & \ddots & \ddots & x & a_{n-2} \\
0 & \cdots & 0 & -1 & x + a_{n-1}
\end{vmatrix}
$$

Expanding along the last column shows that $p_A(x)$ equals

$$
(-1)^{n+1}a_0 \cdot (-1)^{n-1} + (-1)^{n+2}a_1 \cdot (-1)^{n-2}x + \cdots + (-1)^{2n-1}a_{n-2} \cdot (-1)x^{n-2} + (-1)^{2n}(a_{n-1} + x) \cdot x^{n-1}
$$

$$
= a_0 + a_1x + \cdots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1} + x^n,
$$
as expected. To verify that $q_A = p_A$, it now suffices to show that $q_A$ cannot have degree $m < n$. Suppose the contrary, i.e., that $q_A$ has the form $q_A(x) = x^m + c_{m-1}x^{m-1} + \cdots + c_1x + c_0$ with $m < n$. Then

$$
0 = q_A(A)(e_1) = (A^m + c_{m-1}A^{m-1} + \cdots + c_1A + c_0 I)(e_1) = e_{m+1} + c_{m-1}e_m + \cdots + c_1e_2 + c_0e_1,
$$

which contradicts the linear independence of the basis vectors $e_{m+1}, e_m, \ldots, e_2, e_1$. \qed
4 Exercises

Ex.1: Prove that if $A \in \mathcal{M}_n$ satisfies $A^p = 0$ for some integer $p \geq 1$, then it satisfies $A^n = 0$.

Ex.2: Exercise 2 p. 129

Ex.3: Exercise 2 p. 139

Ex.4: Exercise 6 p. 140

Ex.5: Exercise 7 p. 140

Ex.6: Exercise 8 p. 140

Ex.7: Exercise 17 p. 141

Ex.8: Exercise 9 p. 149

Ex.9: Exercise 13 p. 150