Kronecker coefficients and noncommutative super Schur functions

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September 2016
The Kronecker problem

$M_\lambda$ denotes the irreducible representation of the symmetric group corresponding to the partition $\lambda$.

The **Kronecker coefficient** $g_{\lambda\mu\nu}$ is the multiplicity of $M_\nu$ in the tensor product $M_\lambda \otimes M_\mu$, where $\lambda, \mu, \nu$ are partitions of $n$.

**Kronecker problem**

Find a positive combinatorial formula for the Kronecker coefficients $g_{\lambda\mu\nu}$. 
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Outline

- Formula for Kronecker coefficients when one of the shapes is a hook
- Noncommutative Schur functions and switchboards
- Noncommutative super Schur functions for the Kronecker problem
- Lascoux’s heuristic for Kronecker coefficients
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Colored Yamanouchi words

- \{1, 2, \ldots, N\} = alphabet of *unbarred letters*
- \{\overline{1}, \overline{2}, \ldots, \overline{N}\} = alphabet of *barred letters*
- \mathcal{A} = alphabet of barred and unbarred letters ordered by 
  \[1 < \overline{1} < 2 < \overline{2} < \cdots < N < \overline{N}\]
- A *colored word* is a word in the alphabet \(\mathcal{A}\).

\(w^{\text{bret}} := \) the ordinary word formed from \(w\) by shuffling the barred letters to the right, reversing this subword of barred letters and removing their bars.

\(w\) is *Yamanouchi* of content \(\lambda\) if \(w^{\text{bret}}\) is Yamanouchi of content \(\lambda\).

**Example**

\[w = 2\overline{1}21\overline{3}\overline{1}21\]
\[w^{\text{bret}} = 21211321\]

These words are Yamanouchi of content \((4, 3, 1)\).
Colored Yamanouchi words

- \{1, 2, \ldots, N\} = alphabet of \textit{unbarred letters}
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- A \textit{colored word} is a word in the alphabet \(\mathcal{A}\).

\(w^\text{brgt} :=\) the ordinary word formed from \(w\) by shuffling the barred letters to the right, reversing this subword of barred letters and removing their bars. \(w\) is \textit{Yamanouchi} of content \(\lambda\) if \(w^\text{brgt}\) is Yamanouchi of content \(\lambda\).

Example

\[
\begin{align*}
    w &= 2\overline{1}21\overline{3}\overline{1}21 \\
    w^\text{brgt} &= 21211321
\end{align*}
\]

These words are Yamanouchi of content \((4, 3, 1)\).
A *colored tableau* is a tableau with entries in $\mathcal{A}$ such that

- each row and column is weakly increasing,
- the unbarred letters in each column are strictly increasing,
- the barred letters in each row are strictly increasing.

For a colored tableau $T$, the word $\text{barread}(T)$ is obtained by reading the diagonals of $T$ one by one, starting from the southwest corner. In each diagonal, read the unbarred entries in the direction $\searrow$, followed by the barred entries in the direction $\swarrow$.

**Example**

$$\text{barread}\left(\begin{array}{cccc}
1 & 1 & 3 & 4 \\
\overline{2} & 3 & 4 & \overline{4} \\
3 & \overline{3} & \overline{4} & 5
\end{array}\right) = 3 \overline{2} 3 \overline{3} 3 1 \overline{4} 5 4 1 \overline{3} \overline{4} 4 \overline{6}.$$
A colored tableau is a tableau with entries in $A$ such that

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For a colored tableau $T$, the word $\text{barread}(T)$ is obtained by reading the diagonals of $T$ one by one, starting from the southwest corner. In each diagonal, read the unbarred entries in the direction $\swarrow$, followed by the barred entries in the direction $\searrow$.

**Example**

$$\text{barread} \begin{pmatrix} 1 & 1 & 3 & 4 & 6 \\ 2 & 3 & 4 & \bar{4} \\ 3 & 3 & 4 & 5 \end{pmatrix} = 3 \overline{2} \overline{3} 3 1 \overline{4} 5 4 1 \overline{3} \overline{4} \overline{4} 6.$$
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For a colored tableau $T$, the word $\text{barread}(T)$ is obtained by reading the diagonals of $T$ one by one, starting from the southwest corner. In each diagonal, read the unbarred entries in the direction ↘, followed by the barred entries in the direction ↙.

**Example**

\[
\begin{pmatrix}
1 & 1 & 3 & 4 & 6 \\
\bar{2} & 3 & 4 & 4 \\
3 & \bar{3} & 4 & 5 \\
\end{pmatrix}
\]

\[
\text{barread}
\begin{pmatrix}
1 & 1 & 3 & 4 & 6 \\
\bar{2} & 3 & 4 & 4 \\
3 & \bar{3} & 4 & 5 \\
\end{pmatrix}
\] 

$= 3 \bar{2} \bar{3} \bar{3} \bar{1} \bar{4} 5 4 1 \bar{3} \bar{4} \bar{4} 6$. 

Colored tableaux

A colored tableau is a tableau with entries in $A$ such that

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For a colored tableau $T$, the word $\text{barread}(T)$ is obtained by reading the diagonals of $T$ one by one, starting from the southwest corner. In each diagonal, read the unbarred entries in the direction $\searrow$, followed by the barred entries in the direction $\nearrow$.

Example

$$\text{barread}\left(\begin{array}{cccc}
1 & 1 & 3 & 4 \\
2 & 3 & 4 & 4 \\
3 & 3 & 4 & 5 \\
\end{array}\right) = 3 \bar{2} \bar{3} 3 1 \bar{4} 5 4 \bar{1} \bar{3} \bar{4} \bar{4} 6.$$
A **colored tableau** is a tableau with entries in $\mathcal{A}$ such that

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For a colored tableau $T$, the word $\text{barread}(T)$ is obtained by reading the diagonals of $T$ one by one, starting from the southwest corner. In each diagonal, read the unbarred entries in the direction $\rightarrow$, followed by the barred entries in the direction $\downarrow$.

**Example**

$$\text{barread}\left(\begin{array}{cccc}
1 & 1 & 3 & 4 \\
2 & 3 & 4 & 4 \\
3 & 3 & 4 & 5 \\
\end{array}\right) = 3 \bar{2} \bar{3} 3 1 \bar{4} 5 \bar{4} \bar{3} \bar{4} \bar{6}.$$
A *colored tableau* is a tableau with entries in $A$ such that

- each row and column is weakly increasing,
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- the barred letters in each row are strictly increasing.

For a colored tableau $T$, the word $\text{barread}(T)$ is obtained by reading the diagonals of $T$ one by one, starting from the southwest corner. In each diagonal, read the unbarred entries in the direction $\uparrow\downarrow$, followed by the barred entries in the direction $\downarrow\uparrow$.

**Example**

\[
\begin{array}{cccccc}
1 & 1 & 3 & 4 & 6 \\
2 & 3 & 4 & 4 \\
3 & 3 & 4 & 5 \\
\end{array}
\]

$\text{barread} \left( \begin{array}{cccccc}
1 & 1 & 3 & 4 & 6 \\
2 & 3 & 4 & 4 \\
3 & 3 & 4 & 5 \\
\end{array} \right) = 3\ 2\ 3\ 3\ 1\ 4\ 5\ 4\ 1\ 3\ 4\ 4\ 6.$
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Example

\[
\begin{pmatrix}
1 & 1 & 3 & 4 & 6 \\
2 & 3 & 4 & 4 \\
3 & 3 & 4 & 5 \\
\end{pmatrix}
\]

\[
\text{barread}
\begin{pmatrix}
1 & 1 & 3 & 4 & 6 \\
2 & 3 & 4 & 4 \\
3 & 3 & 4 & 5 \\
\end{pmatrix}
= 3 \overline{2} \overline{3} \overline{3} \overline{1} \overline{4} \overline{5} \overline{4} \overline{1} \overline{3} \overline{4} \overline{4} \overline{6}.
\]
Colored tableaux

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Example

\[
\text{barread}\left(\begin{array}{cccc}
1 & 1 & 3 & 4 \\
2 & 3 & 4 & 4 \\
3 & 3 & 4 & 5 \\
\end{array}\right) = 3 \, 2 \, 3 \, 3 \, 1 \, 4 \, 5 \, 4 \, 1 \, 3 \, 4 \, 4 \, 6.
\]
A \textit{colored tableau} is a tableau with entries in \( A \) such that
- each row and column is weakly increasing,
- the unbarred letters in each column are strictly increasing,
- the barred letters in each row are strictly increasing.

For a colored tableau \( T \), the word \( \text{barread}(T) \) is obtained by reading the diagonals of \( T \) one by one, starting from the southwest corner. In each diagonal, read the unbarred entries in the direction \( \nwarrow \), followed by the barred entries in the direction \( \searrow \).

\textbf{Example}

\[
\begin{array}{cccccc}
1 & 1 & 3 & 4 & 6 \\
2 & 3 & 4 & 4 & \\
3 & 3 & 4 & 5 & \\
\end{array}
\]

\[
\begin{array}{cccccc}
\bar{1} & 1 & \bar{3} & 4 & 6 \\
2 & 3 & 4 & \bar{4} & \\
3 & 3 & 4 & 5 & \\
\end{array}
\]

\[
\text{barread}\left(\begin{array}{cccccc}
1 & 1 & 3 & 4 & 6 \\
2 & 3 & 4 & 4 & \\
3 & 3 & 4 & 5 & \\
\end{array}\right) = 3\bar{2}3\bar{3}31\bar{4}5\bar{4}13\bar{4}4\bar{6}.
\]
Kronecker coefficients for one hook shape

\( \mu(d) \) denotes the hook partition \((n - d, 1^d)\) = \(\begin{array}{c} n - d \\ \hline \end{array}\)

**Theorem (R. Liu)**

For any partitions \(\lambda, \nu\) of \(n\) and \(d \leq n - 1\), the Kronecker coefficient \(g_{\lambda \mu(d) \nu}\) is the number of colored tableaux \(T\) such that

- \( \text{barread}(T) \) is Yamanouchi of content \(\lambda\),
- \( T \) has exactly \(d\) barred letters,
- \( T \) has shape \(\nu\),
- the northeast corner of \(T\) is unbarred.

\[
\begin{array}{cccc}
1 & 1 & 1 & 3 \\
\bar{1} & 2 & \bar{2} & 4 \\
\bar{1} & 3 \\
\bar{1} & 4
\end{array}
\]

\(\lambda = (6, 2, 2, 2)\)

\(\mu(d) = (8, 1, 1, 1, 1)\)

\(\nu = (4, 4, 2, 2)\)
Kronecker coefficients for one hook shape

\[ \mu(d) \text{ denotes the hook partition } (n - d, 1^d) = d \{\begin{array}{c}
\end{array}\} \]

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- \( T \) has shape \( \nu \),
- the northeast corner of \( T \) is unbarred.

\[
\begin{array}{cccc}
1 & 1 & 1 & 3 \\
\bar{1} & 2 & \bar{2} & 4 \\
\bar{1} & 3 \\
\bar{1} & 4 \\
\end{array}
\]

\( \lambda = (6, 2, 2, 2) \)
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Kronecker coefficients for one hook shape

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- the northeast corner of \(T\) is unbarred.

\[
\begin{array}{cccc}
1 & 1 & 1 & 3 \\
\bar{1} & 2 & \bar{2} & 4 \\
\bar{1} & 3 \\
\bar{1} & 4 \\
\end{array}
\]

\(\lambda = (6, 2, 2, 2)\)

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\( \mu(d) \) denotes the hook partition \((n - d, 1^d)\) = \(d\) \begin{array}{l}\hline\mid n - d \mid \hline\end{array}\)

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For any partitions \(\lambda, \nu\) of \(n\) and \(d \leq n - 1\), the Kronecker coefficient \(g_{\lambda \mu(d) \nu}\) is the number of colored tableaux \(T\) such that

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- \(T\) has exactly \(d\) barred letters,
- \(T\) has shape \(\nu\),
- the northeast corner of \(T\) is unbarred.

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\bar{1} & 2 & 2 \\
\bar{1} & 3 & \\
\bar{1} & 4 & \\
\end{array}
\]

\[
\lambda = (6, 2, 2, 2) \\
\mu(d) = (8, 1, 1, 1, 1) \\
\nu = (4, 4, 2, 2)
\]
Kronecker coefficients for one hook shape

The colored tableaux $T$ and $\text{barread}(T)$ such that

- $\text{barread}(T)$ is Yamanouchi of content $(6, 2, 2, 2)$,
- $T$ has exactly 4 barred letters,
- $T$ has shape $(4, 4, 2, 2)$,
- the northeast corner of $T$ is unbarred.

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\bar{1} & \bar{2} & 3 & 3 \\
\bar{1} & 2 \\
4 & 4 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 1 & 1 & 3 \\
\bar{1} & 2 & 2 & 4 \\
\bar{1} & 3 \\
\bar{1} & 4 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\bar{1} & 2 & 2 & 3 \\
\bar{1} & 3 \\
4 & 4 \\
\end{array}
\]

$g(6, 2, 2, 2)(8, 1, 1, 1)(4, 4, 2, 2) = 3$
Noncommutative Schur functions are a powerful tool for solving positivity problems. They have led to positive combinatorial formulae for

- the Schur expansion of Stanley symmetric functions and stable Grothendieck polynomials.
- the Schur expansion of LLT polynomials indexed by 3-tuples of skew shapes and transformed Macdonald polynomials indexed by shapes with 3 columns.
- Kronecker coefficients for one hook shape and two arbitrary shapes.
Two words of the same length are related by a *switch* in position $i$ if

- their entries match to the left of position $i−1$ and to the right of $i+1$;
- their entries in positions $i−1, i, i+1$ are related in one of these ways:

\[
\begin{align*}
\cdots & \text{bac} \cdots & \text{Knuth}\text{-rotation} & \cdots & \text{bca} \cdots \\
\cdots & \text{acb} \cdots & \text{Knuth}\text{-rotation} & \cdots & \text{cab} \cdots \\
\cdots & \text{bab} \cdots & \text{braid} & \cdots & \text{bba} \cdots \\
\cdots & \text{aba} \cdots & \text{idempotent} & \cdots & \text{baa} \cdots
\end{align*}
\]

A *switchboard* is an edge-labeled graph on a vertex set of words such that

- each edge labeled $i$ corresponds to a switch in position $i$;
- if a word $w$ appearing in the switchboard can in principle be switched in position $i$, then $w$ is incident to exactly one edge labeled $i$. 
Switchboards

Two words of the same length are related by a switch in position $i$ if

- their entries match to the left of position $i-1$ and to the right of $i+1$;
- their entries in positions $i-1$, $i$, $i+1$ are related in one of these ways:

$$
\begin{array}{c}
\cdots \text{bac} \cdots \quad \text{Knuth} \quad \cdots \text{bca} \cdots \\
\text{rotation} \quad \text{rotation} \\
\cdots \text{acb} \cdots \quad \text{Knuth} \quad \cdots \text{cab} \cdots \\
\end{array}
$$

$$
\begin{array}{c}
\cdots \text{bab} \cdots \quad \text{Knuth} \quad \cdots \text{bba} \cdots \\
\text{braid} \quad (b = a + 1) \\
\cdots \text{aba} \cdots \quad \text{Knuth} \quad \cdots \text{baa} \cdots \\
\text{idempotent} \quad (b = a + 1)
\end{array}
$$

A switchboard is an edge-labeled graph on a vertex set of words such that

- each edge labeled $i$ corresponds to a switch in position $i$;
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Switchboards

2134 \[ \overset{2}{\longrightarrow} \] 2314 \[ \overset{3}{\longrightarrow} \] 2341

2143 \[ \overset{2}{\longrightarrow} \] \[ \overset{3}{\longrightarrow} \] 2413

Figure: A switchboard whose 3-edges are Knuth switches.
Switchboards

2134 \sim 2 \quad 2314 \quad 2341
\sim 3 \quad \sim 3
2143 \sim 2 \quad 2413

Figure: A switchboard whose 3-edges are rotation switches.
Switchboards

Figure: Different switchboards on the same set of vertices.
The switchboard on $S_4$ with only Knuth switches

\[
\begin{align*}
2134 & \quad 2 \quad 2314 \quad 3 \quad 2341 \quad \frac{2}{3} \quad 3421 \quad 2 \quad 3241 \quad 3 \quad 3214 \\
3124 & \quad 2 \quad 1324 \quad 3 \quad 1342 \quad \frac{2}{3} \quad 2431 \quad 2 \quad 4231 \quad 3 \quad 4213 \\
4123 & \quad 2 \quad 1423 \quad 3 \quad 1243 \quad \frac{2}{3} \quad 1432 \quad 2 \quad 4132 \quad 3 \quad 4312 \\
1234 & \quad 2413 \quad \frac{2}{3} \quad 2143 \quad 4321 \\
3412 & \quad \frac{2}{3} \quad 3142
\end{align*}
\]
The switchboard on $S_4$ with only rotation switches

1234

2134 $\bar{2}$ 1324 $\bar{3}$ 1243

3124 $\bar{2}$ 2314 $\bar{3}$ 2143 $\bar{2}$ 1423 $\bar{3}$ 1342

1432 $\bar{2}$ 3142 $\bar{3}$ 3214 4123 $\bar{2}$ 2413 $\bar{3}$ 2341

2431 $\bar{2}$ 3241 $\bar{3}$ 3412 $\bar{2}$ 4132 $\bar{3}$ 4213

3421 $\bar{2}$ 4231 $\bar{3}$ 4312

4321
The symmetric function associated to a switchboard

- Let \( x = (x_1, x_2, \ldots) \) be commuting variables.
- \( \text{Des}(w) = \{i \in \{1, \ldots, n-1\} \mid w_i > w_{i+1}\} \) denotes the descent set of a word \( w = w_1 \cdots w_n \).
- Gessel’s fundamental quasisymmetric function is
  \[
  Q_{\text{Des}(w)}(x) = \sum_{1 \leq i_1 \leq \cdots \leq i_n, \ j \in \text{Des}(w) \Rightarrow i_j < i_{j+1}} x_{i_1} \cdots x_{i_n}.
  \]
- For a switchboard \( \Gamma \), define
  \[
  F_\Gamma(x) = \sum_{w \in \text{Vert}(\Gamma)} Q_{\text{Des}(w)}(x).
  \]

Theorem (B.-Fomin)

*For any switchboard \( \Gamma \), the function \( F_\Gamma(x) \) is symmetric in \( x_1, x_2, \ldots \).*
The symmetric function associated to a switchboard

- Let \( x = (x_1, x_2, \ldots) \) be commuting variables.
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  \[
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  \]
  
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The symmetric function associated to a switchboard

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- Gessel’s fundamental quasisymmetric function is
  \[
  Q_{\text{Des}(w)}(x) = \sum_{1 \leq i_1 \leq \cdots \leq i_n} \sum_{\substack{j \in \text{Des}(w) \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.
  \]
- For a switchboard \( \Gamma \), define
  \[
  F_\Gamma(x) = \sum_{w \in \text{Vert}(\Gamma)} Q_{\text{Des}(w)}(x).
  \]

**Theorem (B.-Fomin)**

*For any switchboard \( \Gamma \), the function \( F_\Gamma(x) \) is symmetric in \( x_1, x_2, \ldots \).*
The symmetric function associated to a switchboard

\[ 132 \quad ^2 \quad 312 \]

\[
\begin{align*}
\times_1 \times_1 \times_2 \\
\times_1 \times_1 \times_3 \\
\times_1 \times_2 \times_3 \\
\times_2 \times_2 \times_3 \\
\end{align*}
\]

\[
\begin{align*}
\times_1 \times_2 \times_2 \\
\times_1 \times_2 \times_3 \\
\times_1 \times_3 \times_3 \\
\times_2 \times_3 \times_3 \\
\end{align*}
\]

**Figure:** A switchboard \( \Gamma \) with \( F_\Gamma = s_{21} \). The sum of the outlined monomials is \( s_{21}(x_1, x_2, x_3) \).

\[ 2134 \quad ^2 \quad 2314 \quad ^3 \quad 2143 \quad ^2 \quad 1423 \quad ^3 \quad 1243 \]

\[
\begin{align*}
\times_1 \times_2 \times_2 \times_2 \\
\times_1 \times_2 \times_2 \times_3 \\
\times_1 \times_2 \times_3 \times_3 \\
\times_2 \times_3 \times_3 \times_3 \\
\end{align*}
\]

\[
\begin{align*}
\times_1 \times_1 \times_2 \times_2 \\
\times_1 \times_1 \times_2 \times_3 \\
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**Figure:** A switchboard \( \Gamma' \) with \( F_{\Gamma'} = s_{31} + s_{22} \). The sum of the outlined monomials is \( s_{31}(x_1, x_2, x_3) + s_{22}(x_1, x_2, x_3) \).
The symmetric function associated to a switchboard

\[ 132^2 312 \]

\[
\begin{array}{c}
\times_1 \times_1 \times_2 \\
\times_1 \times_1 \times_3 \\
\times_1 \times_2 \times_3 \\
\times_2 \times_2 \times_3
\end{array}
\begin{array}{c}
\times_1 \times_2 \times_2 \\
\times_1 \times_2 \times_3 \\
\times_1 \times_3 \times_3 \\
\times_2 \times_3 \times_3
\end{array}
\]

**Figure:** A switchboard \( \Gamma \) with \( F_\Gamma = s_{21} \). The sum of the outlined monomials is \( s_{21}(x_1, x_2, x_3) \).

\[ 2134^2 2314^3 2143^2 1423^3 1243 \]

\[
\begin{array}{c}
\times_1 \times_2 \times_2 \times_2 \\
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Noncommutative Schur functions

Let $\mathcal{U} = \mathbb{Z}\langle u_1, u_2, \ldots, u_N \rangle$. We identify the monomials in $\mathcal{U}$ with words in the alphabet $[N]$ and write $312$, $cab$, etc. for words/monomials.

Noncommutative elementary symmetric functions:

$$e_k(u) = \sum_{N \geq i_1 > i_2 > \cdots > i_k \geq 1} u_{i_1} u_{i_2} \cdots u_{i_k}$$

Noncommutative Schur functions $\mathcal{J}_\lambda(u)$:

- $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition.
- $\lambda'$ is the conjugate partition.
- $t = \lambda_1$ is the number of parts of $\lambda'$.

$$\mathcal{J}_\lambda(u) = \sum_{\pi \in S_t} \text{sgn}(\pi) e_{\lambda'_1 + \pi(1) - 1}(u) e_{\lambda'_2 + \pi(2) - 2}(u) \cdots e_{\lambda'_t + \pi(t) - t}(u).$$
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Schur expansion of $F_{\gamma}(x)$

- For $\gamma = \sum_{\text{words } w} \gamma_w w \in \mathcal{U}$, define $F_{\gamma}(x) = \sum_{\text{words } w} \gamma_w Q_{\text{Des}(w)}(x)$.
- We consider $\mathcal{U}$ to be endowed with the symmetric bilinear form for which the monomials/words form an orthonormal basis.
- Note that any element of $\mathcal{U}/I$ has a well-defined pairing with any element of $I^\perp$, for any ideal $I$ of $\mathcal{U}$.

**Theorem (Fomin-Greene)**

Let $I$ be an ideal of $\mathcal{U}$ such that the $e_k(u)$ commute pairwise in $\mathcal{U}/I$. Then for any $\gamma \in I^\perp$,

$$F_{\gamma}(x) = \sum_{\lambda} s_{\lambda}(x) \langle \hat{J}_{\lambda}(u), \gamma \rangle.$$
Let $\mathcal{U}/I_S$ be the quotient of $\mathcal{U}$ by the relations
\[
\begin{align*}
    b(ca - ac) &= (ca - ac)b \quad \text{for} \ a < b < c, \\
    caa &= aca, \quad cca = cac \quad \text{for} \ c - a > 1, \\
    bab + baa &= bba + aba \quad \text{for} \ b = a + 1.
\end{align*}
\]

**Theorem (A. N. Kirillov, B.-Fomin)**

The $e_k(u)$ commute pairwise in $\mathcal{U}/I_S$.

**Proposition**

For a set of words $W$ of the same length, the following are equivalent:
- $\sum_{w \in W} w \in I_S^\perp$;
- $W$ is the vertex set of a switchboard.

**Corollary**

For any switchboard $\Gamma$, $F_{\Gamma}(x) = \sum_{\lambda} s_{\lambda}(x) \langle \mathcal{J}_\lambda(u), \sum_{w \in \text{Vert}(\Gamma)} w \rangle.$
Recipe for positive combinatorial formulae

We obtain the following recipe for producing positive combinatorial formulae for the coefficients in the Schur expansions of many classes of symmetric functions:

1. Find a suitable ideal $I \supset I_S$ such that symmetric functions of interest are of the form $F_\Gamma(x)$ with $\gamma = \sum_{w \in \text{Vert}(\Gamma)} w \in I^\perp$.

2. Find a monomial positive expression for $\tilde{J}_\lambda(u)$ in $U/I$.

3. Apply (*) to obtain a positive formula for the Schur expansion of $F_\Gamma$.

$$F_\Gamma(x) = F_\gamma(x) = \sum_\lambda s_\lambda(x) \langle \tilde{J}_\lambda(u), \gamma \rangle \quad (*)$$
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### Noncommutative super Schur functions

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Recipe for positivity with $I = I_{\text{Kron}}$, $\gamma = \sum_{w \in CYW_{\lambda,d}} w$

$CYW_{\lambda,d} = \text{set of colored Yamanouchi words of content } \lambda \text{ having exactly } d \text{ barred letters}$

**Proposition**

*For any partitions $\lambda, \nu$ of $n$ and $d \leq n$,*

$$g_{\lambda \mu(d)} \nu + g_{\lambda \mu(d-1)} \nu = \left( \text{the coefficient of } s_\nu(x) \text{ in } F_{CYW_{\lambda,d}}(x) \right).$$

**Fact:**

$$\sum_{w \in CYW_{\lambda,d}} w \in (I_{\text{Kron}})^\perp$$

where $I_{\text{Kron}}$ is the ideal of $\overline{U}$ corresponding to the relations

- $(xz - zx)y = y(xz - zx)$ for $x, y, z \in A$,
- $xz = zx$ for $x, z \in A$, $x < z - 1$,
- $yyx = yxy$, $zyy = yzy$ for $x, z \in A$, $y$ unbarred, $x < y < z$,
- $xxy = yxy$, $yyz = yzy$ for $x, z \in A$, $y$ barred, $x < y < z$,

where for a barred letter $\bar{a}$, we define $\bar{a} - 1$ to be the barred letter $\bar{a - 1}$. 
Recipe for positivity with \( I = I_{\text{Kron}} \), \( \gamma = \sum_{w \in CYW_{\lambda, d}} w \)

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xyz = zyx, \quad zyx = yzx \quad \text{for } x, z \in A, y \text{ unbarred, } x < y < z
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xzx = zxx, \quad zyx = yzx \quad \text{for } x, z \in A, y \text{ barred, } x < y < z
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The set $CYW_{(3,2),2}$

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A super switchboard on the vertex set $\text{CYW}_{(3,2),2}$
Main theorem: monomial positivity of $\tilde{J}_\nu(u)$

**Theorem (B.-Liu)**

In the algebra $\overline{U}/l_{Kron}$, the noncommutative super Schur functions have the following monomial positive expression:

$$\tilde{J}_\nu(u) = \sum_{T \in CT(\nu)} \text{barread}(T) \quad \text{in} \quad \overline{U}/l_{Kron}.$$  

$CT(\nu)$ denotes the set of colored tableaux of shape $\nu$. 

The words of the form $\bar{\text{read}}(T)$ are outlined
Colored tableaux $T$ such that $\text{barread}(T) \in \text{CYW}_{(3,2),2}$
Colored tableaux $T$ such that $\text{barread}(T) \in \text{CYW}_{(3,2),2}$

$$g(3,2) (3,1,1) (3,1,1) + g(3,2) (4,1) (3,1,1) = 3$$
Colored tableaux $T$ such that $\text{barread}(T) \in \text{CYW}_{(3,2),2}$

\[ g(3,2) (3,1,1) (3,1,1) = 2 \quad g(3,2) (4,1) (3,1,1) = 1 \]
Theorem (B.-Liu)

For any $\gamma \in (I_{\text{Kron}})^\perp$, the function $F_\gamma(x)$ is symmetric and

$$F_\gamma(x) = \sum_\nu s_\nu(x) \langle \tilde{J}_\nu(u), \gamma \rangle.$$ 

Corollary

For any set of colored words $W$ such that $\sum_{w \in W} w \in (I_{\text{Kron}})^\perp$, the coefficient of $s_\nu(x)$ in $F_W(x)$

$$= \langle \tilde{J}_\nu(u), \sum_{w \in W} w \rangle$$

$$= \left| \left\{ T \in CT(\nu) : \text{barread}(T) \in W \right\} \right|.$$ 

Corollary (R. Liu)

For any partitions $\lambda, \nu$ of $n$ and $d \leq n$,

$$g_{\lambda \mu(d)} \nu + g_{\lambda \mu(d-1)} \nu = \left| \left\{ T \in CT(\nu) : \text{barread}(T) \in CYW_{\lambda,d} \right\} \right|.$$
Lascoux’s heuristic for Kronecker coefficients

\( \Gamma_\lambda = \) the set of permutations obtained by standardizing the Yamanouchi words of content \( \lambda \).

For partitions \( \lambda \) and \( \mu \), define the multiset of permutations

\[
\Gamma_\lambda \circ \Gamma_\mu := \{ u \circ v \mid u \in \Gamma_\lambda, v \in \Gamma_\mu \},
\]

where \( \circ \) denotes ordinary composition of permutations.

\[\begin{array}{c|ccc}
\Gamma_\lambda \circ \Gamma_\mu \text{ for } \lambda = (3,1), \mu = (2,1,1) \\
\hline
\circ & 4312 & 4132 & 1432 \\
\hline
4123 & 3241 — 3421 & 4321 & 1 \ 4 \ 3 \ 2 \\
1423 & 3214 & 3124 — 1324 & 2 \ 3 \ 4 \ 1 \\
1243 & 3412 — 3142 & 1342 & 1 \ 2 \ 3 \ 4 \\
\end{array}\]
Lascoux’s heuristic for Kronecker coefficients

\[ \Gamma_\lambda = \text{the set of permutations obtained by standardizing the Yamanouchi words of content } \lambda. \]

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\[
\begin{array}{cccc}
\Gamma_\lambda \circ \Gamma_\mu & & & \\
\circ & 4312 & 4132 & 1432 \\
\{ & 4123 & \begin{array}{c} 3241 \longrightarrow 3421 \ 4321 \end{array} & 1423 & \begin{array}{c} 3214 \ 3124 \longrightarrow 1324 \end{array} & 1243 & \begin{array}{c} 3412 \longrightarrow 3142 \ 1342 \end{array} \\
\end{array}
\]

where
Lascoux’s heuristic for Kronecker coefficients

Theorem (Lascoux)

If \( \lambda \) and \( \mu \) are hook shapes, then \( \Gamma_\lambda \circ \Gamma_\mu \) is a union of Knuth equivalence classes, and the Kronecker coefficient \( g_{\lambda \mu \nu} \) is the number of these classes with insertion tableau of shape \( \nu \).

Though this rule no longer holds outside the hook-hook case, it seems to approximate Kronecker coefficients amazingly well for any three partitions and therefore gives a useful heuristic.
Lascoux’s heuristic for Kronecker coefficients

\[ g_{\lambda \mu} (3,2,1) \]
Lascoux’s heuristic for Kronecker coefficients

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\[
g_{\lambda \mu}(3,2,1) = \left| \left\{ w \in \Gamma_{\lambda} \circ \Gamma_{\mu} : Q(w) = \begin{array}{c} 1 \ 2 \ 3 \\ 4 \ 5 \ 6 \end{array} \right\} \right|
\]
### Lascoux's heuristic for Kronecker coefficients

\[
g_{\lambda \mu}(3,3,1) = \begin{array}{ccccccccccccccccc}
7 & 61 & 52 & 511 & 43 & 421 & 4111 & 331 & 322 & 3211 & 31^4 & 2221 & 221^3 & 21^5 & 1^7 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
61 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
52 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 0 & 1 & 1 & 0 & 0 \\
511 & 0 & 0 & 1 & 0 & 1 & 3 & 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \\
43 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\
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4111 & 0 & 0 & 1 & 1 & 1 & 3 & 2 & 2 & 2 & 3 & 1 & 1 & 1 & 0 & 0 \\
331 & 1 & 1 & 2 & 1 & 1 & 3 & 2 & 1 & 2 & 3 & 2 & 1 & 1 & 1 & 0 \\
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31^4 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 3 & 0 & 1 & 1 & 0 & 0 \\
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221^3 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 \\
21^5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1^7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Lascoux’s heuristic for Kronecker coefficients

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Lascoux’s heuristic for Kronecker coefficients

When only $\mu$ is a hook, $\Gamma_\lambda \circ \Gamma_\mu$ is not in general a union of Knuth equivalence classes, but

**Proposition**

*If $\mu$ is a hook, the quasisymmetric function*

$$F_{\Gamma_\lambda \circ \Gamma_\mu}(x) := \sum_{w \in \Gamma_\lambda \circ \Gamma_\mu} Q_{\text{Des}(w)}(x)$$

*is equal to* $\sum_\nu g_{\lambda \mu \nu} s_\nu(x)$.

\[
\text{CYW}_{\lambda, d} \xrightarrow{\text{standardize}} (\Gamma_\lambda \circ \Gamma_\mu(d)) \sqcup (\Gamma_\lambda \circ \Gamma_\mu(d-1)).
\]

The theory of noncommutative super Schur functions can then be used to go from this quasisymmetric function expansion to the Schur function expansion, thereby obtaining a formula for $g_{\lambda \mu \nu}$. 
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