Cyclage, catabolism, and the affine Hecke algebra

by

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University of California, Berkeley
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Abstract

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This thesis consists of four papers and an introduction. All four papers use canonical bases to attack difficult conjectures in algebraic combinatorics surrounding the subtle tableau combinatorics of cyclage and catabolism. Our main result is to exhibit \( q \)-analogues \( R_\lambda \) of the Garsia-Procesi modules, endowed with canonical bases coming from the extended affine Hecke algebra. We show that \( R_\lambda \) has cells naturally in bijection with the set of \( \lambda \)-catabolizable tableaux.

Paper I expands on the theory of inducing \( W \)-graphs began by Howlett and Yin in \([17, 18]\), carefully working out the combinatorics of cells in type \( A \). It then applies this to give two \( W \)-graph versions of tensoring with the \( S_n \) defining representation \( V \). The corresponding \( W \)-graph versions of the projection \( V \otimes V \otimes - \to S^2 V \otimes - \) are worked out and determined combinatorially in terms of cells.
Papers II-IV relate to the extended affine Hecke algebra $\hat{H}$ of type A. Paper II gives an algorithm for computing catabolizability of standard tableaux that was motivated by the cellular picture. This algorithm leads to two new characterizations of catabolizability and strengthens and simplifies proofs of some of its known properties.

In Paper III we show that any canonical basis element in the lowest two-sided cell of an extended affine Hecke algebra factors as the product of a symmetric function in the Bernstein generators and the canonical basis elements of what we call primitive elements, which are in bijection with elements of the associated finite Weyl group. The type A case of this result is used in Paper IV and was our starting point for using the canonical basis of $\hat{H}$ to better understand Garsia-Procesi modules. Paper IV identifies the cellular Garsia-Procesi modules $R_\lambda$. Important for this cellular picture is a subalgebra $\hat{H}^+$ of $\hat{H}$. Its cells are labeled by positive affine tableaux, tableaux filled with positive integer entries having distinct residues mod $n$. We show how these are ideal objects for studying cyclage posets. We present an array of conjectures that realize $k$-atoms and atom copies as cellular subquotients of $R_\lambda$ and identify the positive affine tableaux comprising their cells.

Professor Mark Haiman
Dissertation Committee Chair
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Chapter 1

Introduction

This thesis consists of four papers and this introduction. In this introduction, we discuss how these papers fit into the literature and summarize the main result of each.

This work lies at the intersection of algebraic combinatorics and representation theory. The interplay between these fields we have found to be strikingly powerful: combinatorics offers concrete and intuitive structures that are easy to manipulate and ideal for gathering experimental data, while algebra offers guidance about which manipulations are natural as well as elegant methods of proof. Many of the results and conjectures in these papers were discovered with the help of computer experimentation. The back-and-forth between making conjectures from experimental data and using theory to prove these conjectures and decide which computations to perform we have found to be quite fruitful.
The central object of this work is the polynomial ring $R = \mathbb{C}[y_1, \ldots, y_n]$, thought of as a graded representation of the symmetric group $S_n$ with the action given by permuting variables. The central problem of this work is, roughly:

Determine how a decomposition of this representation into irreducibles is compatible with multiplication by the $y_i$.

We investigate several precise versions of this problem. Though basic-sounding, it is at the heart of several important open problems in algebraic combinatorics, and, as we hope to demonstrate, has a deep connection to the theory of canonical bases.

We now give an account of the background in algebraic combinatorics and canonical basis theory upon which our work builds.

### 1.1 Algebraic combinatorics in subquotients of $R$

#### 1.1.1 Garsia-Procesi modules

A starting point for our work and the first evidence that $R$ is rich with algebraic and combinatorial structure are certain quotients of $R$ known as Garsia-Procesi modules. For each partition $\lambda \vdash n$, the irreducible $S_n$-module $V_\lambda$ occurs with multiplicity one in degree $n(\lambda) = \sum_i (i - 1)\lambda_i$ of $R$, and not at all in degree less than $n(\lambda)$; this copy of $V_\lambda \subseteq (R)_{n(\lambda)}$ is spanned by Garnir polynomials, so we refer to it as the Garnir representation of shape $\lambda$ or, more briefly, $G_\lambda$. There is a unique maximal homogeneous $S_n$-invariant ideal $I_\lambda \subseteq R$ not containing $G_\lambda$. The Garsia-Procesi modules are
the corresponding quotients $R_{\lambda} = R/I_{\lambda}$ (see [8, 13]). For $\lambda$ the single column $1^n$, the Garnir representation is the $\mathbb{C}$-span of the Vandermonde determinant, $I_{1^n}$ is the ideal generated by symmetric polynomials without constant term, and $R_{1^n}$ is the ring of coinvariants.

It is classically known that the ring of coinvariants is a graded version of the regular representation of $S_n$. The rings $R_{\lambda}$ first turned up in the work of Springer in the late 70’s as the cohomology rings of certain varieties now known as Springer fibers. Almost 15 years later, the work of Tanisaki, Garsia-Procesi, and Bergeron-Garsia led to the more elementary description of $R_{\lambda}$ above.

Now that we have a definition of these somewhat mysterious graded $S_n$-modules, it is natural to ask

What are the characters of the $R_{\lambda}$ as graded modules?

Equivalently, one can ask for their Frobenius series $\mathcal{F}_{R_{\lambda}}(z; t)$, the symmetric function in infinitely many variables $z_1, z_2, z_3, \ldots$ with coefficients in $\mathbb{C}[t]$ which encodes this character. Precisely,

$$\mathcal{F}_{R_{\lambda}}(z; t) = \sum_{\mu \vdash n} t^{d}(\text{char}(R_{\lambda})_d, \text{char} V_{\mu}) s_{\mu}(z),$$

(1.1)

where $s_{\mu}(z)$ is the Schur function of shape $\mu$. The Frobenius series $\mathcal{F}_{R_{\lambda}}(z; t)$ were worked out by Hotta and Springer in terms of Green polynomials [16] shortly after Springer’s study of the cohomology rings $R_{\lambda}$; we introduce the Hall-Littlewood polynomials, close relatives of Green polynomials, below. Later, Garsia and Procesi
gave an elementary proof of Hotta and Springer’s result using a description of the $R_\lambda$ similar to the one given above [8].

1.1.2. Hall-Littlewood polynomials

Hall-Littlewood polynomials were around for some time before the developments of Hotta and Springer. They first appeared implicitly in the mid 50’s in a series of papers by Green about characters of the groups $GL_n(\mathbb{F}_q)$ (see [23]). These polynomials have shown up in a remarkable number of places—in addition to $GL_n(\mathbb{F}_q)$ characters and the examples mentioned above, they also have combinatorial descriptions in terms of cocharge and catabolism by Lascoux and Schützenberger [27, 28] and in terms of the rigged configurations of Kirillov and Reshetikhin, and appear as certain Kazhdan-Lusztig polynomials (see [23, 13, 24]).

Define the $t$-integer $[k]_t$ to be $\frac{t^k - t^{-k}}{t - t^{-1}}$. To define the Hall-Littlewood polynomials, first define the symmetric functions $Q_\lambda(z; t)$ in $n$ variables $z_1, \ldots, z_n$ by

$$Q_\lambda(z; t) = (1 - t)^{\ell(\lambda)} [n - \ell(\lambda)]! \sum_{w \in S_n} w \left( z^\lambda \frac{\prod_{i<j} (1 - t z_j / z_i)}{\prod_{i<j} (1 - z_j / z_i)} \right).$$

(1.2)

The functions $Q_\lambda(z; t)$ can then be defined formally in infinitely many variables $z_1, z_2, \ldots$. Let $Z$ be the formal sum $z_1 + z_2 + \cdots$. The transformed Hall-Littlewood polynomials $H_\lambda(z; t)$ are given by $H_\lambda(z; t) = Q_\lambda[Z/(1 - t); t]$, where $[Z/(1 - t)]$ denotes the plethystic substitution $Z \rightarrow Z/(1 - t)$ (see [13, §3.3]). The cocharge variant transformed Hall-Littlewood polynomials are $\tilde{H}_\lambda(z; t) := t^{n(\lambda)} H_\lambda(z; t^{-1})$. The coeffi-
coefficients of the expansion of the $\tilde{H}_\lambda(z;t)$ in terms of Schur functions are the cocharge Kostka-Foulkes polynomials $\tilde{K}_{\mu\lambda}$:

$$\tilde{H}_\lambda(z; t) = \sum_\mu \tilde{K}_{\mu\lambda}(t)s_\mu(z). \quad (1.3)$$

We can now state the precisely the result of Hotta and Springer.

**Theorem 1.1.1.** The Frobenius series of the Garsia-Procesi modules are the cocharge variant transformed Hall-Littlewood polynomials, i.e., $\mathcal{F}_{R_\lambda}(z; t) = \tilde{H}_\lambda(z; t)$.

Of the many interpretations of Hall-Littlewood polynomials, the one above (1.2) from symmetric function theory is probably the most straightforward way to define them. However, from this definition, it is not clear that the polynomials $\tilde{K}_{\mu\lambda}(t)$ have nonnegative coefficients. Our work focuses on two interpretations of Hall-Littlewood polynomials which make this positivity property obvious: as the Frobenius series of the Garsia-Procesi modules, and the combinatorial description of Lascoux in terms catabolizability.

### 1.1.3. Cocyclage and catabolism

The starting point for the combinatorial side of our work are the cyclage and catabolism operations originating in [27, 28] (see also [37]).

The *cocharge labeling* $v^{cc}$ of a standard word $v$ is the (non-standard) word obtained from $v$ by reading the numbers of $v$ in increasing order; labeling the 1 of $v$ with a 0, and if the $i$ of $v$ is labeled by $k$, then labeling the $i+1$ of $v$ with a $k$ (resp. $k+1$) if
the \( i + 1 \) in \( v \) appears to the right (resp. left) of \( i \). For example, the cocharge labeling of 614352 is 302120. The cocharge labeling \( T^{cc} \) of a standard Young tableau \( T \) is the insertion tableau of the cocharge labeling of any word inserting to \( T \). The sum of the numbers in the cocharge labeling of a standard word \( v \) (resp. standard tableau \( T \)) is the cocharge of \( v \) (resp. \( T \)) or \( \text{cocharge}(v) \) (resp. \( \text{cocharge}(T) \)). Cocharge of semistandard words and tableaux are more subtle notions, which we do not define here. In Paper IV, we give a new way of understanding this statistic.

For a semistandard word \( w \) and number \( a \neq 1 \), \( aw \) (resp. \( wa \)) is a corotation (resp. rotation) of \( wa \) (resp. of \( aw \)). There is a cocyclage from the tableau \( T \) to the tableau \( T' \), written \( T \xrightarrow{cc} T' \), if there exist words \( u, v \) such that \( v \) is the corotation of \( u \) and \( P(u) = T \) and \( P(v) = T' \). Rephrasing this condition solely in terms of tableaux, \( T \xrightarrow{cc} T' \) if there exists a corner square \((r, c)\) of \( T \) and uninserting the square \((r, c)\) from \( T \) yields a tableau \( Q \) and number \( a \) such that \( T' \) is the result of column-inserting \( a \) into \( Q \).

For a composition \( \eta \) of \( n \), let \( \mathcal{T}(\eta) \) be the set of semistandard tableaux of content \( \eta \). If \( \eta \) is a partition, then the cocyclage poset \( \text{CCP}(\mathcal{T}(\eta)) \) is the poset on the set \( \mathcal{T}(\eta) \) generated by the relation \( \xrightarrow{cc} \). The relations \( \xrightarrow{cc} \) of \( \text{CCP}(\mathcal{T}(\eta)) \) are understood to be colored by the additional datum \((r, c)\), the outer corner removed in performing cocyclage. For \( \eta \) not a partition, the cocyclage poset \( \text{CCP}(\mathcal{T}(\eta)) \) is defined in terms of \( \text{CCP}(\mathcal{T}(\eta_+)) \) using reflection operators (see [37]), where \( \eta_+ \) denotes the partition obtained from \( \eta \) by sorting its parts in decreasing order. The cyclage poset on \( \mathcal{T}(\eta) \)
is the dual of the poset CCP(\(T(\eta)\)), i.e. the poset obtained by reversing all relations.

**Theorem 1.1.2** ([28]). The cyclage poset on \(T(\eta)\) is graded, with rank function given by cocharge.

The **catabolism** operation of Lascoux is defined in terms of certain slicing operations: for a skew tableau \(T\) and index \(r\), let \(H_r(T) = P(T_nT_s)\), where \(T_n\) and \(T_s\) are the north and south subtableaux obtained by slicing \(T\) horizontally between its \(r\)-th and \((r + 1)\)-th rows. Here we are thinking of tableaux as being drawn in English notation.

Let \(T\) be a standard tableau. If \(\lambda \subseteq \text{sh}(T)\), then define \(T_\lambda\) to be the subtableau of \(T\) of shape \(\lambda\). If \((m) \subseteq \text{sh}(T)\), then define the \(m\)-catabolism of \(T\), notated \(\text{rcat}_m(T)\), to be the tableau \(H_1(T - T(m))\). The **catabolizability** of \(T\), denoted \(\text{ctype}(T)\), is the partition \(\lambda = (\lambda_1, \hat{\lambda})\) defined inductively as follows: \(\lambda_1\) is the number of 0’s in \(T^{cc}\); \(\hat{\lambda} = \text{ctype}(\text{rcat}_{\lambda_1}(T))\).

A beautiful result of Lascoux [27] (see also [37]) is the right-hand equality of the following theorem.

**Theorem 1.1.3.** There holds

\[
\mathcal{F}_{R_\lambda}(z; t) = \tilde{H}_\lambda(z; t) = \sum_{\substack{T \in \text{SYT} \\
\text{ctype}(T) \succeq \lambda}} t^{\text{cocharge}(T)} S_{\text{sh}(T)}. \tag{1.4}
\]

Here SYT is the set of standard Young tableaux and \(\succeq\) is dominance order on partitions.
Up until now, the only proof of Theorem 1.1.3 relied on going through Hall-Littlewood polynomials, and the right-hand equality of (1.4) was proved by checking that these symmetric functions satisfy the same recurrence. One of our main achievements ([4, Theorem 8.8]) is to give a direct interpretation of the equality (1.4) by showing that $R_\lambda$ has a canonical basis with cells naturally in bijection with the set of tableaux $\{T : \text{ctype}(T) \succeq \lambda\}$. Unfortunately, our current proof of this result relies on heavy machinery.

1.1.4. More recent developments

More recent work suggests that there are other combinatorial mysteries hiding in the ring of coinvariants. Of particular interest is the appearance of the $k$-Schur functions. These have received a lot of attention recently, being studied intensively by Lapointe and Morse, and also Lam, Lascoux, Shimozono, and others (see [25, 26]). Additionally, exciting new developments of Li-Chung Chen, a fellow student of Mark Haiman, give a conjectural generalization and simplification of $k$-atoms. Chen’s atoms are defined in terms of Shimozono-Weyman atoms, which also conjecturally correspond to certain subquotients of $R$.

1.1.5. Shimozono-Weyman atoms

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of $n$, $\eta = (\eta_1, \ldots, \eta_k)$ a composition of $r$, and $l_j = \sum_{i=1}^{j-1} \eta_i$, $j \in [k+1]$ be the partial sums (where the empty sum is understood to be 0). Let $R = (R_1, \ldots, R_k)$ be the sequence of partitions given by
\( R_i = (\lambda_{i+1}, \ldots, \lambda_{i+1}) \). In [37], Shimozono and Weyman study the generalized Hall-Littlewood polynomials \( H_R(z;t) \) and the corresponding generalized Kostka-Foulkes polynomials \( K_{\mu;R}(t) \), which are related by

\[
H_R(z;t) = \sum_{\mu} K_{\mu;R}(t) s_{\mu}(z). \tag{1.5}
\]

The generalized Hall-Littlewood polynomials are defined to be the formal characters of the Euler characteristics of certain \( \mathbb{C}[\mathfrak{gl}_n] \)-modules supported in nilpotent conjugacy class closures. The polynomials \( K_{\mu;R}(t) \) are \( t \)-analogues of the Littlewood-Richardson coefficients \( \langle s_{\mu}(z), s_{R_1}(z)s_{R_2}(z) \cdots s_{R_k}(z) \rangle \).

In [37], Shimozono and Weyman give a generalization of catabolizability to semistandard tableaux that depends on \( R \), and they conjecture that the coefficient of \( t^d \) in \( K_{\mu;R}(t) \) is the number of \( R \)-catabolizable tableaux of charge \( d \). It is then natural to conjecture that the symmetric functions \( H_R(z;t) \) are the Frobenius series of certain submodules of \( R_\lambda \). In Paper IV, we state this as

**Conjecture 1.1.4.** The Frobenius series of \( F^{\text{mod}}(A_{G,\lambda,\eta}^{\text{SWr}}) \) is \( H_R(z;t) \).

Here, \( A_{G,\lambda,\eta}^{\text{SWr}} \) is a set of tableaux closely related to the \( R \)-catabolizable tableaux of [37], and \( F^{\text{mod}} \) takes this set of tableaux to a submodule of the Garsia-Procesi module \( \mathcal{R}_\lambda \) in a natural way. We define the *Shimozono-Weyman atom* to be the set of tableaux \( A_{G,\lambda,\eta}^{\text{SWr}} \).

1.1.6. The \( k \)-atoms of Lascoux, Lapointe, and Morse, and Chen’s atoms
Definition 1.1.5. If $\theta = \Theta/\nu$ is a skew shape with $|\theta| = n$ and $\lambda, \mu \vdash n$, then $\lambda$ is skew-linked to $\mu$ by $\theta$, written $\lambda \stackrel{\theta}{\rightarrow} \mu$, if $\text{row}(\theta) = \lambda$ and $\text{column}(\theta) = \mu'$. See Figure 1.1.

Let $S_\mu$ be the parabolic subgroup $S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_{\ell(\mu)}}$ of $S_n$ and $e_\mu^+$ the trivial module for $\mathbb{C}S_\mu$. Recall that $V_\lambda$ denotes the $S_n$ irreducible of shape $\lambda$.

Proposition 1.1.6 ([5]). The following are equivalent.

(a) There is a skew shape $\theta = \Theta/\nu$, such that $\lambda \stackrel{\theta}{\rightarrow} \mu$,

(b) There exists a non-negative integer $d(\mu, \lambda)$ such that in the $S_n$-module $R \otimes_{\mathbb{C}} (\text{Ind}_{S_\mu}^{S_n} e_\mu^+)$, $V_\lambda$ occurs with multiplicity 1 in degree $d(\mu, \lambda)$ and this is the unique occurrence of any $V_\nu$ with $\nu \subseteq \lambda$ in degree less than or equal to $d(\mu, \lambda)$. 
Let $R \star S_n = \mathbb{C}(\mathbb{Z}_{\geq 0} \rtimes S_n)$ be the monoid algebra of $\mathbb{Z}_{\geq 0} \rtimes S_n$.

**Definition 1.1.7.** For $\lambda, \mu$ satisfying any (all) of the conditions in Proposition 1.1.6, let $\mathbb{A}_{\mu, \lambda}^{\text{mod}}$ be the minimal quotient of $R \otimes \mathbb{C} (\text{Ind}_{\text{S}_{\mu}^\text{e}}^{\text{S}_{\mu}^\text{e}+ \mu})$ (as an $R \star S_n$-module) containing $V_\lambda$. By the proposition, this determines a unique $R \star S_n$-module. Equivalently, $\mathbb{A}_{\mu, \lambda}^{\text{mod}}$ is the submodule of $R_\lambda$ (as an $R \star S_n$-module) generated by the unique occurrence of $V_\mu$ in degree $n(\lambda) - d(\mu, \lambda)$.

For $\lambda \rightarrow^\theta \mu$ the Li-Chung Chen atom $\mathbb{A}^{\text{Chen}}_{U, G_\lambda}$ is the intersection of $\mathbb{A}^{\text{SWr}}_{G_\lambda, \eta}$ over certain $\ell(\lambda)$-compositions $\eta$ defined in terms of the shape $\theta$, where $U$ is a certain tableau of shape $\mu$. We then have (our version of) Chen’s conjecture

**Conjecture 1.1.8.** The Frobenius series of $\mathbb{A}_{\mu, \lambda}^{\text{mod}}$ is given by tableaux satisfying certain catabolizability conditions, i.e. $\mathbb{F}^{\text{mod}}(\mathbb{A}^{\text{Chen}}_{U, G_\lambda}) = \mathbb{A}_{\mu, \lambda}^{\text{mod}}$.

The super atom $\mathbb{A}^{(k)}_{\lambda}$ of [25], which is a set of tableaux corresponding to the $k$-atom $A^{(k)}_{\lambda}[X; t]$, is conjecturally a special kind of Chen atom. Let us see how this comes about.

A partition $\lambda$ is $k$-bounded if its parts have length $\leq k$. For a $k$-bounded partition $\lambda$ with $r$ parts, the skew shape $\theta^k(\lambda) = \Theta/\nu$ is defined uniquely by the condition $\text{row}(\theta) = \lambda$ and the inductive conditions: $\Theta_r = \lambda_r$, $\nu_r = 0$; $\theta^k(\lambda)_{<1} = \theta^k(\lambda_{<1})$ and $\nu_1$ is the smallest non-negative integer such that the hook lengths of $\theta^k(\lambda)$ are $\leq k$. See (1.6).
Proposition 1.1.9 ([25, Property 33]). For a $k$-bounded partition $\lambda$, $\lambda \overset{g^k}{\longrightarrow} \mu$ for some partition $\mu$ whose conjugate is also $k$-bounded.

In the language of [25], the $k$-conjugate of $\lambda$ is $\mu'$ in the proposition.

Our version of a conjecture of Chen is

Conjecture 1.1.10. For $\mu, \lambda$ as in Proposition 1.1.9, there is a color preserving isomorphism between the cocyclage poset on $A_{U,G,\lambda}^{Chen}$ and the cocyclage poset on $A_{\lambda}^{(k)}$ (see §1.1.3 for the meaning of color preserving).

1.2 Canonical bases

This work began with the goal of finding a combinatorial model for $R_{1^n}$ that would be compatible both with the $S_n$-action and with multiplication in the ring, and would make combinatorics of the Garsia-Procesi modules and other subquotients of $R_{1^n}$ transparent. We obtained essentially the best answer that could be hoped for: a $q$-analogue (or, in our notation, a $u$-analogue) $\mathcal{R}_{1^n}$ of $R_{1^n}$ endowed with a canonical basis whose cells correspond to $S_n$-irreducibles and with cellular quotients $\mathcal{R}_{\lambda}$ that are $u$-analogues of the Garsia-Procesi modules. This is the main topic of Paper IV, and before describing its results in more detail, we give the necessary background in
canonical basis theory.

Canonical bases, also known as Kazhdan-Lusztig bases or crystal bases, originated in the investigations of Kazhdan and Lusztig on singularities of Schubert varieties. Since their introduction in the famous paper [22] of 1979, they have played a central role at the intersection of representation theory, algebraic geometry, and combinatorics. While the application of crystal bases of quantum groups to understand tableau combinatorics is well-established and used prolifically (as we discuss briefly in §1.2.3), the connection between combinatorics and canonical bases of Hecke algebras is less developed. This work strengthens this connection, advancing both canonical basis theory and algebraic combinatorics with several strong theorems and an array of conjectures.

1.2.1. Hecke algebras

We study Iwahori-Hecke algebras, which arise as the algebra functions on a Chevalley group over a finite field that are invariant under the two-sided action of a Borel subgroup, with a multiplication given by convolution [19]. Such algebras can be given an explicit presentation by generators and relations, which we now recall. Let $A = \mathbb{Z}[u, u^{-1}]$ be the ring of Laurent polynomials in the indeterminate $u$. The Hecke algebra $\mathcal{H}(W)$ of the (extended) Coxeter group $(W, S)$ is the free $A$-module with
basis \( \{ T_w : w \in W \} \) and relations generated by

\[
T_u T_v = T_{uv} \quad \text{if} \ uv = u \cdot v \text{ is a reduced factorization,}
\]

\[
(T_s - u)(T_s + u^{-1}) = 0 \quad \text{if} \ s \in S.
\]

At \( u = 1 \), \( \mathcal{H} \) specializes to the group algebra \( \mathbb{Z}W \). In the case that \( W \) is a Weyl group of a Chevalley group over \( \mathbb{F}_q \), the parameter \( u^2 \) equals \( q \).

1.2.2. The canonical basis of \( \mathcal{H}(W) \)

In [22], Kazhdan and Lusztig introduce \( W \)-graphs as a combinatorial structure for describing an \( \mathcal{H} \)-module with a special basis. A \( W \)-graph consists of a vertex set \( \Gamma \), an edge weight \( \mu(\delta, \gamma) \in \mathbb{Z} \) for each ordered pair \( (\delta, \gamma) \in \Gamma \times \Gamma \), and a descent set \( L(\gamma) \subseteq S \) for each \( \gamma \in \Gamma \). These are subject to the condition that \( AG \) has a left \( \mathcal{H} \)-module structure given by

\[
C' s \gamma = \begin{cases} 
[2] s \gamma & \text{if} \ s \in L(\gamma), \\
\sum_{(\delta, s \neq L(\delta))} \mu(\delta, \gamma) \delta & \text{if} \ s \notin L(\gamma).
\end{cases}
\]

We will use the same name for a \( W \)-graph and its vertex set.

The bar-involution, \( \overline{\cdot} \), of \( \mathcal{H} \) is the additive map from \( \mathcal{H} \) to itself extending the involution \( \overline{\cdot} : A \rightarrow A \) given by \( \overline{u} = u^{-1} \) and satisfying \( \overline{T_w} = T_{w^{-1}} \). Observe that \( \overline{T_s} = T_{s^{-1}} = T_s + u^{-1} - u \) for \( s \in S \). Some simple \( \overline{\cdot} \)-invariant elements of \( \mathcal{H} \) are \( C'_{\text{id}} := T_{\text{id}} \) and \( C'_s := T_s + u^{-1} = T_{s^{-1}} + u \), \( s \in S \). Let \( A^- \) be the subring \( \mathbb{Z}[u^{-1}] \) of \( A \).

Define the lattice

\[
\mathcal{L} = A^- \{ T_w : w \in W \}.
\]
**Theorem 1.2.1** (Kazhdan-Lusztig [22]). For each \( w \in W \), there is a unique element \( C'_w \in \mathcal{H}(W) \) such that \( C'_w = C_w' \) and \( C'_w \) is congruent to \( T_w \mod u^{-1}\mathcal{L} \). There exist integers \( \mu(x, w) \), \( x, w \in W \) so that \( \{C'_w : w \in W\} \) gives \( \mathcal{H}(W) \) a \( W \)-graph structure. The \( A \)-basis \( \{C'_w : w \in W\} \) of \( \mathcal{H}(W) \) is the canonical basis or Kazhdan-Lusztig basis. The corresponding \( W \)-graph is denoted \( \Gamma_W \).

Let \( \Gamma \) be a \( W \)-graph and put \( E = A\Gamma \). The preorder \( \leq_\Gamma \) (also denoted \( \leq_E \)) on the vertex set \( \Gamma \) is generated by the relations/edges

\[
\delta \xleftarrow{\Gamma} \gamma \quad \text{if there is an } h \in \mathcal{H} \text{ such that } \delta \text{ appears with non-zero coefficient in the expansion of } h\gamma \text{ in the basis } \Gamma.
\] (1.9)

Equivalence classes of \( \leq_\Gamma \) are the left cells of \( \Gamma \) (or of \( E \)). A cellular submodule of \( E \) is a submodule of \( E \) that is spanned by a subset of \( \Gamma \) (and is necessarily a union of left cells). A cellular quotient of \( E \) is a quotient of \( E \) by a cellular submodule, and a cellular subquotient of \( E \) is a cellular submodule of a cellular quotient.

A beautiful result in the original paper [22] of Kazhdan and Lusztig, and the beginning of a powerful connection between combinatorics and representation theory, is that the left cells of \( \mathcal{H}(S_n) \) are in bijection with the set of SYT and the left cell containing \( C'_w \) corresponds to the recording tableau of \( w \) under this bijection.

In Paper I, we further develop this connection between tableau combinatorics and \( S_n \)-graphs for iterated inductions and restrictions of \( \mathcal{H}(S_n) \)-modules.

### 1.2.3. Quantum groups
Canonical bases or *crystal bases* of quantum groups were developed independently by Kashiwara and Lusztig about 10 years after the discovery of their counterparts for Hecke algebras. Crystal bases have a similar flavor to canonical bases of Hecke algebras, but are more difficult to define because there is no simple counterpart to the standard basis of $T$’s. However, crystal bases seem to more readily give rise to tableau combinatorics than their Hecke algebra counterparts.

For instance, crystal bases have been used to

- Prove a version of the Littlewood-Richardson rule for the classical Lie algebras [34].
- Interpret the reflection operators of Lascoux and Schützenberger [28] as operators on type $A_{n-1}$ crystals.
- Show that Kostka-Foulkes polynomials are given by the graded multiplicities of $A_{n-1}$ irreducibles in tensor products of certain finite $\tilde{A}_n$ crystals (see [36]).
- Show the equivalence of a combinatorial formula for the generalized Kostka polynomials $K_{\mu,R}(t)$ and the decomposition of tensor products of certain type $\tilde{A}_n$ crystals into irreducibles in the case that $R$ is a sequence of rectangular partitions [36].

It is likely that there is a version of Schur-Weyl duality which connects our canonical basis for the polynomial ring to a quantum group version of this. We have not yet managed to work this out, and it seems that the necessary version of Schur-Weyl
duality is not in the literature. The work of Feigin and Loktev on current algebras [6], the quantum affine Schur-Weyl duality of Ginzburg, Reshetikhin, and Vasserot [11], and work of Grojnowski and Lusztig [12] contain results that are close to what is needed.

1.2.4. The type A affine Hecke algebra

The object we eventually settled on for providing a canonical basis for the coinvariant ring is the extended affine Hecke algebra \( \widehat{H} = H(W_e) \). Here we use \( W_e, W_a \) to denote the extended and unextended affine Weyl groups of type A. The unextended affine Weyl group \( W_a \) is a Coxeter group with simple reflections \( K = \{ s_0, s_1, \ldots, s_{n-1} \} \) and relations given by the type \( \tilde{A}_n \) Dynkin diagram:

\[
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\
s_i^2 = 1
\]  (1.10)

for all \( i \in [0, n-1] \) and subscripts taken mod \( n \). Let \( Y = \mathbb{Z}^n \), thought of as the weight lattice of the algebraic group \( GL_n \). Also let \( \Pi = \mathbb{Z} \) be the weight lattice quotiented by the root lattice. The group \( W_e \) is equal to both \( Y \rtimes \mathcal{S}_n \) and \( \Pi \rtimes W_a \) and this gives rise to the following two presentations of \( \widehat{H} \). Going back and forth between these presentations is a powerful tool and important in our work.

The algebra \( \widehat{H} \) contains the Hecke algebra \( H(W_a) \) and is isomorphic to the twisted group algebra \( \Pi \rtimes H(W_a) \) generated by \( \Pi \) and \( H(W_a) \) with relations generated by

\[
\pi T_w = T_{\pi w \pi^{-1}} \pi
\]
for $\pi \in \Pi$, $w \in W_a$.

The other presentation of $\widehat{\mathcal{H}}$ is due to Bernstein. For any $\lambda \in Y$ there exist dominant $\mu, \nu \in Y$ such that $\lambda = \mu - \nu$. Define

$$Y^\lambda := T_{y^\mu} (T_{y^\nu})^{-1},$$

which is independent of the choice of $\mu$ and $\nu$. The algebra $\widehat{\mathcal{H}}$ is the free $A$-module with basis $\{Y^\lambda T_w : w \in W_f, \lambda \in Y\}$ and relations generated by

$$T_i Y^\lambda = Y^\lambda T_i \quad \text{if } \lambda_i = \lambda_{i+1},$$

$$T_i^{-1} Y^\lambda T_i^{-1} = Y^{s_i(\lambda)} \quad \text{if } \lambda_i = \lambda_{i+1} + 1,$$

$$(T_i - u)(T_i + u^{-1}) = 0$$

for all $i \in [n - 1]$, where $T_i := T_{s_i}$.

1.2.5. The Kazhdan-Lusztig preorder

The preorder $\leq_E$ of an $\mathcal{H}(W)$-module $E$ coming from a $W$-graph induces a partial order on the cells of $E$, also denoted $\leq_E$. The major barrier to our using canonical bases to understand subquotients of the coinvariants is the difficulty of computing the partial order $\leq_{\mathfrak{A}_n}$. The order $\leq_{\Gamma_W}$ seems to be quite difficult to compute completely even in the simplest cases; it is not even known for the $S_n$-graph $\Gamma_{S_n}$. The paper [39] gives a nice account of what is known about $\leq_{\Gamma_{S_n}}$.

The following result may be used to gain some information about $\leq_{\Gamma_W}$. This proposition originated in the work of Barbasch and Vogan on primitive ideals, and is proven in the generality stated here by Roichman [35] (see also §3.3, Paper I).
Let $J \subseteq S$, let $W_J$ denote the parabolic subgroup of $W$ generated by $J$ and $^JW$ denote the set of minimal coset representatives $\{w \in W : w \text{ minimal in } W_Jw\}$.

**Proposition 1.2.2.** Let $J \subseteq S$ and $E = \text{Res}_{\mathcal{H}(W_J)}\mathcal{H}$. Then for any $x \in ^JW$,

$$A\{C'_{vx} : v \in W_J\} \xrightarrow{\sim} \mathcal{H}(W_J), C'_{vx} \mapsto C'_v \quad (1.11)$$

is an isomorphism of $\mathcal{H}(W_J)$-modules with basis (equivalently, the corresponding map of $W_J$-graphs is an isomorphism).

A key development in the theory of canonical bases is Lusztig’s $a$-invariant [32, 33]. This work relies on the following positivity result of Kazhdan-Lusztig and Beilinson-Bernstein-Deligne-Gabber (see, for instance, [32]). For $h \in \mathcal{H}(W)$ and $z \in W$, let $[C'_z]h$ denote the coefficient of $C'_z$ in the expansion of $h$ in the canonical basis.

**Theorem 1.2.3.** If $(W, S)$ is crystallographic, then the structure coefficients $\beta_{x,y,z} = [C'_z]C'_x C'_y$ are Laurent polynomials in $u$ with non-negative coefficients.

**Definition 1.2.4.** The $a$-invariant of an element $z \in W$, denoted $a(z)$, is

$$a(z) = \max\{\deg_u(\beta_{x,y,z}) : x, y \in W\},$$

where $\deg_u(f)$ is the $u$-degree of $f \in A$; if no maximum exists, define $a(z)$ to be $\infty$.

The $a$-invariant is a powerful tool for understanding the preorder $\leq_{\Gamma_W}$, as the next results demonstrate. If $(W, S)$ is crystallographic, then the following hold:

- The $a$-invariant is constant on two-sided cells of $W$. 
• If $W$ is bounded in the sense of [33, 1.1 (d)] and $x \leq_{rW} z$, then $a(x) \geq a(z)$ and if $x$ and $z$ are not in the same left cell, then $a(x) > a(z)$.

• Assume $W$ is bounded in the sense of [33, 1.1 (d)]. Let $J \subseteq S$, $u, v \in W_J$, and $x, y \in {}^J W$. Then $ux \leq_{\text{Res}, \mathcal{F}(W_J), \mathcal{F}(W)} vy$ and $u, v$ in the same two-sided cell of $\Gamma_{W_J}$ implies that $u$ and $v$ are in the same left cell of $\Gamma_{W_J}$ and $x = y$ [10].

For $W = S_n$, $a(w) = n(\lambda')$, where $\lambda$ is the shape of $P(w)$, i.e., the shape corresponding to the irreducible $S_n$-module of the left cell of $w$.

1.3 Main results

1.3.1. Paper I: $W$-graph versions of tensoring with the $S_n$ defining representation

This paper attempts to construct a combinatorial model for $R$ as a $\mathbb{C}S_n$-module that takes into account multiplication by the $y_i$, along the following lines: decompose the tensor algebra $TV$ into canonically chosen irreducible $\mathbb{C}S_n$-submodules, where $V$ is the degree 1 part of $R$. Define a poset in which an irreducible $E'$ is less than an irreducible $E$ if $E' \subseteq V \otimes E$. Somehow project this picture onto a canonical decomposition of $R$ into $\mathbb{C}S_n$-irreducibles. Lower order ideals of the projected poset would correspond to $\mathbb{C}S_n$-modules that are also $R$-modules. The poset would be controlled by a local rule saying that any sequence $(E, E'), (E', E'')$ of covering relations must satisfy $E'' \subseteq S^2 V \otimes E$. 
The beginning of this approach is carried out in this paper as follows. To obtain a nice decomposition of $TV$ and $R$ into irreducibles, we replace $\mathbb{C}S_n$ with the Hecke algebra $\mathcal{H}(S_n)$ and apply the theory of canonical bases. The functor $V \otimes -$ is replaced by $\mathcal{H}(S_n) \otimes \mathcal{H}(S_n) \langle - \rangle$, $J = \{s_2, \ldots , s_{n-1}\} \subseteq S$. We are naturally led to a construction that takes an $\mathcal{H}(W) \langle - \rangle$-module $E$ coming from a $W \langle - \rangle$-graph and produces a $W$-graph structure on $\mathcal{H}(W) \otimes \mathcal{H}(W) \langle - \rangle E$. This construction of inducing $W$-graphs is due to Howlett and Yin [17]. This paper expands on this theory and carefully works out the combinatorics of cells in type A.

Once this groundwork is laid, we can form an $S_n$-graph version of $TV \otimes E$, $TV$ being the tensor algebra of $V$, for any $\mathcal{H}(S_n)$-module $E$ coming from an $S_n$-graph. We can then try to project this onto an $S_n$-graph version of $SV \otimes E = R \otimes E$. This is even interesting for $T^2V$ and $S^2V$ and is what we focus on in this paper. Define $T^2_{\text{red}}V := \mathbb{Z}\{x_i \otimes x_j : i \neq j\}$ and $S^2_{\text{red}}V := \mathbb{Z}\{x_i \otimes x_j + x_j \otimes x_i : i \neq j\}$. We show that our $S_n$-graph version of $T^2V \otimes E$ has a cellular decomposition into $\tilde{F}^2 := \mathcal{H}(S_n) \otimes \mathcal{H}(S_n) \langle - \rangle E$ and $\mathcal{H}(S_n) \langle - \rangle E$, which at $u = 1$ become $T^2_{\text{red}}V \otimes E$ and $V \otimes E$. There is a canonical map

$$\tilde{F}^2 \xrightarrow{\tilde{\beta}} \mathcal{H}(S_n) \otimes S_{\langle s_2 \rangle} E,$$

specializing at $u = 1$ to the projection $T^2_{\text{red}}V \otimes E \to S^2_{\text{red}}V \otimes E$. The map $\tilde{\beta}$ does not send canonical basis elements to canonical basis elements, but it approximates doing so as $u \to 0$. This partitions the canonical basis of $\tilde{F}^2$ into two parts — the approximate kernel, which we refer to as combinatorial wedge, and the approximate inverse
image of the canonical basis of \( \mathcal{H}(S_n) \otimes_{S_{2n}} E \), which we refer to as combinatorial reduced sym. We determine this partition in terms of cells. The growth diagrams of Fomin (see [38, 7.13]) make a pleasing appearance here.

We also consider the \( S_n \)-graph version \((\widehat{\mathcal{H}}^+ \otimes \mathcal{H}(S_n) - 1)\) of tensoring with \( V \), where \( \widehat{\mathcal{H}}^+ \) is a subalgebra of the extended affine Hecke algebra and the subscript signifies taking the degree 1 part. This mostly parallels the version just described, but there are some interesting differences. Most notably, the combinatorics of this \( S_n \)-graph version of the inclusion \( T_{\text{red}}^2 V \otimes E \rightarrow V \otimes V \otimes E \) is transpose to that of the other.

### 1.3.2. Paper II: An insertion algorithm for catabolizability

In our investigations of the canonical basis of \( \mathcal{R}_{1^n} \), we were naturally led to an insertion-like algorithm that computes \( \text{ctype}(P(w)) \) for any \( w \in S_n \). In this paper, we describe this algorithm and its consequences. A particularly nice one is

**Theorem 1.3.1.** Catabolizability is characterized as the statistic on permutations that is invariant under non-zero (co)rotations, catabolism transformations, and Knuth transformations (and satisfies a certain normalization condition).

Non-zero (co)rotations and catabolism transformations are operations we introduce in this paper, and are defined as follows. A non-zero (co)rotation of a standard word is a (co)rotation (see §1.1.3) such that the rotated number does not have cocharge label 0 (before and after it is (co)rotated). A catabolism transformation of \( w \) is a transformation swapping two adjacent elements \( w_i, w_{i+1} \) of \( w \) provided their
cocharge labels differ by at least two.

The insertion algorithm also strengthens and simplifies the proofs of some of the previously known facts about catabolizability. Additionally, using Theorem 1.3.1 we give the following characterization of catabolizability, reminiscent of the Greene’s Theorem characterization of the shape of the insertion tableau of a permutation.

**Theorem 1.3.2.** There holds

\[
\sum_{i=1}^{k+1} \text{ctype}(w)_i = I_k(\tilde{w}),
\]

where $\tilde{w}$ is an affine word associated to $w$ and $I_k(\tilde{w})$ is the maximal size of what we call a $k$-bounded chain family of $\tilde{w}$.

1.3.3. Paper III: A factorization theorem for affine Kazhdan-Lusztig basis elements

The lowest two-sided cell of the extended affine Weyl group $W_e$ is the set \{ $w \in W_e : w = x \cdot w_0 \cdot z$, for some $x,z \in W_e$ \}, denoted $W(\nu)$. We identify a certain subset of $W(\nu)$, termed primitive elements. If $W_e$ is that associated to a simply connected algebraic group, then the primitive elements are naturally in bijection with elements of the associated finite Weyl group $W_f$. For $\nu$ minimal in its coset $vW_f$ (resp. $W_f v$), define $\tilde{C}'_v$ (resp. $\tilde{C}'_v$) by $\tilde{C}'_v C_w' = C_{v w}'$ (resp. $C_{v w_0}' \tilde{C}'_v = C_C'$). We prove that

**Theorem 1.3.3.** For $w \in W(\nu)$ and with $w = v_1 \cdot w_0 y^\lambda \cdot v_2$, we have the factorization

\[
C_w' = \chi_\lambda(Y) \tilde{C}'_{v_1} C_{w_0}' \tilde{C}'_{v_2},
\]

(1.13)
where $\chi_{\lambda}(Y)$ is the character of the irreducible representation of highest weight $\lambda$ in the Bernstein generators, and $v_1$ and $v_2^{-1}$ are primitive elements.

This theorem gives an expression for any $C'_w, w \in W(\nu)$ in terms of only finitely many canonical basis elements. After completing this paper, we realized that this result was first proved by Xi in [41]. Our proof is significantly different and somewhat longer than Xi’s, however our’s has the advantage of being mostly self-contained, while Xi’s makes use of results of Lusztig from [30] and Cells in affine Weyl groups I-IV and the positivity of Kazhdan-Lusztig coefficients.

1.3.4. Paper IV: Cyclage, catabolism, and the affine Hecke algebra

In order to construct the $u$-analogue $R_{1^n}$, we identify a subalgebra $\hat{H}^+$ of $\hat{H}$ that is a $u$-analogue of the monoid algebra of $\mathbb{Z}_{\geq 0} \rtimes S_n$. Focusing on this subalgebra rather than $\hat{H}$ itself was a crucial step in our goal of relating subquotients of $R$ to tableau combinatorics. In this paper we establish some basic properties of this subalgebra, including a description of its left cells. These cells are naturally labeled by tableaux filled with positive integer entries having distinct residues mod $n$, which we term positive affine tableaux (PAT). Our investigations have thoroughly convinced us that these are excellent combinatorial objects for describing graded $S_n$-modules.

Let $e^+$ be the one-dimensional trivial module for $\mathcal{H}(S_n)$. The left $\hat{H}^+$-module $\hat{H}^+e^+$ may be identified with a cellular submodule of $\hat{H}^+$. This submodule is our desired $u$-analogue of the polynomial ring $R$. 
Here we prove our main result

**Theorem 1.3.4.** The $\hat{\mathfrak{h}}^+$-module $\hat{\mathfrak{h}}^+ e^+$ has a cellular quotient equal to

$$\mathcal{R}_1^n := \hat{\mathfrak{h}}^+ e^+ / (S_n \text{-invariant polynomials in the Bernstein generators } Y^\lambda) e^+$$

with a canonical basis labeled by affine words that are essentially standard words with cocharge labels and left cells labeled by PAT that are essentially SYT with cocharge labels. Multiplication by an element $\pi \in \hat{\mathfrak{h}}^+$ corresponds to corotations of words and on cells corresponds to cocyclage.

We go on to show that $\mathcal{R}_1^n$ contains an $S_n$-graph version of the Garsia-Procesi modules.

**Theorem 1.3.5.** The minimal quotient of $\mathcal{R}_1^n$ containing the cell corresponding to $G_\lambda$ is cellular and consists of those left cells corresponding to $T \in SYT$ such that $\text{ctype}(T) \succeq \lambda$.

The proof of this theorem uses several ingredients:

- The positivity of the structure coefficients of the canonical basis of $\hat{\mathfrak{h}}^+$,

- The identification of certain $C'_w$ as elementary symmetric functions in subsets of Bernstein generators $Y_1, \ldots, Y_n$,

- The $u = 1$ results of Garsia-Procesi and Bergeron-Garsia.

We further conjecture the stronger statement that $\mathcal{R}_1^n$ has cellular subquotients corresponding to $k$-atoms and Chen’s atoms (though, admittedly, we have only computed up to $n = 6$).
Our investigations focused primarily on subquotients of the coinvariants, however it appears that there are many other isomorphic copies of these subquotients in $\widehat{H}^+$, outside of the coinvariants. This notion of atom copies generalizes the atom copies of [25]. We formalize this notion in this paper and give some conjectural descriptions of these copies in terms of a natural generalization of catabolizability to positive affine tableau. Though we believe these copies to be isomorphic as cellular subquotients, they come with genuinely different combinatorics, just as the cocyclage poset on semistandard tableaux is not obviously isomorphic to a subposet of the cocyclage poset on standard tableaux. We believe that a critical problem towards understanding $k$-atoms and catabolizability is to produce a combinatorial structure less rigid than tableaux that makes it obvious that these copies are isomorphic.
Chapter 2

Paper I: $W$-graph versions of tensoring with the $S_n$ defining representation

Abstract

We further develop the theory of inducing $W$-graphs worked out by Howlett and Yin in [17], [18], focusing on the case $W = S_n$. Our main application is to give two $W$-graph versions of tensoring with the $S_n$ defining representation $V$, one being $\mathcal{H} \otimes \mathcal{H}_J$ for $\mathcal{H}, \mathcal{H}_J$ the Hecke algebras of $S_n, S_{n-1}$ and the other $(\widehat{\mathcal{H}}^+ \otimes \mathcal{H}^-)_1$, where $\widehat{\mathcal{H}}^+$ is a subalgebra of the extended affine Hecke algebra and the subscript signifies taking the degree 1 part. We look at the corresponding $W$-graph versions of
the projection \( V \otimes V \otimes - \to S^2V \otimes - \). This does not send canonical basis elements to canonical basis elements, but we show that it approximates doing so as the Hecke algebra parameter \( u \to 0 \). We make this approximation combinatorially explicit by determining it on cells. Also of interest are a combinatorial conjecture stating that the restriction of \( \mathcal{H} \) to \( \mathcal{H}_J \) is “weakly multiplicity-free” for \( |J| = n - 1 \), and a partial determination of the map \( \mathcal{H} \otimes \mathcal{H}_J \to \mathcal{H} \) on canonical basis elements, where \( \beta \) is the counit of adjunction between restriction and induction.

### 2.1 Introduction

The polynomial ring \( R := \mathbb{C}[x_1, \ldots, x_n] \) is well understood as a \( \mathbb{C}S_n \)-module, but how this \( \mathbb{C}S_n \)-module structure is compatible with the structure of \( R \) as a module over itself is not. This work came about from an attempt to construct a combinatorial model for \( R \) as a \( \mathbb{C}S_n \)-module that takes into account multiplication by the \( x_i \). The hope is that such a model would lead to a better understanding of the Garsia-Procesi modules, particularly, the combinatorics of cyclage and catabolism. We also might hope to find modules corresponding to the \( k \)-atoms of Lascoux, Lapointe, and Morse and uncover combinatorics that governs them.

Such a model might look something like this: decompose the tensor algebra \( TV \) into canonically chosen irreducible \( \mathbb{C}S_n \)-submodules, where \( V \) is the degree 1 part of \( R \). Define a poset in which an irreducible \( E' \) is less than an irreducible \( E \) if \( E' \subseteq V \otimes E \). Somehow project this picture onto a canonical decomposition of \( R \) into
CSₙ-irreducibles. Lower order ideals of the projected poset would correspond to CSₙ-modules that are also R-modules. The poset would be controlled by a local rule saying that any sequence \((E, E'), (E', E'')\) of covering relations must satisfy \(E'' \subseteq S^2V \otimes E\).

The main results of this paper are a first step towards this approach; further work will appear in [4]. To obtain a nice decomposition of \(TV\) and \(R\) into irreducibles, we replace \(\mathbb{C}S_n\) with the Hecke algebra \(\mathcal{H}\) of \(W = S_n\) and apply the theory of canonical bases. The functor \(V \otimes -\) is replaced by \(\mathcal{H} \otimes_{\mathcal{H}_J} -\), \(J = \{s_2, \ldots, s_{n-1}\} \subseteq S\), \(S\) the simple reflections of \(W\). We are naturally led to a construction that takes an \(\mathcal{H}_J\)-module \(E\) coming from a \(W_J\)-graph and produces a \(W\)-graph structure on \(\mathcal{H} \otimes_{\mathcal{H}_J} E\). This construction of inducing \(W\)-graphs, found independently by the author, is due to Howlett and Yin [17]. We spend a good deal of this paper (§2.2 – §2.4) developing this theory, proving some basic results of interest for their own sake as well as for this application.

Once this groundwork is laid, we can form a \(W\)-graph version of \(TV \otimes E\), \(TV\) being the tensor algebra of \(V\), for any \(\mathcal{H}\)-module \(E\) coming from a \(W\)-graph. We can then try to project this onto a \(W\)-graph version of \(SV \otimes E = R \otimes E\). This is even interesting for \(T^2V\) and \(S^2V\) and is what we focus on in this paper. Define \(T^2_{\text{red}}V := \mathbb{Z}\{x_i \otimes x_j : i \neq j\}\) and \(S^2_{\text{red}}V := \mathbb{Z}\{x_i \otimes x_j + x_j \otimes x_i : i \neq j\}\). We show in Proposition 2.7.10 that our \(W\)-graph version of \(T^2V \otimes E\) has a cellular decomposition into \(\tilde{F}^2 := \mathcal{H} \otimes_{\mathcal{H}_J \setminus S^2} E\) and \(L \mathcal{H} \otimes_{\mathcal{H}_J} E\), which at \(u = 1\) become \(T^2_{\text{red}}V \otimes E\) and \(V \otimes E\).
There is a canonical map (2.44)

\[ \widetilde{F}^2 \xrightarrow{\tilde{\beta}} \mathcal{H} \otimes_{S_{\text{red}}} E, \]

specializing at \( u = 1 \) to the projection \( T_{\text{red}}^2 V \otimes E \to S_{\text{red}}^2 V \otimes E \). The map \( \tilde{\beta} \) does not send canonical basis elements to canonical basis elements, but it approximates doing so as the Hecke algebra parameter \( u \to 0 \) (Corollary 2.8.9). This partitions the canonical basis of \( \tilde{F}^2 \) into two parts—the approximate kernel, which we refer to as combinatorial wedge, and the approximate inverse image of the canonical basis of \( \mathcal{H} \otimes_{S_{\text{red}}} E \), which we refer to as combinatorial reduced sym. Theorem 2.9.1 determines this partition in terms of cells.

We also consider a \( W \)-graph version of tensoring with \( V \) coming from the extended affine Hecke algebra. This mostly parallels the version just described, but there are some interesting differences. Most notably, the combinatorics of this \( W \)-graph version of the inclusion \( T_{\text{red}}^2 V \otimes E \to V \otimes V \otimes E \) is transpose to that of the other; compare Theorems 2.9.1 and 2.9.3.

This paper is organized mainly in order of decreasing generality. We begin in §2.2 by introducing the Hecke algebra, \( W \)-graphs, and the inducing \( W \)-graph construction. We reformulate some of this theory using the formalism of IC bases as presented in [21]. This has the advantage of avoiding explicit calculations involving Kazhdan-Lusztig polynomials, or rather, hides these calculations in the citations of [17], [18]. This allows us to focus more on cells and cellular subquotients. In §2.3 we specialize to the case where \( W \)-graphs come from iterated induction from the regular
representation. In this case we prove that all left cells are isomorphic to those occurring in the regular representation of $W$ (Theorem 2.3.5). Next, in §2.4, we review the combinatorics of cells in the case $W = S_n$. As was first observed in [35], there is a beautiful connection between the Littlewood-Richardson rule and the cells of an induced module $\mathcal{H} \otimes_{\mathcal{H}_J} E$ (Proposition 2.4.1). The combinatorics of the cells of the restriction $\text{Res}_{\mathcal{H}_J} \mathcal{H}$ is less familiar; see Conjecture 2.3.8. Sections 2.5 and 2.6 give a nice result about how canonical basis elements behave under the projection $\mathcal{H} \otimes_J \mathcal{H} \to \mathcal{H}$. The remaining sections 2.7, 2.8, and 2.9 contain our main results just discussed.

2.2 IC bases and inducing $W$-graphs

2.2.1. We will use the following notational conventions in this paper. If $A$ is a ring and $S$ is a set, then $AS$ is a free $A$-module with basis $S$ (possibly endowed with some additional structure, depending on context). Elements of induced modules $\mathcal{H} \otimes_{\mathcal{H}_J} E$ will be denoted $h \boxtimes e$ to distinguish them from elements of a tensor product over $\mathbb{Z}$, $F \otimes_{\mathbb{Z}} E$, whose elements will be denoted $f \otimes e$. The symbol $[n]$ is used for the set $\{1, \ldots, n\}$ and also for the $u$-integer (defined below), but there should be no confusion between the two.

2.2.2. Let $W$ be a Coxeter group and $S$ its set of simple reflections. The length $\ell(w)$ of $w \in W$ is the minimal $l$ such that $w = s_1 \ldots s_l$ for some $s_i \in S$. If $\ell(uv) =$
\( \ell(u) + \ell(v) \), then \( uv = u \cdot v \) is a reduced factorization. The notation \( L(w) = \{ s \in S : sw < w \} \), \( R(w) = \{ s \in S : ws < w \} \) will be used for the left and right descent sets of \( w \).

For any \( J \subseteq S \), the parabolic subgroup \( W_J \) is the subgroup of \( W \) generated by \( J \). Each left (resp. right) coset \( wW_J \) (resp. \( W_Jw \)) contains an unique element of minimal length called a minimal coset representative. The set of all such elements is denoted \( W^J \) (resp. \( J^W \)). For any \( w \in W \), define \( w^J, Jw \) by

\[
    w = w^J \cdot Jw, \quad w^J \in W^J, \quad Jw \in W_J. \quad (2.1)
\]

Similarly, define \( w_J, J^w \) by

\[
    w = w_J \cdot J^w, \quad w_J \in W_J, \quad J^w \in J^W. \quad (2.2)
\]

2.2.3. Let \( A = \mathbb{Z}[u, u^{-1}] \) be the ring of Laurent polynomials in the indeterminate \( u \), \( A^- \) (resp. \( A^+ \)) be the subring \( \mathbb{Z}[u^{-1}] \) (resp. \( \mathbb{Z}[u] \)), and \( \overline{\cdot} : A \to A \) be the involution given by \( \overline{u} = u^{-1} \). The Hecke algebra \( \mathcal{H} \) of \( W \) is the free \( A \)-module with basis \( \{ T_w : w \in W \} \) and relations generated by

\[
    T_u T_v = T_{uv} \quad \text{if} \ uv = u \cdot v \text{ is a reduced factorization,} \quad (2.3)
\]

\[
    (T_s - u)(T_s + u^{-1}) = 0 \quad \text{if} \ s \in S.
\]

For each \( J \subseteq S \), \( \mathcal{H}_J \) denotes the subalgebra of \( \mathcal{H} \) with \( A \)-basis \( \{ T_w : w \in W_J \} \), which is also the Hecke algebra of \( W_J \).

The involution, \( \overline{\cdot} \), of \( \mathcal{H} \) is the additive map from \( \mathcal{H} \) to itself extending the involution \( \overline{\cdot} \) on \( A \) and satisfying \( \overline{T_w} = T_{w^{-1}} \). Observe that \( \overline{T_s} = T_s^{-1} = T_s + u^{-1} - u \) for
$s \in S$. Some simple $\tau$-invariant elements of $\mathcal{H}$ are $C'_{\text{id}} := T_{\text{id}}$ and $C'_s := T_s + u^{-1} = T_s^{-1} + u$, $s \in S$. The $\tau$-invariant $u$-integers are $[k] := \frac{u^k - u^{-k}}{u - u^{-1}} \in A$.

### 2.2.4

Before introducing $W$-graphs and the Kazhdan-Lusztig basis, we will discuss a slightly more general setup for canonical bases. The presentation here follows Du [21]. This formalism originated in [22] and was further developed by Lusztig and Kashiwara (see the references in [21]).

Given any $A$-module $E$ (no Hecke algebra involved), we can try to construct a canonical basis or IC basis from a standard basis and involution $\cdot : E \to E$. Let \{\(t_i : i \in I\}\) be an $A$-basis of $E$ (the standard basis) for some index set $I$ and assume the involution $\cdot$ intertwines the involution $\cdot$ on $A$: $\overline{at} = \overline{a} \overline{t}$ for any $a \in A$, $t \in E$.

Define the lattice $\mathcal{L}$ to be $A^{-}\{t_i : i \in I\}$. If there exists a unique $\tau$-invariant basis \{\(c_i : i \in I\}\) of the free $A^{-}$-module $\mathcal{L}$ such that $c_i \equiv t_i \mod u^{-1}\mathcal{L}$, then \{\(c_i : i \in I\}\) is an IC basis of $E$, denoted

$$IC_E(\{t_i : i \in I\}, \overline{\cdot}). \quad (2.4)$$

**Theorem 2.2.1** (Du [21]). *With the notation above, if $(I, \prec)$ is a poset such that for all $j \in I$, $\{i \in I : i \prec j\}$ is finite and $\overline{t_j} \equiv t_j \mod A\{t_i : i \prec j\}$, then the IC basis $IC_E(\{t_i : i \in I\}, \overline{\cdot})$ exists.*

In the remainder of this paper, $\overline{\cdot}$ will be clear from context so will be omitted from the $IC(\cdot)$ notation. An observation that will be used in §2.2.7 and §2.3 is that this construction behaves well with taking lower order ideals.
Proposition 2.2.2. With the notation of Theorem 2.2.1, if $I'$ is a lower order ideal of $I$ and $E' := A\{t_i : i \in I'\}$, then

$$\mathcal{IC}_{E'}(\{t_i : i \in I'\}) = \{c_i : i \in I'\} \subseteq \mathcal{IC}_{E}(\{t_i : i \in I\})$$

Proof. The poset $I'$ and the involution $\tau$ restricted to $E'$ satisfy the necessary hypotheses so that Theorem 2.2.1 applies. Label the resulting IC basis by $d_i$, $i \in I'$ and put $L' = A - \{t_i : i \in I'\}$. Then $d_i \equiv t_i \mod u - 1 L'$ for $i \in I'$ certainly implies $d_i \equiv t_i \mod u - 1 L$. Uniqueness of the IC basis then implies $d_i = c_i$ ($i \in I'$).

We now come to the main construction studied in this paper. Let $E$ be an $H_J$-module with an involution $\tau: E \rightarrow E$ intertwining $\tau$ on $H_J$ ($\overline{he} = \overline{h} \overline{e}$ for all $h \in H_J$ and $e \in E$). Suppose $\Gamma$ is a $\tau$-invariant $A$-basis of $E$. Put $\widetilde{E} = H \otimes_{H_J} E$. We will apply Theorem 2.2.1 to $\widetilde{E}$ with standard basis $\widetilde{T} := \{\widetilde{T}_z : z \in W^J \times \Gamma\}$, where $\widetilde{T}_{w,\gamma} := T_w \boxtimes \gamma$. The lattice $\mathcal{L}$ is then $A^{-1} \mathcal{T}$. Define the involution $\overline{\tau}$ on $\widetilde{E}$ from the involutions on $E$ and $H$:

$$\overline{h \boxtimes e} = \overline{h} \boxtimes \overline{e}, \text{ for every } h \in H, e \in E. \quad (2.5)$$

It is easy to check (and is done in [17]) that the definition of $\overline{\tau}: \widetilde{E} \rightarrow \widetilde{E}$ is sound, that it’s an involution and intertwines the involution $\tau$ on $H$.

Let $<$ be the partial order on $W^J \times \Gamma$ generated by the rule: $(w', \gamma') < (w, \gamma)$ if $\widetilde{T}_{w',\gamma'}$ appears with non-zero coefficient in $(\overline{T_w - T_{w'}}) \boxtimes \gamma$ expanded in the basis $\mathcal{T}$. Since $\overline{T_w - T_w}$ is an $A$-linear combination of $T_x$ for $x < w$, it is easy to see that $\overline{T_{w,\gamma} - \overline{T}_{w,\gamma}}$ ($w \in W^J, \gamma \in \Gamma$) is an $A$-linear combination of $\{\overline{T}_{x,\delta} : x < w, \delta \in \Gamma\}$.
the definition of $\preceq$ is sound. To see that $D_{w,\gamma} := \{(w', \gamma') : (w', \gamma') \preceq (w, \gamma)\}$ is finite, induct on $\ell(w)$. The set $D_{w,\gamma}$ is the union of $\{(w, \gamma)\}$ and $D_{w',\gamma'}$ over those $(w', \gamma')$ such that $\tilde{T}_{w',\gamma'}$ appears with non-zero coefficient in $(\tilde{T}_w - T_w) \otimes \gamma$, each of which is finite by induction.

Thus Theorem 2.2.1 applies and we obtain a canonical basis $\Lambda = IC_{\tilde{E}}(\tilde{T}) = \{\tilde{C}_{w,\gamma} : w \in W^J, \gamma \in \Gamma\}$ of $\tilde{E}$. This is one way of proving the following theorem that is Theorem 5.1 in [17] (there they use the basis $C_{w,\gamma}$ that is $\equiv \tilde{T}_{w,\gamma} \mod uA^+\tilde{T}$).

**Theorem 2.2.3** (Howlett, Yin [17]). There exists a unique $\gamma$-invariant basis $\Lambda = \{\tilde{C}_{w,\gamma} : w \in W^J, \gamma \in \Gamma\}$ of $\tilde{E}$ such that $\tilde{C}_{w,\gamma} \equiv \tilde{T}_{w,\gamma} \mod u^{-1}\mathcal{L}$.

Applied to $J = \emptyset$ and $\Gamma$ the free $A$-module of rank one, this yields the usual Kazhdan-Lusztig basis $\Gamma_W := \{C'_{w} : w \in W\}$ of $\mathcal{H}$.

2.2.5. In [22], Kazhdan and Lusztig introduce $W$-graphs as a combinatorial structure for describing an $\mathcal{H}$-module with a special basis. A $W$-graph consists of a vertex set $\Gamma$, an edge weight $\mu(\delta, \gamma) \in \mathbb{Z}$ for each ordered pair $(\delta, \gamma) \in \Gamma \times \Gamma$, and a descent set $L(\gamma) \subseteq S$ for each $\gamma \in \Gamma$. These are subject to the condition that $A\Gamma$ has a left $\mathcal{H}$-module structure given by

$$C'_s\gamma = \begin{cases} [2]_{\gamma} & \text{if } s \in L(\gamma), \\ \sum_{\{\delta \in \Gamma : a \in L(\delta)\}} \mu(\delta, \gamma)\delta & \text{if } s \notin L(\gamma). \end{cases} \quad (2.6)$$

We will use the same name for a $W$-graph and its vertex set. If an $\mathcal{H}$-module $E$ has an $A$-basis $\Gamma$ that satisfies (2.6) for some choice of descent sets, then we say that
Γ gives $E$ a $W$-graph structure, or $Γ$ is a $W$-graph on $E$.

It is convenient to define two $W$-graphs $Γ, Γ'$ to be isomorphic if they give rise to isomorphic $H$-modules with basis. That is, $Γ ≅ Γ'$ if there is a bijection $α : Γ → Γ'$ of vertex sets such that $L(α(γ)) = L(γ)$ and $μ(α(γ), α(γ)) = μ(δ, γ)$ whenever $L(δ) ⊆ L(γ)$.

Given a $W$-graph $Γ$, we always have an involution

\[ τ : AΓ \to AΓ, \text{ with } τγ = γ \text{ for every } γ ∈ Γ, \] (2.7)

and extended $A$-semilinearly using the involution on $A$. It is quite clear from (2.6) (and checked in [17]) that this involution intertwines $τ$ on $H$.

2.2.6. Now let $Γ$ be a $W_J$-graph, $E = AΓ$, and $τ : E → E$ be as just mentioned in (2.7). Then we are in the setup of §2.2.4 except $Γ$ is a $W_J$-graph instead of any $τ$-invariant basis of $E$. Maintaining the notation of §2.2.4, let $Λ = IC_{\tilde{E}}(\tilde{T}) = \{ \tilde{C}_{w,γ} : w ∈ W^J, γ ∈ Γ \}$. As would be hoped, $Λ$ gives $\tilde{E}$ a $W$-graph structure.

Define $\tilde{P}_{x,δ,w,γ}$ by the formula

\[ \tilde{C}_{w,γ} = \sum_{(x,δ) ∈ W^J \times Γ} \tilde{P}_{x,δ,w,γ} \tilde{T}_{x,δ}, \] (2.8)
For every \((x, \delta), (w, \gamma) \in W^J \times \Gamma\) define

\[
\mu(x, \delta, w, \gamma) = \begin{cases} 
\text{coefficient of } u^{-1} \text{ in } \tilde{P}_{x,\delta,w,\gamma} & \text{if } x < w, \\
\mu(\delta, \gamma) & \text{if } x = w, \\
1 & \text{if } x = sw, x > w, s \in S, \delta = \gamma, \\
0 & \text{otherwise.}
\end{cases}
\]

(2.9)

Also define \(L(w, \gamma) = L(w) \cup \{s \in S : sw = wt, t \in L(\gamma)\}\). Now we can state the main result of Howlett and Yin.

**Theorem 2.2.4** ([17, Theorem 5.3]). With \(\mu\) and \(L\) as defined above, \(\Lambda\) gives \(\tilde{E} = \mathcal{H} \otimes_{W_J} \Gamma\) a \(W\)-graph structure.

We will often abuse notation and refer to a module when we really mean the \(W\)-graph on that module, but there should be no confusion as there will never be more than one \(W\)-graph structure on a given module. We will use the notation \(\mathcal{H} \otimes_{W_J} \Gamma\) to mean the \(\Lambda\) in this theorem, in case we want refer to its vertex set or to emphasize the \(W\)-graph rather than the module.

**Remark 2.2.5.** A \(W\)-graph is **symmetric** if it is isomorphic to a \(W\)-graph with \(\mu(x, w) = \mu(w, x)\) for all vertices \(x, w\). The \(W\)-graph \(\Gamma_W\) on the regular representation of \(\mathcal{H}\) is symmetric. The \(W\)-graph \(\Lambda\) defined above is symmetric if and only if \(\Gamma\) is symmetric, although this is not obvious from the definition of \(\mu\) (2.9). In [17], the \(W\)-graph for \(\Lambda\) is defined so that it is clearly symmetric, and then it is proved later that it is isomorphic to the \(W\)-graph \(\Lambda\) defined here.
2.2.7. Let $\Gamma$ be a $W$-graph and put $E = A\Gamma$. The preorder $\leq_\Gamma$ on the vertex set $\Gamma$ is generated by

$$\delta \leq_\Gamma \gamma$$

if there is an $h \in \mathcal{H}$ such that $\delta$ appears with non-zero coefficient in the expansion of $h\gamma$ in the basis $\Gamma$.

Equivalence classes of $\leq_\Gamma$ are the left cells of $\Gamma$, or just cells since we will almost exclusively work with left cells. Sometimes we will speak of the cells of $E$ or the preorder on $E$ to mean that of $\Gamma$, when the $W$-graph $\Gamma$ is clear from context. A cellular submodule of $E$ is a submodule of $E$ that is spanned by a subset of $\Gamma$ (and is necessarily a union of cells). A cellular quotient of $E$ is a quotient of $E$ by a cellular submodule, and a cellular subquotient of $E$ is a cellular submodule of a cellular quotient. We will abuse notation and sometimes refer to a cellular subquotient by its corresponding union of cells.

We will give one result about cells in the full generality of §2.2.6 before specializing $W$ and the $W_J$-graph $\Gamma$. Let $D$ be a cellular submodule of $E$ spanned by a subset $\Gamma_D$ of $\Gamma$ and $p : E \to E/D$ the projection. Put $\Gamma_{E/D} = p(\Gamma \setminus \Gamma_D)$. The $W_J$-graph $\Gamma$ yields $W_J$-graphs $\Gamma_D$ on $D$ and $\Gamma_{E/D}$ on $E/D$. The involution $\cdot$ on $E$ restricts to one on $D$ and projects to one on $E/D$; elements of $\Gamma_D$ (resp. $\Gamma_{E/D}$) are fixed by the involution $\cdot$ on $D$ (resp. $E/D$). Since $\mathcal{H}$ is a free right $\mathcal{H}_J$-module, we have the exact sequence

$$0 \longrightarrow \mathcal{H} \otimes_J D \longrightarrow \mathcal{H} \otimes_J E \longrightarrow \mathcal{H} \otimes_J E/D \longrightarrow 0,$$  

where the shorthand $\mathcal{H} \otimes_J E := \mathcal{H} \otimes_{\mathcal{H}_J} E$ will be used here and from now on.
In other words, inducing commutes with taking subquotients. It is also true that inducing and taking canonical bases commutes with taking cellular subquotients:

**Proposition 2.2.6.** With the notation above and that of §2.2.6, let
\[ \tilde{T}_D = \{ \tilde{T}_{w,\gamma} : w \in W^J, \gamma \in \Gamma_D \} \] and \[ \tilde{T}_{E/D} = \{ T_w \boxtimes \gamma : w \in W^J, \gamma \in \Gamma_{E/D} \} . \]

Then
\[ (i) \quad \mathcal{IC}_{\mathcal{H} \otimes J D} \left( \tilde{T}_D \right) = \{ \tilde{C}_{w,\gamma} : w \in W^J, \gamma \in \Gamma_D \} \subseteq \mathcal{IC}_{\tilde{E}}(\tilde{T}) , \]
\[ (ii) \quad \mathcal{IC}_{\mathcal{H} \otimes J E/D} \left( \tilde{T}_{E/D} \right) = \{ \tilde{p}(\tilde{C}_{w,\gamma}) : w \in W^J, \gamma \in \Gamma \backslash \Gamma_D \} \subseteq \tilde{p}\left( \mathcal{IC}_{\tilde{E}}(\tilde{T}) \right) . \]

In particular, \( \mathcal{H} \otimes J D \) (resp. \( \mathcal{H} \otimes J E/D \)) is a cellular submodule (resp. quotient) of \( \mathcal{H} \otimes J E \).

**Proof.** Statement (i) is actually a special case of Proposition 2.2.2. From the definition of \( \prec \) in §2.2.4 we can see that \( W^J \times \Gamma_D \) is a lower order ideal of \( W^J \times \Gamma \).

We prove (ii) directly. The lattice \( L_{E/D} := A^{-\tilde{T}_{E/D}} \) is the quotient \( L / L_D = \tilde{p}(L) \). Therefore, given \( w \in W^J \) and \( \gamma \in \Gamma \backslash \Gamma_D \), we have
\[ \tilde{p}(\tilde{C}_{w,\gamma}) = \tilde{p}(T_w \boxtimes \gamma + u^{-1}x) \equiv \tilde{p}(T_w \boxtimes \gamma) = T_w \boxtimes p(\gamma) , \tag{2.12} \]
where \( x \) is some element of \( L \) and the congruence is \( \mod u^{-1} L_{E/D} \). By definition, \( p(\gamma) \in \Gamma_{E/D} \) so \( \tilde{p}(\tilde{C}_{w,\gamma}) \) is the element of \( \mathcal{IC}_{\mathcal{H} \otimes J E/D} \left( \tilde{T}_{E/D} \right) \) congruent to \( T_w \boxtimes p(\gamma) \mod u^{-1} L_{E/D} \).

This proposition is essentially [18, Theorem 4.3], though the proof here is different. It also appears in [9, Theorem 1] in the case that \( \Gamma = \Gamma_{W_J} \) (the usual \( W_J \)-graph on \( \mathcal{H}_J \)) but in the generality of unequal parameters.
2.3 Iterated induction from the regular representation

In this paper we will primarily be interested in the case where $E$ is obtained by some sequence of inductions and restrictions of the regular representation of a Hecke algebra, or subquotients of such modules. In this section, let $\tilde{E}$ denote $\mathcal{H}_1 \otimes_J E$, where $E = A\Gamma, \Gamma = \Gamma_{W_2}$ unless specified otherwise.

2.3.1. Suppose we are given Coxeter groups $W_1, W_2$ with simple reflections $S_1, S_2$ and a set $J$ with inclusions $i_k : J \to S_k, k = 1, 2$ such that $(W_1)_{i_1(J)} \cong (W_2)_{i_2(J)}$ as Coxeter groups. Define the set

$$W_1 \times^J W_2 := \{(w_1, w_2) : w_1 \in W_1, w_2 \in W_2\} / \langle (w_1 w, w_2) \sim (w_1, w w_2) : w \in W_J \rangle,$$

(2.13)

where $W_J := W_{1,J} \cong W_{2,J}$. The set $W_1 \times^J W_2$ can also be identified with any of $W_1 \times^J W_2, W_1^J \times W_2, W_1^J \times^J W_2$. These sets label canonical basis elements of Hecke algebra modules obtained by inducing from the regular representation just as a Coxeter group labels the canonical basis elements of its regular representation.

The material that follows in this subsection is somewhat tangent from our main theme, but we include it for completeness. We omit the details of proofs.

The set $W_1 \times^J W_2$ comes with a left action by $W_1$, a length function, and a partial order generalizing the Bruhat order, as described in the following proposition.
Proposition 2.3.1. Let \((w_1, w_2) \in W_1 \times W_2\). The set \(W_1 \times W_2\) comes equipped with

(i) A left action by \(W_1\): \(x \cdot (w_1, w_2) = (xw_1, w_2)\),

(ii) a length function: \(\ell(w_1, w_2) = \ell(w_1) + \ell(w_2)\) whenever \(w_1 \in W_1^J\),

(iii) a partial order: \((w'_1, w'_2) \leq (w_1, w_2)\), whenever there exists \((w''_1, w''_2) \sim (w'_1, w'_2)\) such that \(w''_i \leq w_i, w'_i, w''_i \in W_i, i = 1, 2, \) and \(w_1 \in W_1^J\).

Proposition 2.3.2. The \(W_1\)-graph \(\tilde{E}\) is bipartite in the sense of [18, Definition 3.1]. Moreover, if \(z, z' \in W_1 \times W_2\), and \(\ell(z) - \ell(z')\) is even (resp. odd), then \(\tilde{P}_{z',z}\) involves only even (resp. odd) powers of \(u\).

Proof. This follows from [18, Proposition 3.2].

Proposition 2.3.3. The \(W_1\)-graph \(\tilde{E}\) is ordered in the sense of [18, Definition 1.1]. Stronger, \(W_1 \times W_2\) has a partial order from Proposition 2.2 of [18] using the Bruhat order on \(W_2\), and this agrees with \(\leq\) of Proposition 2.3.1. Therefore if \(z, z' \in W_1 \times W_2\) and \(\tilde{P}_{z',z} \neq 0\), then \(z' \leq z\).

Proof. Showing the partial orders from [18] and Proposition 2.3.1 are equal takes some work. The rest is a citation of results in [18].

2.3.2. A similar definition to that in the previous subsection can be given for \(W_1^{J_1} \times \ldots \times W_{J_{d-1}} \times W_d\). To work with these sets, introduce the following notation. A representa-
tive \((w_1, \ldots, w_d)\) of an element of \(W_1^{J_1} \times \ldots \times W_d^{J_{d-1}}\) is \(i\)-stuffed if

\[
w_1 \in W_1^{J_1}, \ldots, w_{i-1} \in W_{i-1}^{J_{i-1}}, w_i \in W_i, w_{i+1} \in J_{i+1}, \ldots, J_{d-1}W_d. \tag{2.14}
\]

It is convenient to represent the element \(z \in W_1^{J_1} \times \ldots \times W_d\), somewhat redundantly, in stuffed notation: \(z = (z_1, z_2, \ldots, z_d)\), where \(z_i\) is the \(i\)-th component of the \(i\)-stuffed expression for \(z\).

2.3.3. The main ideas in this subsection also appear in [10, §4] where they are used to adapt Lusztig’s \(a\)-invariant to give results about the partial order on the cells of \(\text{Res}_J \Gamma_W\).

For any \(X \subseteq W_1 \times W_2\), define the shorthands

\[
TT(X) := \{T_{w_1} \boxtimes T_{w_2} : (w_1, w_2) \in X\},
\]

\[
TC(X) := \{T_{w_1} \boxtimes C'_{w_2} : (w_1, w_2) \in X\}, \tag{2.15}
\]

\[
CT(X) := \{C'_{w_1} \boxtimes T_{w_2} : (w_1, w_2) \in X\}.
\]

The construction from §2.2.4 applied to \(\Gamma_W\) gives the IC basis \(\text{IC}_{\tilde{E}}(TC(W_1^{J_1} \times W_2))\) of \(\tilde{E}\). The next proposition shows that the same canonical basis can be constructed from two other standard bases, and this will be used implicitly in what follows.

**Proposition 2.3.4.** The standard bases

\[
TC(W_1^{J_1} \times W_2), TT(W_1^{J_1} \times W_2) = TT(W_1 \times JW_2), CT(W_1 \times JW_2)
\]

of \(\tilde{E} = \mathcal{H}_1 \otimes_J \mathcal{H}_2\) have the same \(A^-\)-span, denoted \(\mathcal{L}\). Moreover,

\[
T_{w_1} \boxtimes C'_{w_2} \equiv T_{w_1} \boxtimes T_{w_2} = T_{w_1} \boxtimes T_{w_2} \equiv C'_{w_1} \boxtimes T_{w_2} \mod u^{-1}\mathcal{L}
\]
for every $w_1 \in W_1^J, v \in W_J, w_2 \in J^J W_2$. Therefore, the corresponding IC bases are identical:

$$\IC\tilde{E}(TC(W_1^J \times W_2)) = \IC\tilde{E}(TT(W_1^J \times W_2)) = \IC\tilde{E}(CT(W_1 \times J^J W_2))$$

(and these will be denoted $\Lambda = \{C'_{w_1,w_2} : (w_1, w_2) \in W_1 \times W_2\}$).

**Proof.** The lattices $A^-\{T_{w_2} : w_2 \in W_2\}$ and $A^-\{C'_{w_2} : w_2 \in W_2\}$ are equal by the definition of an IC basis (§2.2.4). Thus $A^-TC(W_1^J \times W_2) = A^- TT(W_1^J \times W_2)$ and similarly $A^- TT(W_1 \times J^J W_2) = A^- CT(W_1 \times J^J W_2)$. The remaining statements are clear. \qed

Now given any lower order ideal $I$ in $J^J W_2$, define $D_I = A^- CT(W_1 \times I)$, thought of as an $\mathcal{H}_1$-submodule of $\tilde{E}$. Applying Proposition 2.2.2 to $D_I \subseteq \tilde{E}$ with poset $W_1 \times J^J W_2$ and lower ideal $W_1 \times D_I$ shows that $D_I$ has canonical basis $\{C'_{w_1,w_2} : w_1 \in W_1, w_2 \in I\}$ (Proposition 2.3.4 is used implicitly). The next theorem now comes easily.

Let $D_{\leq x} = D_{\{w : w \in J^J W_2, w \leq_x\}}$ and $D_{< x} = D_{\{w : w \in J^J W_2, w < x\}}$. Recall that $\Gamma_{W_1}$ is the usual $W_1$-graph of the regular representation of $\mathcal{H}_1$.

**Theorem 2.3.5.** The module $\tilde{E}$ (with $W_1$-graph structure $\Lambda$) has a filtration with cellular subquotients that are isomorphic as $W_1$-graphs to $\Gamma_{W_1}$. In particular, the left cells of $\Lambda$ are isomorphic to those occurring in $\Gamma_{W_1}$.

**Proof.** For any $x \in J^J W_2$, the map $\pi : D_{\leq x} \to \mathcal{H}_1$ given by $\pi(D_{< x}) = 0$ and $C'_w \boxtimes T_x \mapsto C'_w$ is an $\mathcal{H}_1$-module homomorphism. Hence the exact sequence

$$0 \to D_{< x} \to D_{\leq x} \xrightarrow{\pi} \mathcal{H}_1 \to 0.$$ (2.16)
Moreover, \( \pi(\tilde{C}'_{w,x}) = C'_w \), which is clear when viewing the \( \tilde{C}'_{w,x} \) as being constructed from the standard basis \( CT(W_1 \times J W_2) \). This gives an isomorphism of \( W_1 \)-graphs 
\( D_{\leq x}/D_{< x} \cong \mathcal{H} \).

Letting \( \mathcal{H} \) be the Hecke algebra of \( (W, S) \) and setting \( \mathcal{H}_1 = \mathcal{H}_J, \mathcal{H}_2 = \mathcal{H}, \)
\( J \subseteq S \), we obtain

**Corollary 2.3.6.** The left cells of \( \text{Res}_J \mathcal{H} \) are isomorphic as \( W_J \)-graphs to the left cells of the regular representation of \( \mathcal{H}_J \).

This corollary is implied by results from [18, §5], but the method of proof is different. It is also a consequence of [35, Theorem 5.2].

By the same methods we can check that the canonical basis construction for induced modules is well-behaved for nested parabolic subgroups.

**Proposition 2.3.7.** Let \( \mathcal{H} \) be the Hecke algebra of \( (W, S) \), \( J_2 \subseteq J_1 \subseteq S \), \( E \) a left \( \mathcal{H}_{J_2} \)-module with involution \( \tau \) intertwining that of \( \mathcal{H}_{J_2} \), and \( \Gamma \) a \( \tau \)-invariant basis of \( E \) (like the setup in §2.2.4). Let \( \Lambda_{J_1} = IC_{\mathcal{H}_{J_1} \otimes E}(\{ \tilde{T}_{w,\gamma} : w \in W_{J_1}^{J_2}, \gamma \in \Gamma \}) \). Then, putting \( \tilde{E} = \mathcal{H}_{J_2} \otimes E \), we have

\[
IC_{\tilde{E}}(\{ T_{w, \gamma} \otimes \gamma : w \in W^{\cdot J_2}, \gamma \in \Gamma \}) = IC_{\tilde{E}}(\{ T_{w, \delta} \otimes \delta : w \in W^{\cdot J_1}, \delta \in \Lambda_{J_1} \}). \tag{2.17}
\]

**Proof.** By the same argument as in Proposition 2.3.4, the right-hand side of (2.17) can also be constructed from the standard basis \( \{ T_{v_1} \otimes T_{v_2} \otimes \gamma : v_1 \in W^{\cdot J_1}, v_2 \in W_{J_1}^{J_2}, \gamma \in \Gamma \} \). It remains to check that \( W^{\cdot J_1} \times W_{J_1}^{J_2} = W^{\cdot J_2} \) by \( (v_1, v_2) \mapsto v_1 v_2 \). As \( v_1 \) ranges...
over left coset representatives of \( W_{J_1} \) and \( v_2 \) over left coset representatives of \( W_{J_2} \) inside \( W_{J_1} \), \( v_1v_2 \) ranges over left coset representatives of \( W_{J_2} \) in \( W \) (true for any pair of nested subgroups in a group). To see that \( v_1v_2 \) is a minimal coset representative, let \( x \in W_{J_2} \); then \( v_2 \cdot x \) is a reduced factorization and \( v_2x \in W_{J_1} \) (and \( v_1 \) minimal in \( v_1W_{J_1} \)) implies \( v_1 \cdot v_2x \) is a reduced factorization and thus so is \( v_1 \cdot v_2 \cdot x \).

\[ \square \]

### 2.3.4. The set of cells of a \( W \)-graph \( \Gamma \) is denoted \( \mathcal{C}(\Gamma) \). We will describe the cells of \( \mathcal{H}_1 \otimes_{J_1} \cdots \otimes_{J_{d-1}} \mathcal{H}_d \) using the results of the previous subsection \( \S 2.3.3 \).

Let \( \Upsilon \) be a cell of \( \mathcal{H}_1 \otimes_{J} \mathcal{H}_2 \). By Theorem 2.3.5 and its proof, \( \Upsilon = \{ C'_{w_1,x_2} : w_1 \in \Upsilon' \} \) for some cell \( \Upsilon' \) of \( \Gamma_{W_1} \) and \( x_2 \in J_{W_2} \). We say that \( \Upsilon' \) is the local label of \( \Upsilon \). By Theorem 2.3.5, the cells \( \Upsilon \) and \( \Upsilon' \) are isomorphic as \( W_1 \)-graphs so that the isomorphism type of a cell is determined by its local label. Thus \( \mathcal{C}(\mathcal{H}_1 \otimes_{J} \mathcal{H}_2) \) has a description via the bijection \( \mathcal{C}(\mathcal{H}_1 \otimes_{J} \mathcal{H}_2) \cong \mathcal{C}(\mathcal{H}_1) \times J_{W_2} \), \( \Upsilon \mapsto (\Upsilon', x_2) \), taking a cell to its local label and an element of \( J_{W_2} \). Unfortunately, from this description it is difficult to determine the cells of a cellular subquotient \( \mathcal{H}_1 \otimes_{J} \mathcal{A} \Gamma \) of \( \mathcal{H}_1 \otimes_{J} \mathcal{H}_2 \) for some \( \Gamma \in \mathcal{C}(H_2) \) (this is a cellular subquotient of \( \mathcal{H}_1 \otimes_{J} \mathcal{H}_2 \) by Proposition 2.2.6).

Essentially the same argument used in Theorem 2.3.5 yields a similar expression for the general case:

\[
\mathcal{C}(\mathcal{H}_1 \otimes_{J_1} \cdots \otimes_{J_{d-1}} \mathcal{H}_d) \cong \mathcal{C}(\mathcal{H}_d) \times J_{1W_2} \times \cdots \times J_{d-1W_d},
\]

(2.18)

taking a cell to its local label and a tuple of right coset representatives. This of course has the same drawback of it being difficult to identify the subset of cells obtained by
taking a cellular subquotient of $H_d$. We now address this deficiency.

Put $\widetilde{E}^k = H_{d-k} \otimes J_{d-k} \ldots \otimes J_{d-1} H_d$. The collection of cells $\bigcup_{k=0}^{d-1} \mathcal{C}(\widetilde{E}^k)$ can be pictured as vertices of an acyclic graph $G$ (see Figure 2.1 of §2.7.3). The subset $\mathcal{C}(\widetilde{E}^k)$ of vertices is the $k$th level of $G$. There is an edge between $\Upsilon^k$ of level $k$ and $\Upsilon^{k+1}$ of level $k + 1$ if $\Upsilon^{k+1} \in \mathcal{C}(H_{d-(k+1)} \otimes J_{d-(k+1)} \Upsilon^k)$. Here we are using Proposition 2.2.6 to identify $H_{d-(k+1)} \otimes J_{d-(k+1)} \Upsilon^k$ with a cellular subquotient of $\widetilde{E}^{k+1}$. Note that from a vertex of level $k + 1$ there is a unique edge to a vertex of level $k$ since the cells of a module $\widetilde{E}^k$ are the composition factors of a composition series for $\widetilde{E}^k$, thereby yielding composition factors for the induced module of $\widetilde{E}^{k+1} = H_{d-(k+1)} \otimes J_{d-(k+1)} \widetilde{E}^k$.

A vertex $\Upsilon^k$ in the $k$-th level of $G$ has a unique path to a vertex $\Upsilon^0$ in the 0-th level. The local labels $(\Gamma^k, \ldots, \Gamma^0)$ of the vertices in this path is the local sequence of $\Upsilon^k$ (where $\Gamma^i$ is the local label of the vertex in the $i$-th level).

The cell of $\widetilde{E}^{d-1}$ containing $C'_z$, $z \in W_1 \times J_1 \times \ldots \times W_d$ is the end of a path with local labels $(\Gamma_1, \ldots, \Gamma_d)$, where $\Gamma_i \in \mathcal{C}(\Gamma_{W_i})$ is the cell containing $C'_{z_i}$ and $(z_1, \ldots, z_d)$ is stuffed notation for $z$.

A local sequence $(\Gamma^{d-1}, \ldots, \Gamma^0)$ does not in general determine a cell of $\widetilde{E}^{d-1}$ uniquely. For instance, the cells of $H_J \otimes_J H$ with $J = \emptyset$ are just single canonical basis elements of $H$, so a local sequence does not determine a cell unless the cells of $H$ are of size 1. We say that the tuple $(\widetilde{E}^{d-1}, \ldots, \widetilde{E}^0)$ is weakly multiplicity-free if there is at most one cell of $\widetilde{E}^{d-1}$ with local sequence $(\Gamma^{d-1}, \ldots, \Gamma^0)$ for all $\Gamma^i \in \mathcal{C}(\Gamma_{W_{d-i}})$. Pure induction $(H \otimes_J H, J)$ is trivially weakly multiplicity-free.
since the local label of a cell in $\mathcal{H} \otimes_J \mathcal{H}_J = \mathcal{H}$ is the same thing as the cell itself. It is not hard to see that $(\widetilde{E}^{d-1}, \ldots, \widetilde{E}^0)$ is weakly multiplicity-free if and only if the restriction $(\mathcal{H}_j \otimes_J \mathcal{H}_{i+1}, \mathcal{H}_{i+1})$ is for all $i$.

We have seen that the restriction $(\mathcal{H} \otimes_J \mathcal{H}, \mathcal{H})$ is not always weakly multiplicity-free, but a natural question is whether it always is for $J$ of size $|S| - 1$. This fails for $W$ of type $B_2$ and $B_3$ for all choices of $J$ (and presumably for $B_n$, $n > 3$). This failure may only be because cells in type $B$ do not always correspond to irreducible modules, so this question should be investigated in the unequal parameter setting.

We conjecture the following for type $A$.

**Conjecture 2.3.8.** If $\mathcal{H}$ is the Hecke algebra of $(W, S) = (S_n, \{s_1, \ldots, s_{n-1}\})$ and $|J| = |S| - 1$, then the restriction $(\mathcal{H}_j \otimes_J \mathcal{H}, \mathcal{H})$ is weakly multiplicity-free.

This conjecture was verified for $n = 10$, $J = S \setminus \{s_5\}$ using Magma, and for $n = 16$ and a few arbitrary choices of a cell $\Gamma$, we checked that $(\mathcal{H}_j \otimes_J \Gamma, \Gamma)$ is weakly multiplicity-free. Strangely, it does not seem to be amenable to typical RSK, jeu de taquin style combinatorics. See §2.4.3 for more about the combinatorics involved here.

### 2.4 Tableau combinatorics

2.4.1. We will make the description of cells from the previous section combinatorially explicit in the case $W = S_n$. In this section fix $S = \{s_1, \ldots, s_{n-1}\}$ and $\mathcal{H}$ the
Hecke algebra of type $A_{n-1}$. As is customary, we will think of an element of $S_n$ as a word of length $n$ in the numbers $1, \ldots, n$. We want to maintain the convention used thus far of looking only at left $\mathcal{H}$-modules, however tableau combinatorics is a little nicer if a right action is used. To get around this, define the word associated to an element $w = s_{i_1} s_{i_2} \ldots s_{i_k} \in W$ to be $w^{-1}(1) w^{-1}(2) \ldots w^{-1}(n)$, where (to be completely explicit) $w^{-1}(i) = s_{i_k} s_{i_{k-1}} \ldots s_{i_1}(i)$ and $s_j$ transposes $j$ and $j+1$. The left descent set of $w \in S_n$ is $\{s_i : w^{-1}(i) > w^{-1}(i+1)\}$.

The RSK algorithm gives a bijection between $S_n$ and pairs of standard Young tableau (SYT) of the same shape sending $w \in S_n$ to the pair $(P(w), Q(w))$, written $w \xrightarrow{\text{RSK}} (P(w), Q(w))$, where $P(w)$ and $Q(w)$ are the insertion and recording tableaux of the word of $w$ (which is equal to $w^{-1}(1) w^{-1}(2) \ldots w^{-1}(n)$ by our convention). As was shown in [22], the left cells of $\mathcal{H}$ are in bijection with the set of SYT and the cell containing $C'_w$ corresponds to the insertion tableau of $w$ under this bijection. The cell containing those $C'_w$ such that $w$ has insertion tableau $P$ is the cell labeled by $P$. Note that the shape of the tableau labeling a cell is the transpose of the usual convention for Specht modules, i.e. the trivial representation is labeled by the tableau of shape $1^n$, sign by the tableau of shape $n$.

For the remainder of this paper let $r \in \{1, \ldots, n-1\}$, $J_r = \{s_1, \ldots, s_{r-1}\}$, $J'_{n-r} = \{s_{r+1}, \ldots, s_{n-1}\}$, and $J = J_r \cup J'_{n-r}$.

2.4.2. Let $\Gamma$ be a cell of $W_J$ labeled by a pair of insertion tableaux $(T, T') \in T_{1^r0^{n-r}} \times T_{0^r1^{n-r}}$, where $T_\alpha$ is the set of tableau with $\alpha_i$ entries equal to $i$. Here we are using
the easy fact, proven carefully in [35], that a cell of $\Gamma_{W_1 \times W_2}$ is the same as a cell of $\Gamma_{W_1}$ and one of $\Gamma_{W_2}$. We will describe the cells of $\mathcal{H} \otimes J A \Gamma$.

For any $w \in W$, in the notation of §2.2.2, $Jw = (a, b) \in W_J \times W_{J'}$, where $a$ (resp. $b$) is the permutation of numbers $1, \ldots, r$ (resp. $r + 1, \ldots, n$) obtained by taking the subsequence of the word of $w$ consisting of those numbers. For example, if $n = 6$, $w = 436125$, and $r = 3$, then $a = 312$ and $b = 465$.

The induced module $\mathcal{H} \otimes J A \Gamma$ has canonical basis $\{C'_w : P(Jw) = (T, T')\}$, where we define $P(a, b)$ for $(a, b) \in W_J \times W_{J'}$ to be $(P(a), P(b))$. For any tableau $P$, let $jdt(P)$ denote the unique straight-shape tableau in the jeu de taquin equivalence class of $P$. From the most basic properties of insertion and jeu de taquin it follows that if $Jw = (a, b)$, then $P(w)_{\leq r} = P(a)$, $P(w)_{> r} = jdt(P(b))$, where $P_{\leq r}$ (resp. $P_{> r}$) is the (skew) subtableau of $P$ with entries $1, \ldots, r$ (resp. $r + 1, \ldots, n$). See, for instance, [7, A1.2] for more on this combinatorics. We now have the following description of cells.

**Proposition 2.4.1.** With $\Gamma$ labeled by $T, T'$ as above, the cells of $\mathcal{H} \otimes J A \Gamma \subseteq \mathcal{H}$ are those labeled by $P$ such that $P_{\leq r} = T$, $jdt(P_{> r}) = T'$.

**Example 2.4.2.** Let $n = 6$, $r = 3$, and $T, T' = (\begin{array}{c} 1 \end{array}, \begin{array}{c} 2 \end{array})$. Then the cells of $\mathcal{H} \otimes J A \Gamma$ are labeled by

\[
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
1
\end{array} \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array},
\end{array}
\end{array}
\]

This is, of course, the Littlewood-Richardson rule. The combinatorics of the Littlewood-Richardson rule matches beautifully with the machinery of canonical
bases. This version of the Littlewood-Richardson is due to Schützenberger and its connection with canonical bases was also shown in [35].

Let $V_\lambda$ be the Specht module corresponding to the partition $\lambda$, and put $\mu = \text{sh}(T)$, $\nu = \text{sh}(T')$. It was established in [22] that all left cells of $\mathcal{H}$ isomorphic at $u = 1$ to $V_\lambda$ are isomorphic as $W$-graphs. This, together with the fact that the $W$-graph of Theorem 2.2.4 depends only on the isomorphism type of the $W_J$-graph $\Gamma$, shows that the multiplicity of $V_\lambda$ in $\text{Ind}_{W_J}^W(V_\mu \boxtimes V_\nu)$ is given by the combinatorics above and is independent of the chosen insertion tableaux $T, T'$.

2.4.3. Let $\Gamma$ be a cell of $\mathcal{H}$ labeled by $P$ with $\text{sh}(P) = \lambda$. We will describe the cells of $\text{Res}_J A \Gamma$.

For any $w \in W$, $w_J = (a, b) \in W_r \times W_{n-r}$, where $a$ (resp. $b$) is the permutation of numbers $1, \ldots, r$ (resp. $r + 1, \ldots, n$) with the same relative order as $w^{-1}(1) w^{-1}(2) \cdots w^{-1}(r)$ (resp. $w^{-1}(r + 1) \cdots w^{-1}(n)$). For example, if $n = 6$, $w = 436125$, and $r = 3$, then $a = 213$ and $b = 456$.

Specifying a cell $\Upsilon$ of $\text{Res}_J \mathcal{H}$ is equivalent to giving $x \in J^W$ and $(T, T') \in T_{r0^n-r} \times T_{0r^{1n-r}}$. Under this correspondence, $\Upsilon = \{C'_w : P(w_J) = (T, T'), Jw = x\}$.

Given $\mu \vdash r, \nu \vdash n - r$, define

$$
\mu \sqcup \nu = (\nu_1 + \mu_1, \nu_2 + \mu_1, \ldots, \nu_\ell(\nu) + \mu_1, \mu_1, \mu_2, \ldots, \mu_\ell(\mu)),
$$

where $\ell(\mu)$ is the number of parts of $\mu$.

Expressing the tableaux on $1, \ldots, r$ and $r + 1, \ldots, n$ that label the cells of $\text{Res}_J A \Gamma$
in terms of $P$ is tricky: first define the set

$$X := \{(T, T') : |T| = r, |T'| = n - r, jdt(TT') = P\}, \quad (2.20)$$

where $TT'$ is the tableau of shape $\mu \sqcup \nu/\rho$ ($\text{sh}(T) = \mu$, $\text{sh}(T') = \nu$, $\rho = \mu_1^\ell(\nu)$) obtained by adding $T'$ to the top right of $T$. The multiset of local labels of the cells of $\text{Res}_J A\Gamma$ (Conjecture 2.3.8 says this is actually a set) is obtained by projecting each element of $X$ onto the set $T_{\ell 0n-\ell} \times T_{\ell 0n-\ell}$ by replacing the entries of $T$ (resp. $T'$) by $1, \ldots, r$ (resp. $r+1, \ldots, n$) so that relative order is preserved.

**Example 2.4.3.** If $n = 6$, $r = 3$, and $P = \begin{array}{c|c|c}
\hline
3 & 6 & 4 \\
\hline
1 & 2 & 5 \\
\hline
\end{array}$, then $X$ is

$$\left\{(\begin{array}{c|c|c}
\hline
3 & 6 & 4 \\
\hline
1 & 2 & 5 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array}), (\begin{array}{c|c|c}
\hline
1 & 6 & 4 \\
\hline
2 & 5 & 3 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array}), (\begin{array}{c|c|c}
\hline
1 & 3 & 4 \\
\hline
2 & 5 & 6 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array}), (\begin{array}{c|c|c}
\hline
1 & 4 & 6 \\
\hline
2 & 5 & 3 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array}), (\begin{array}{c|c|c}
\hline
1 & 3 & 4 \\
\hline
2 & 5 & 6 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array}), (\begin{array}{c|c|c}
\hline
1 & 3 & 4 \\
\hline
2 & 5 & 6 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array})\right\}.$$

Hence the cells of $\text{Res}_J A\Gamma$ have local labels

$$\left\{(\begin{array}{c|c|c}
\hline
3 & 6 & 4 \\
\hline
1 & 2 & 5 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array}), (\begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array}), (\begin{array}{c|c|c}
\hline
1 & 3 & 4 \\
\hline
2 & 5 & 6 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array}), (\begin{array}{c|c|c}
\hline
1 & 4 & 6 \\
\hline
2 & 5 & 3 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array}), (\begin{array}{c|c|c}
\hline
1 & 3 & 4 \\
\hline
2 & 5 & 6 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array}), (\begin{array}{c|c|c}
\hline
1 & 3 & 4 \\
\hline
2 & 5 & 6 \\
\hline
\end{array}, \begin{array}{c|c|c}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
\end{array})\right\}. $$

A slightly better description of the cells of $\text{Res}_J A\Gamma$ is as follows. Fix $\mu \vdash r$, $\nu \vdash n - r$ such that $\lambda \subseteq \mu \sqcup \nu$, and $B$ a tableau of the rectangle shape $\rho := \mu_1^\ell(\nu)$. Now consider the jeu de taquin growth diagrams with lower left row corresponding to $P$, lower right row corresponding to $B$, and the partition at the top equal to $\mu \sqcup \nu$ (see, e.g., [7, A1.2]). The upper right row of such a growth diagram necessarily corresponds to some $TT'$ such that $jdt(TT') = P$, and the upper left row corresponds to some $A$ such that $jdt(A) = B$. Since a growth diagram is constructed uniquely from either of
its sides, we obtain the bijection

$$\{(T, T') : \text{sh}(T) = \mu, \text{sh}(T') = \nu, \text{jdt}(TT') = P\}$$

$$\cong \{A : \text{sh}(A) = \mu \cup \nu/\rho, \text{jdt}(A) = B\}. \quad (2.21)$$

From an $A$ in the set above, one obtains the corresponding $(T, T')$ as follows: perform jeu de taquin to $P$ in the order specified by the entries of $A$ to obtain a tableau of shape $\mu \cup \nu/\rho$; split this into a tableau of shape $\mu$ and one of shape $\nu$. This can be used to give another description of the set $X$. This description has the advantage that the same choice of $B$ can be used for all tableau $P$ of shape $\lambda$.

2.4.4. If $r = 1$ or $r = n - 1$, then restricting and inducing are multiplicity-free. Therefore, we only need to keep track of the shapes of the tableaux rather than the tableaux themselves, except at the first step $\mathfrak{C}(\mathcal{H}_d)$, in order to determine a cell of $\mathcal{H}_1 \otimes \mathcal{J}_1 \ldots \otimes \mathcal{J}_{d-1} \mathcal{H}_d$. However, it is often convenient for working concrete examples to keep track of all tableaux.

If $r = 1$ or $r = n - 1$, then the cells of $\text{Res}_J A \Gamma$, with $\Gamma$ labeled by $P$, can be described explicitly. If $r = 1$ (resp. $r = n - 1$), they are labeled by the tableaux obtained from $P$ by column-uninserting (resp. row-uninserting) an outer corner and replacing the entries of the result with $2, \ldots, n$ (resp. $1, \ldots, n - 1$) so that relative order is preserved.

We will work with both $r = 1$ and $r = n - 1$ in this paper because tableau combinatorics is easier with $r = n - 1$, but $r = 1$ is preferable for our work in §2.7.
and beyond. It is therefore convenient to be able to go back and forth between these two conventions.

On the level of algebras, this is done by replacing any $\mathcal{H}_K$-module by the $\mathcal{H}_{w_0 K w_0}$-module obtained by twisting by the isomorphism $\mathcal{H}_{w_0 K w_0} \cong \mathcal{H}_K, T_{s_i} \mapsto T_{s_{n-i}}$, where $w_0$ is the longest element of $W$. Combinatorially, this corresponds to replacing a word $x_1 x_2 \cdots x_n$ with

$$x' := n + 1 - x_1 \ n + 1 - x_2 \ \cdots \ n + 1 - x_n.$$  

The local label of a cell changes from $T$ to evac($T$), where $T \mapsto$ evac($T$) is the Schützenberger involution (see, e.g., [7, A1.2]). More precisely, the local label $(T, T') \in T_{1/0^{n-j}} \times T_{0/1^{n-j}}$ of a cell of an $\mathcal{H}_{S \setminus s_j}$-module becomes $(\text{evac}(T')^*, \text{evac}(T)^*)$, where evac($T')^*$ (resp. evac($T)^*$) is obtained from evac($T'$) by adding a constant to all entries so that evac($T')^* \in T_{1^{n-j}0^j}$ (resp. evac($T)^* \in T_{0^{n-j}1^j}$).

2.4.5. In this subsection we give a combinatorial description of cells of a certain submodule of $\text{Res}_K \hat{\mathcal{H}}$, where $\hat{\mathcal{H}}$ is the extended affine Hecke algebra of type $A$. We digress to introduce this object. See [41], [14] for a more thorough introduction.

First of all, everything we have done so far for Coxeter groups also holds for extended Coxeter groups. An extended Coxeter group, defined from a Coxeter group $(W, S)$ and an abelian group $\Pi$ acting by automorphisms on $(W, S)$, is the semi-direct product $\Pi \ltimes W$, denoted $W_e$. The length function and partial order on $W$ extend to $W_e$: $\ell(\pi v) = \ell(v)$, and $\pi v \leq \pi' v'$ if and only if $\pi = \pi'$ and $v \leq v'$, where $\pi, \pi' \in \Pi$.  

The definitions of left and right descent sets, reduced factorization, the \( \tau \)-involution, and definition of the Hecke algebra (2.3) of \( \S 2.2 \) carry over identically. The Hecke algebra elements \( T_\pi \) for \( \pi \in \Pi \) will be denoted simply by \( \pi \); note that these are \( \tau \)-invariant.

Although it is possible to allow parabolic subgroups to be extended Coxeter groups, we define a parabolic subgroup of \( W_e \) to be an ordinary parabolic subgroup of \( W \) to simplify the discussion (this is the only case we will need later in the paper). With this convention, each coset of a parabolic subgroup \( W_{e,J} \) contains a unique element of minimal length.

In the generality of extended Coxeter groups, a \( W_e \)-graph \( \Gamma \) must satisfy \( \pi \gamma \in \Gamma \) for all \( \pi \in \Pi \), \( \gamma \in \Gamma \) in addition to (2.6). The machinery of IC bases carries over without change. Everything we have done so far holds in this setting; the only thing that needs some comment is Theorem 2.2.4. Presumably the proof carries over without change, however it is also easy to deduce this from Theorem 2.2.4 for ordinary Coxeter groups: use the fact that \( \tilde{P}_{x,\delta,\pi v,\gamma} = \tilde{P}_{x,\delta,v,\gamma} \) to deduce that with the definition (2.9) for \( \mu \), \( \mu(\pi x, \delta, \pi v, \gamma) = \mu(x, \delta, v, \gamma) \) \( (x, v \in W; \pi \in \Pi) \); the identity \( \tilde{C}_{\pi v, \gamma} = \pi \tilde{C}_{v, \gamma} \) together with the theorem for ordinary Coxeter groups give it for extended Coxeter groups.

Let \( W, W_n \) be the Weyl groups of type \( A_{n-1}, \tilde{A}_{n-1} \) respectively. Put

\[
K_j = \{ s_0, s_1, \ldots, \hat{s}_j, \ldots, s_{n-1} \}.
\]

Let \( Y \cong \mathbb{Z}^n \), \( Q \cong \mathbb{Z}^{n-1} \) be the weight lattice, root lattice of \( GL_n \). The extended affine
Weyl group $W_e$ is both $Y \rtimes W$ and $\Pi \rtimes W_a$ where $\Pi \cong Y/Q \cong \mathbb{Z}$. For $\lambda \in Y$, let $y^\lambda$ be the corresponding element of $W_e$ and let $y_i = y^{\epsilon_i}$, where $\epsilon_1, \ldots, \epsilon_n$ is the standard basis of $Y$. Also let $\pi$ be the generator of $\Pi$ such that $s_i \pi = \pi s_i - 1$, where subscripts are taken mod $n$. The isomorphism $Y \rtimes W \cong \Pi \rtimes W_a$ is determined by

$$y_i \mapsto s_i \ldots s_1 \pi s_n - 1 \ldots s_i,$$

and the condition that $W \hookrightarrow Y \rtimes W \cong \Pi \rtimes W_a$ identifies $W$ with $W_{aK_0}$ via $s_i \mapsto s_i$, $i \in [n]$.

Another description of $W_e$, due to Lusztig, is as follows. The group $W_e$ can be identified with the group of permutations $w : \mathbb{Z} \to \mathbb{Z}$ satisfying $w(i + n) = w(i) + n$ and $\sum_{i=1}^{n}(w(i) - i) \equiv 0 \mod n$. The identification takes $s_i$ to the permutation transposing $i + kn$ and $i + 1 + kn$ for all $k \in \mathbb{Z}$, and takes $\pi$ to the permutation $k \mapsto k + 1$ for all $k \in \mathbb{Z}$. We can then express an element $w$ of $W_e$ in window notation as the sequence of numbers $w^{-1}(1) \ldots w^{-1}(n)$, also referred to as just the word of $w$. For example, if $n = 4$ and $w = \pi^2 s_2 s_0 s_1$, then the word of $w$ is -3203.

Let $Y^+ = \mathbb{Z}_{\geq 0}^n$ and $W_e^+ = Y^+ \rtimes W$. There is a corresponding subalgebra $\widehat{\mathcal{H}}^+$ of $\widehat{\mathcal{H}}$, equal to both $A\{T_w : w \in W_e^+\}$ and $A\{C'_w : w \in W_e^+\}$ [4]. Let $\Gamma$ be a $W$-graph and put $E = A\Gamma$. The positive, degree $d$ part of $\text{Res}_{\mathcal{H}} \widehat{\mathcal{H}} \otimes_{\mathcal{H}} E$ is

$$(\widehat{\mathcal{H}}^+ \otimes_{\mathcal{H}} E)_d :=$$

$$A\{C_{y^\lambda v, \gamma} : \lambda \in Y^+, |\lambda| = d, v \in W \text{ such that } y^\lambda v \in W_{eK_0}, \gamma \in \Gamma\}.$$ (2.23)

**Proposition 2.4.4.** $(\widehat{\mathcal{H}}^+ \otimes_{\mathcal{H}} E)_d$ is a cellular submodule of $\text{Res}_{\mathcal{H}} \widehat{\mathcal{H}} \otimes_{\mathcal{H}} E$.  

Proof. The $A$-basis above can be rewritten as

$$\{ \pi^d \tilde{C}_{w, \gamma} : \pi^d w \in W^+, w \in W_a, \gamma \in \Gamma \}.$$ 

It is easy to see this is left stable by the action of $\mathcal{H}$, given that $\hat{\mathcal{H}}^+$ is a subalgebra of $\hat{\mathcal{H}}$ containing $\mathcal{H}$.

Let $\Gamma$ be a cell of $\mathcal{H}$ labeled by $T$ and $\hat{E}^1 = \left( \hat{\mathcal{H}}^+ \otimes \mathcal{H} \mathcal{A} \Gamma \right)_1$. We now return to give a combinatorial description of the cells of $\hat{E}^1$. The restriction $\text{Res}_{\mathcal{H}^p \hat{\mathcal{H}} \otimes \mathcal{H} E} \mathcal{H} \otimes \mathcal{H} E$ is not weakly multiplicity-free, so we have to use the description (2.18). In this case, we have found it most convenient to use a hybrid of the description in (2.18) and local labels, which we now describe.

Given $x \in W_e$, define $P(x)$ to be the insertion tableau of the word of $x$. Since $x_{K_0}$ is the permutation of $1, \ldots, n$ with the same relative order as the word of $x$, $P(x_{K_0})$ is obtained from $P(x)$ by replacing the entries with $1, \ldots, n$ and keeping relative order the same.

Let $a_k = s_{k-1} \ldots s_1$ for $k \in \{2, \ldots, n\}$, $a_1 = 1$ be the minimal left coset representatives of $W_{J_n'}$. Then $\hat{E}^1 = A\{ \tilde{C}'_{a_k \pi, w} : k \in [n], P(w) = T \}$. In this case, define the local label of the cell containing $\tilde{C}'_{a_k \pi, w}$ to be $P(a_k \pi w)$. A caveat to this is that if we then form some induced module $\mathcal{H}_1 \otimes_{J_1} \hat{E}^1$, it is good to convert the local labels of $\hat{E}^1$ to be the tableaux $P((a_k \pi w)_{K_0})$ before computing local labels of $\mathcal{H}_1 \otimes_{J_1} \hat{E}^1$ (see Figure 2.2 of §2.7.3).

Combinatorially, the cells of $\hat{E}^1$ may be described as follows. Let $w \in W$ with
\[ P(w) = T \] and define \( Q = Q(w) \). Let \( w^*_{J_{n-1}} \) be the word obtained from \( w \) by deleting its last number (see Example 2.4.6). Then \( w^*_{J_{n-1}} \xrightarrow{\text{RSK}} (T^-, Q_{\leq n-1}) \), where \( T^- \) is obtained from \( T \) by uninserting the square \( Q \setminus Q_{\leq n-1} \); let \( c \) be the number uninserted.

Write \( a_k \pi w \) in window notation, which is \( w^*_{J_{n-1}} \) with a \( c - n \) inserted in the \( k \)-th spot.

Let \( Q^+ \) be the tableau obtained by column-inserting \( k \) into the tableau obtained from \( Q_{\leq n-1} \) by replacing entries with \( \{1, \ldots, k-1, k+1, \ldots, n\} \) and keeping the same relative order. We have \( a_k \pi w \xrightarrow{\text{RSK}} (T^+, Q^+) \), where \( T^+ \) is \( jdt(T^-, Q^+ \setminus Q_{\leq n-1}) \) with the number \( c - n \) added to the top left corner (so that the resulting tableau has a straight-shape). This implies the following result about the cells of \( \hat{E}^1 \).

**Proposition 2.4.5.** The local labels of the cells of \( \hat{E}^1 \) are those tableaux obtained from \( T \) by uninserting some outer corner then performing jeu de taquin to some inner corner, and finally filling in the missing box in the top left with a \( c - n \), where \( c \) is the entry bumped out in the uninsertion.

**Example 2.4.6.** For the element \( (a_3 \pi, 346512) \in W_e K_0 \times W \), the insertion and recording tableaux discussed above are

\[
\begin{align*}
  a_3 \pi w & \quad w^*_{J_{n-1}} & \quad w \\
  34-4651 & 34651 & 346512 \\
\end{align*}
\]

\[
\begin{array}{c|c|c|c}
  & \text{\( P \)} & \text{\( Q \)} & \text{\( P \)} \\
  \hline
  & 34-465 & 346 & 365 \\
  \hline
  & 34-4 & 346 & 365 \\
\end{array}
\]
2.5 Computations of some $C'_w$

Suppose in what follows that $r = n - 1$. Let $b_k := s_k \ldots s_{n-1}$ for $k \in [n - 1]$ and $b_n = 1$ be the elements of $W^J$. It is possible to write down explicitly an element from each cell of $\mathcal{H} \otimes_J \mathcal{H}$ in terms of the canonical basis of $\mathcal{H}$. This is the content of the following theorem, which we include mainly for its application in the next section. It is quite interesting for its own sake, however, given that it does not seem to be known how to write down an element from each cell of $\mathcal{H}$ in terms of the $T$’s.

**Proposition 2.5.1.** Let $\Gamma$ be a $W_J$-graph, and $\gamma \in \Gamma$ satisfying $K := \{s_k, \ldots, s_{n-2}\} \subseteq L(\gamma)$. Then

$$\tilde{C}'_{b_k,\gamma} = \frac{1}{[n-k]!} C'_{b_kw_0K} \boxtimes \gamma = (T_{b_k} + u^{-1}T_{b_{k+1}} + \ldots + u^{k-n}T_{b_n}) \boxtimes \gamma.$$ 

**Proof.** The right hand equality follows from $K \subseteq L(\gamma)$ and the well-known identity

$$C'_{b_kw_0K} = C'_{w_0K \cup s_{n-1}} = (T_{b_k} + u^{-1}T_{b_{k+1}} + \ldots + u^{k-n}T_{b_n}) C'_{w_0K}$$

($w_0K$ is the longest element of $W_K$). Once this is known we have produced an element that is both $\pi$-invariant (being equal to $\frac{1}{[n-k]!} C'_{b_kw_0K} \boxtimes \gamma$) and belongs to $T_{b_k} \boxtimes \gamma + u^{-1} \mathcal{L}$ (being equal to $(T_{b_k} + u^{-1}T_{b_{k+1}} + \ldots + u^{k-n}T_{b_n}) \boxtimes \gamma$).

**Theorem 2.5.2.** Let $\Upsilon$ be the cell of $\mathcal{H} \otimes_J \mathcal{H}$ determined by $\lambda^{(1)}, \mu, P$, where $\lambda^{(1)}, \mu, \text{sh}(P)$ are partitions of $n, n - 1$, and $n$ respectively satisfying $\mu \subseteq \lambda^{(1)}, \text{sh}(P)$.

Then $\Upsilon$ contains an element

$$\tilde{C}'_{b_{k'},w} = (T_{b_{k'}} + u^{-1}T_{b_{k'+1}} + \ldots + u^{-k+1}T_{b_n}) \boxtimes C'_w,$$
where the $k$-th row of $\lambda^{(1)}$ contains the square $\lambda^{(1)}/\mu$, $k' = n + 1 - k$, and $w$ satisfies $\{s_{k'}, \ldots, s_{n-2}\} \subseteq L(w)$.

Proof. To construct a desired $w$, let $Q$ be any tableau of shape $\lambda^{(2)}$ such that $Q < n$ has a $k' - 1 + r$ in the last box of the $r$-th row for $r \in \{1, \ldots, k - 1\}$ (see Example 2.5.3) and $Q \geq n$ is the square $\lambda^{(2)}/\mu$. Define $w$ by $w \xrightarrow{\text{RSK}} (P, Q)$.

Consider the element $(b_{k'}, w) = z \in W \times W$, which is $(b_{k'} w_J, w)$ in stuffed notation. Now $Q(b_{k'} w_J) = P(w^{-1} s_{n-1} \ldots s_{k'}) = Q^*_{\leq n} \xleftarrow{\lambda^{(1)}} k'$, where $Q^*_{\leq n}$ is $Q_{\leq n}$ with 1 added to all numbers $\geq k'$, and $T \xleftarrow{a}$ denotes the row-insertion of $a$ into $T$. By construction of $Q_{\leq n}$, the bumping path of inserting $k'$ into $Q^*_{\leq n}$ consists of the last square in rows $1, \ldots, k$, the last square in the $k$-th row being the newly added square. Therefore, $\tilde{C}'_w$ is contained in $\Upsilon$ because the shape of $Q^*_{\leq n} \xleftarrow{k'}$ is $\lambda^{(1)}$.

Remembering our convention for the word of $w$, the left descent set $L(w)$ can be read off from $Q$: it is the set of $s_i$ such that $i + 1$ occurs in a row below the row containing $i$. In particular, $K := \{s_{k'}, \ldots, s_{n-2}\} \subseteq L(w)$. The theorem follows from Proposition 2.5.1.

Example 2.5.3. If $n = 9$, $k = 4$, $\mu = (3, 2, 2, 1)$, and $P = \begin{array}{cccc} 5 & 6 & 7 & 8 \\ 3 & 4 & 9 & 10 \\ 2 & 1 & 11 & 12 \\ 0 & 0 & 0 & 0 \end{array}$, we could choose $Q = \begin{array}{cccc} 5 & 6 & 7 & 8 \\ 3 & 4 & 9 & 10 \\ 2 & 1 & 11 & 12 \\ 0 & 0 & 0 & 0 \end{array}$ or any tableau with the given numbers in bold. Then,

$$b'_{k'} w_J, w_J, w = 473219865, 47321865, 473219658,$$

and $Q_{\leq n} = \begin{array}{cccc} 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 \\ 2 & 1 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{array}$, $Q(b'_{k'} w_J) = \begin{array}{cccc} 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 \\ 2 & 1 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{array}$. 

2.6 Canonical maps from restricting and inducing

2.6.1. The functor $\mathcal{H} \otimes_J - : \mathcal{H}_J\text{-Mod} \to \mathcal{H}\text{-Mod}$ is left adjoint to $\text{Res}_J : \mathcal{H}\text{-Mod} \to \mathcal{H}_J\text{-Mod}$. Let $\alpha$ (resp. $\beta$) denote the unit (resp. counit) of the adjunction so that $\alpha(F) \in \text{Hom}_{\mathcal{H}_J\text{-Mod}}(F, \text{Res}_J \mathcal{H} \otimes_J F)$ corresponds to $\text{Id}_{\mathcal{H} \otimes_J F}$ (resp. $\beta(E) \in \text{Hom}_{\mathcal{H}\text{-Mod}}(\mathcal{H} \otimes_J \text{Res}_J E, E)$ corresponds to $\text{Id}_{\text{Res}_J E}$). The unit (resp. counit) is a natural transformation from the identity functor on $\mathcal{H}_J\text{-Mod}$ to the functor $\text{Res}_J \mathcal{H} \otimes_J -$ (resp. from the functor $\mathcal{H} \otimes_J \text{Res}_J$ to the identity functor on $\mathcal{H}\text{-Mod}$). We will omit the argument $F$ or $E$ in the notation for the unit and counit when there is no confusion. Explicitly, $\alpha : F \to \text{Res}_J \mathcal{H} \otimes_J F$ is given by $f \mapsto 1 \otimes f$, and $\beta : \mathcal{H} \otimes_J E \to E$ is given by $h \otimes e \mapsto he$. It is clear from these formulas that the unit and counit intertwine the involution $\bar{\gamma}$.

2.6.2. The unit behaves in a simple way on canonical basis elements.

**Proposition 2.6.1.** Let $F = A\Gamma$ be any $\mathcal{H}_J$-module coming from a $W_J$-graph $\Gamma$. The map $\alpha : F \to \text{Res}_J \mathcal{H} \otimes_J F$ takes canonical basis elements to canonical basis elements. Therefore $\text{im}(\alpha)$ is a cellular submodule isomorphic to $A\Gamma$ as a $W_J$-graph.

**Proof.** The elements $\bar{C}_1, \gamma = \alpha(\gamma) (\gamma \in \Gamma)$ are canonical basis elements and are an $A$-basis for the image of $\alpha$. \qed
2.6.3. Again, restrict to the case where $W$ and $\mathcal{H}$ are of type $A_{n-1}$, $S = \{s_1, \ldots, s_{n-1}\}$, and $J = S \setminus s_{n-1}$.

We are not able to give a good description of where the counit $\beta$ takes canonical basis elements in general, but we have a partial result along these lines, assuming the following conjecture.

**Conjecture 2.6.2.** Let $\Lambda$ be the $W_1$-graph on $\mathcal{H}_1 \otimes J_1 \ldots \otimes J_{d-1} \mathcal{H}_d$ with $\mathcal{H}_1$ of type $A$. If $y \leq_{\Lambda} z, y, z \in \Lambda$ and $y, z$ are in cells with local labels of shape $\lambda, \mu$ respectively, then $\lambda < \mu$ in dominance order.

Let $\Gamma$ be a cell of $\mathcal{H}$ and $\tau : A\Gamma \to A\Gamma$ an $\mathcal{H}$-module homomorphism. We want to conclude that $\tau$ is multiplication by some constant $c \in A$. This can be seen, for instance, by tensoring with $\mathbb{C}$ over $A$ using any map $A \to \mathbb{C}$ that does not send $u$ to a root of unity; Schur’s Lemma applies as $\mathcal{H} \otimes_A \mathbb{C} \cong \mathbb{C} \mathcal{S}_n$ and $A\Gamma \otimes_A \mathbb{C}$ is irreducible. Thus $\tau \otimes_A \mathbb{C} = a \text{Id}$, $a \in \mathbb{C}$ for infinitely many specializations of $u$ implies $\tau = c \text{Id}$, $c \in A$.

Let $X_\lambda$ be the two-sided cell of $\mathcal{H}$ consisting of the cells labeled by tableaux of shape $\lambda \vdash n$. Let $\Gamma$ be a cell of $\mathcal{H} \otimes_J X_\lambda$ with local sequence $P_1, P_2$ both of shape $\lambda$. Conjecture 2.6.2 implies $A\Gamma$ is a submodule of $(\mathcal{H} \otimes_J X_\lambda)/X$, where $X$ is the cellular submodule consisting of those cells of dominance order $< \lambda$. By a similar argument to the one above, $\beta(X) = 0$. Therefore the map $\mathcal{H} \otimes_J X_\lambda \xrightarrow{\beta} X_\lambda$ gives rise to a map $A\Gamma \xrightarrow{\beta} X_\lambda$. 
Letting $\Upsilon$ be a cell of $X_\lambda$, we have

$$A \Gamma \xrightarrow{\beta} X_\lambda \xrightarrow{p} A \Upsilon \cong A \Gamma.$$  \hfill (2.25)

The map $p$ is a cellular quotient map by [33, Corollary 1.9] and the rightmost isomorphism of $W$-graphs comes from the fact that any two cells with the same local label are isomorphic as $W$-graphs ($\S$2.3.4). We now can state the main application of Theorem 2.5.2.

**Corollary 2.6.3.** Assuming Conjecture 2.6.2 and with the notation above, if the square $P_1 \setminus (P_1)_{<n}$ lies in the $k$-th row, then the composition of the maps in (2.25) is $[k] \text{Id}$ if $\Upsilon$ is labeled by $P_2$ and 0 otherwise.

**Proof.** By the discussion above, this composition must be $c \text{Id}$ for some $c \in A$. Apply Theorem 2.5.2 with $\lambda^{(1)} = \lambda$, $\mu = \text{sh}((P_1)_{<n})$, $P = P_2$, noting that from the construction of $w$ in the proof, $\{k', \ldots, n-1\} \subseteq L(w)$ in this case. Therefore

$$\beta(\tilde{C}_{b_{k'},w}) = (T_{b_{k'}} + u^{-1}T_{b_{k'+1}} + \ldots + u^{-k+1}T_{b_n}) C'_w = [k]C'_w.$$  \hfill (2.26)

It is tempting to conjecture that $\beta(\tilde{C}_{b_{k'},w})$ is a constant times a canonical basis element of $H$, where $(b_{k'}, w)$ is as constructed in Theorem 2.5.2, but this is false in general. The following counterexample was found using Magma.

**Example 2.6.4.** Let $n = 6$, $k = 2$, $k' = 4$, $w = 521634 \xrightarrow{\text{RSK}} (134,265)$. Then,

$$\tilde{C}_{b_{k'},w} = (T_{b_4} + u^{-1}T_{b_5} + u^{-2}) \otimes C'_{521634} \xrightarrow{\beta} [2]C'_{521643} + [2]C'_{321654}.$$  \hfill (2.27)
The element \((b_{k'}, w) \in W \times W\) is \((421653, 521634)\) in stuffed notation and the cell containing it has local label \(\text{BB} \text{BB} \text{BB}\). The labels of the cells containing \(521643, 321654\) are \(\text{BB} \text{BB} \text{BB}\), respectively.

2.7 Some \(W\)-graph versions of tensoring with the defining representation

Let \(V\) denote the \(n\)-dimensional defining representation of \(S_n\): \(V = \mathbb{Z}\{x_1, \ldots, x_n\}, s_i(x_j) = x_{s_i(j)}\). In this section, we will explore three \(W\)-graph versions of tensoring with \(V\). We then look at \(W\)-graphs corresponding to tensoring twice with \(V\) and show that these decompose into a reduced and non-reduced part. We make a habit of checking what our \(W\)-graph constructions become at \(u = 1\) in order to keep contact with our intuition for this more familiar case.

2.7.1. In what follows, \(E\) denotes an \(\mathcal{H}\)-module or \(\mathbb{Z}S_n\)-module, depending on context. A useful observation, and indeed, what motivated us to study inducing \(W\)-graphs is that \(V \otimes E \cong \mathbb{Z}S_n \otimes_{\mathbb{Z}S_{n-1}} E\) for any \(\mathbb{Z}S_n\)-module \(E\). This is well-known, but the proof is instructive.

**Proposition 2.7.1.** Given a finite group \(G\), a subgroup \(K\), and a \(\mathbb{Z}(G)\)-module \(E\),
there is a \((\mathbb{Z}G\text{-module})\) isomorphism, natural in \(E\)

\[
\mathbb{Z}G \otimes_{\mathbb{Z}K} E \cong (\mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbb{Z}) \otimes_{\mathbb{Z}} E, \tag{2.28}
\]

\(g \boxtimes e \rightarrow (g \boxtimes 1) \otimes ge,\)

where \(\mathbb{Z}\) denotes the trivial representation of \(K\).

**Proof.** The expressions \(gk \boxtimes k^{-1}e\) and \(g \boxtimes e\) \((k \in K)\) are sent to the same element so this map is well-defined. Similarly, its inverse \((g \boxtimes 1) \otimes e \mapsto g \boxtimes g^{-1}e\) is well-defined. These maps clearly intertwine the action of \(G\). \(\square\)

Maintain the notation \(W = S_n, J_{n-1} = \{s_1, \ldots, s_{n-2}\}, J'_{n-1} = \{s_2, \ldots, s_{n-1}\}\) of the previous sections. Recall that \(b_k = s_k \ldots s_{n-1}\) for \(k \in [n-1]\), \(b_n = 1\) are the minimal left coset representatives of \(W_{J_{n-1}}\), and \(a_k = s_{k-1} \ldots s_1\) for \(k \in \{2, \ldots, n\}\), \(a_1 = 1\) the minimal left coset representatives of \(W'_{J'_{n-1}}\).

**Corollary 2.7.2.** For the inclusions \(S_{n-1} = W_{J'_{n-1}} \hookrightarrow W = S_n\) and \(S_{n-1} = W_{J_{n-1}} \hookrightarrow W = S_n\), we have \(\mathbb{Z}S_n \otimes_{\mathbb{Z}S_{n-1}} E \cong V \otimes_{\mathbb{Z}} E\) for any \(\mathbb{Z}S_n\)-module \(E\).

**Proof.** Put \(G = S_n\). If \(K = W_{J'_{n-1}}\), then \(\mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbb{Z} \cong V\) by \(a_i \boxtimes 1 \mapsto x_i\). If \(K = W_{J_{n-1}}\), then \(\mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbb{Z} \cong V\) by \(b_i \boxtimes 1 \mapsto x_i\). \(\square\)

The Hecke algebra is not a Hopf algebra in any natural way, so it is not clear what a Hecke algebra analogue of \(F \otimes E\) should be for \(F, E \in \mathbb{Z}S_n\)-modules. If \(F = V\), however, then \(\mathcal{H} \otimes_J E\) is a \(u\)-analogue of \(\mathbb{Z}S_n \otimes_{\mathbb{Z}S_{n-1}} E \cong V \otimes E\), where \(r\) is either \(n-1\) or \(1\) (and \(J = J_r \cup J'_{n-r}\)). These choices for \(r\) give isomorphic representations at \(u = 1\), but do not give isomorphic \(W\)-graphs in general.
Example 2.7.3. Let $e^+$ be the trivial representation of $\mathcal{H}$. Then compare $\mathcal{H} \otimes J_{n-1} e^+$ (first row) with $\mathcal{H} \otimes J_n e^+$ (second row) for $n = 4$:

\[
\begin{array}{cccccc}
\tilde{C}_{a_4,e^+}^{1,2,3} & \tilde{C}_{a_3,e^+}^{1,2} & \tilde{C}_{a_2,e^+}^{1,3} & \tilde{C}_{a_1,e^+}^{2,3} \\
\tilde{C}_{b_4,e^+}^{1,2,3} & \tilde{C}_{b_3,e^+}^{2,3} & \tilde{C}_{b_2,e^+}^{1,3} & \tilde{C}_{b_1,e^+}^{1,2} \\
\end{array}
\]

Evidently, these are not isomorphic as $W$-graphs.

In this paper $W$-graphs are drawn with the following conventions: vertices are labeled by canonical basis elements and descent sets appear as superscripts; an edge with no arrow indicates that $\mu = 1$ and neither descent set contains the other; an edge with an arrow indicates that $\mu = 1$ and the descent set of the arrow head strictly contains that of the arrow tail; no edge indicates that $\mu = 0$ or the descent sets are the same.

For the remainder of this paper, let $J = J'_{n-1}$ ($r = 1$) since this is preferable for comparing $\mathcal{H} \otimes J E$ with $(\hat{\mathcal{H}}^+ \otimes \mathcal{H})_1$ (see §2.7.2, below). See §2.4.4 for how to go back and forth between the $J'_{n-1}$ and $J_{n-1}$ pictures.

2.7.2. There is another $u$-analogue of tensoring with $V$ that comes from the extended affine Hecke algebra $\hat{\mathcal{H}}$. See §2.4.5 for a brief introduction to this algebra.

The module $\hat{\mathcal{H}}^+ \otimes \mathcal{H} E$ is a $u$-analogue of $ZW^+_{e} \otimesZW^+_{e \mathcal{K}_0} E$, which, together with the following proposition, shows that $(\hat{\mathcal{H}}^+ \otimes \mathcal{H})_1$ is a $u$-analogue of $V \otimes E$. 
Proposition 2.7.4. The correspondence

\[
\text{Res}_{\mathcal{Z}S_n} Z W_e^+ \otimes_{ZW_{k_0}^+} E \cong \mathbb{Z}[x_1, \ldots, x_n] \otimes_{\mathbb{Z}} E,
\]

is a degree-preserving isomorphism of $\mathbb{Z}S_n$-modules, natural in $E$, where $S_n$ acts on the polynomial ring by permuting the variables.

Proof. Recalling that $W_e = Y \rtimes W$ with $W$ acting on $Y$ by permuting the coordinates, we have $s_i(y^\lambda \boxtimes e) = s_i y^\lambda s_i \boxtimes s_i e = y^{s_i(\lambda)} \boxtimes s_i e$ and $s_i(x^\lambda \otimes e) = x^{s_i(\lambda)} \otimes s_i e$.

Example 2.7.5. To compare with the $W$-graphs in Example 2.7.3, here is the $W$-graph on $(\hat{\mathcal{H}}^+ \otimes_{\mathcal{H}} e^+)_1$. In this case it is isomorphic to the $W$-graph on $\mathcal{H} \otimes J_{n-1} e^+$, but this is not true in general as can be seen by comparing Figures 2.1 and 2.2.

The general relationship between $(\hat{\mathcal{H}}^+ \otimes_{\mathcal{H}} E)_1$ and $\mathcal{H} \otimes J_{n-1} E$ can be explained as a special case of a $W$-graph version of Mackey’s formula due to Howlett and Yin [18, §5], which we now recall.

Let $\Gamma$ be a $W_I$-graph, and $K, I \subseteq S$. Put $F = A\Gamma$. Let $^{KW_I}$ be the set of minimal double coset representatives \{d : d of minimal length in $W_K d W_I$\}. For each $d \in ^{KW_I}$, the $d$-subgraph of (the $W_K$-graph on) $\text{Res}_{K, \mathcal{H}} \otimes I F$ is \{${\mathcal{C}}_{w,\gamma}^d : w \in W_K^L, L = K \cap dId^{-1}, \gamma \in \Gamma$\}.

For any $d \in ^{KW_I}$, let $L = K \cap dId^{-1}$. Then $d^{-1}Ld = d^{-1}Kd \cap I \subseteq I$ so the restriction $\text{Res}_{d^{-1}Ld} F$ makes sense. This $W_{d^{-1}Ld}$-graph naturally gives rise to a $W_L$-
graph, denoted $d\Gamma$, obtained by conjugating descent sets by $d$. Explicitly, the descent set of a vertex $d\gamma$ of $d\Gamma$ is

$$L(d\gamma) = \{dsd^{-1} : s \in L(\gamma) \subseteq I \text{ and } dsd^{-1} \in K\} \subseteq L.$$  \hspace{1cm} (2.30)

The edge weights of $d\Gamma$ are the same as those of $\Gamma$: $\mu(d\delta, d\gamma) = \mu(\delta, \gamma)$ for all $\delta, \gamma \in \Gamma$.

**Theorem 2.7.6** (Howlett, Yin [18]). The $d$-subgraphs of $\text{Res}_K \mathcal{H} \otimes_I F$ partition its canonical basis. Each $d$-subgraph is a union of cells and is isomorphic to $\mathcal{H}_K \otimes_L dF$ ($L = K \cap dId^{-1}$) as a $W_K$-graph via the correspondence $\tilde{C}_{w,d,\gamma} \leftrightarrow \tilde{C}'_{w,d,\gamma}$, $w \in W^L_K$.

**Remark 2.7.7.** It is probably the case that each $d$-subgraph is a cellular subquotient rather than just a union of cells, however this is not proven in [18]. This issue does not come up, however, because in the applications in this paper we can easily show that the $d$-subgraph is a cellular subquotient and sometimes the stronger statement that it is a cellular quotient or submodule.

In the present application, put $K = I = \{s_1, \ldots, s_{n-1}\}$. Then $\pi \in ^KW^I$ and the $\pi$-subgraph of $\text{Res}_\mathcal{H} \hat{\mathcal{H}} \otimes \mathcal{H} E$ is $\{C'_{ak,\pi,\gamma} : k \in [n], \gamma \in \Gamma\}$ since $K \cap \pi I \pi^{-1} = J'_{n-1}$. This is isomorphic as a $W$-graph to $\mathcal{H} \otimes_{J'_{n-1}} \pi E$. The $W_{J'_{n-1}}$-graph $\pi E$ is just $\text{Res}_{J'_{n-1}} E$, with each element of its descent sets shifted up by one. We have proved the following.

**Proposition 2.7.8.** The $W$-graphs $(\hat{\mathcal{H}}^+ \otimes \mathcal{H} E)_1$ and $\mathcal{H} \otimes_{J'_{n-1}} \pi E$ are isomorphic.

**Remark 2.7.9.** Though this suggests that the $W$-graph versions of $V \otimes E$, $(\hat{\mathcal{H}}^+ \otimes \mathcal{H} E)_1$ and $\mathcal{H} \otimes_{J'_{n-1}} E$, behave in essentially the same way, some care must be taken.
At \( u = 1 \), \( \mathcal{H} \otimes_{J_{n-1}} \pi E \) is not isomorphic to \( V \otimes \pi E \) using Proposition 2.7.1 since \( \pi E \) is only a \( \mathbb{Z}W \)-module, not a \( \mathbb{Z}W \)-module. Thus \( (\mathcal{H}^+ \otimes \mathcal{H} E)|_{u=1} \) and \( (\mathcal{H} \otimes_{J_{n-1}} E)|_{u=1} \) are only isomorphic to \( V \otimes E \) by the rather different looking routes Proposition 2.7.4 and Corollary 2.7.2.

2.7.3. Let \( \Gamma \) be a \( W \)-graph, and put \( E = A\Gamma, F = \text{Res}_J A\Gamma, \tilde{E}^2 := \mathcal{H} \otimes_J \mathcal{H} \otimes_J A\Gamma \).

We will show that \( \tilde{E}^2 \) decomposes into what we call a reduced and non-reduced part. Towards this end, consider the exact sequence

\[
0 \rightarrow F \xrightarrow{\alpha} \text{Res}_J \mathcal{H} \otimes_J F \xrightarrow{\tau} \mathcal{H}_J \otimes_{J_{n-2}} \text{Res}_{J_{n-2}} F \xrightarrow{\gamma} 0.
\]

By Proposition 2.6.1 the image of \( \alpha \) is a cellular submodule. The map \( \tau \) induces an isomorphism of \( W_J \)-graphs \( \text{Res}_J \mathcal{H} \otimes_J F/\text{im}(\alpha) \cong \mathcal{H}_J \otimes_{J_{n-2}} \text{Res}_{J_{n-2}} F \); given that the sequence is exact, this is equivalent to taking canonical basis elements to canonical basis elements or to 0. That \( \tau \) is an isomorphism can be seen directly by observing that it takes standard basis elements of \( \mathcal{H} \otimes_J F \) to standard basis elements of \( \mathcal{H}_J \otimes_{J_{n-2}} F \) or to 0, takes the lattice \( A^- \mathcal{H} \otimes_J \Gamma \) to the lattice \( A^- \mathcal{H}_J \otimes_{J_{n-2}} \Gamma \), and intertwines the involutions \( \gamma \).

This decomposition also comes from another application of the \( W \)-graph version of Mackey’s formula (Theorem 2.7.6). For this application, put \( K = I = J \) (\( = \{s_2, \ldots, s_{n-1}\} \)). Then \( K^W J = \{1, s_1\} \). The 1-subgraph of \( \text{Res}_J \mathcal{H} \otimes_J F \) is \( \tilde{C}_1 : \gamma \in \Gamma \) and the \( s_1 \)-subgraph is \( \tilde{C}_1 s_1 : \gamma \in \Gamma \). These are
Figure 2.1: The \( W \)-graph on \( H \otimes H'_{n-1} \) and the graph \( G \) of §2.3.4. The vertices of the tree \( G \) are marked by local labels. Each cell in the \( W \)-graph corresponds to the path from a leaf to the root that is its local sequence.
Figure 2.2: The $W$-graph on $(\mathcal{H}^+ \otimes \mathcal{H}_+ (\mathcal{H}^+ \otimes \mathcal{H}_+ e^+)_1)_1$ and the graph $G$ of §2.3.4, with the labeling conventions of §2.4.5.
isomorphic as $W_J$-graphs to $\mathcal{H}_J \otimes J F = F$ and $\mathcal{H}_J \otimes J_{n-2} \text{Res}_{J_{n-2}} F$ respectively (since we have $d^{-1} L d = L$ for all $d$, $d F$ and $F$ are identical).

Next, tensor (2.31) with $\mathcal{H}$ to obtain

$$0 \rightarrow \mathcal{H} \otimes J A \Gamma \rightarrow \mathcal{H} \otimes J \mathcal{H} \otimes J A \Gamma \rightarrow \mathcal{H} \otimes J_{n-2} A \Gamma \rightarrow 0. \quad (2.32)$$

Put $\widetilde{F}^2 = \mathcal{H} \otimes J_{n-2} A \Gamma$. The quotient $\widetilde{F}^2$ (resp. the submodule $\mathcal{H} \otimes J A \Gamma$) is the reduced (resp. non-reduced) part of $\widetilde{E}^2$.

**Proposition 2.7.10.** The submodule and quotient of $\widetilde{E}^2$ given by (2.32) are cellular and the maps in (2.32) take canonical basis elements to canonical basis elements or to 0.

*Proof.* This follows from the application of Theorem 2.7.6 described above and Proposition 2.2.6.

**Example 2.7.11.** The non-reduced part of $\widetilde{E}^2$ for $E = e^+$ is the bottom row of the $W$-graph in Figure 2.1. The cells comprising it are labeled “non-red” below the tree.

**2.7.4.** Let us determine what the decomposition of $\widetilde{E}^2$ into reduced and non-reduced parts becomes at $u = 1$.

**Proposition 2.7.12.** At $u = 1$, (2.32) becomes

$$0 \rightarrow V \otimes E \rightarrow V \otimes V \otimes E \rightarrow T^2_{\text{red}} V \otimes E \rightarrow 0. \quad (2.33)$$

where $i \neq j$ and $T^2_{\text{red}} V := \mathbb{Z}\{x_i \otimes x_j : i \neq j, i, j \in [n] \} \subseteq V \otimes V$. 


To see this, first define \( a_{k,l} = s_{k-1} \ldots s_1 s_{l-1} \ldots s_2 \) for \( k \in [n], \ l \in \{2, \ldots, n\} \); then

\[
a_{k,l} \cdot s_1 = a_{l,k+1} \text{ if } k < l,
\]

\[
W^{I_{n-2}} = \{a_{k,l} : k \in [n], \ l \in \{2, \ldots, n\}\},
\]

\[
W^{S\backslash s_2 s_1} = \{a_{k,l} : k \geq l > 1\}, \text{ and}
\]

\[
W^{S\backslash s_2} = \{a_{k,l} : k < l\}.
\]

Apply Corollary 2.7.2 twice to obtain

\[
\tilde{E}^2_{|u=1} \cong \mathbb{Z}S_n \otimes \mathbb{Z}S_{n-1} V \otimes E \cong V \otimes V \otimes E
\]

\[
a_k \circledast a_l \circledast \gamma \mapsto a_k \circledast (x_l \otimes a_l(\gamma)) \mapsto \begin{cases} x_k \otimes x_k \otimes a_k a_l(\gamma) & \text{if } l = 1, \\ x_k \otimes x_l \otimes a_k a_l(\gamma) & \text{if } k < l, \\ x_k \otimes x_{l-1} \otimes a_k a_l(\gamma) & \text{if } k \geq l > 1. \end{cases}
\]

\[
(2.35)
\]

**Proof of Proposition 2.7.12.** The interesting part of the calculation is the following diagram

\[
\begin{array}{ccc}
\tilde{E}^2_{|u=1} & \cong & \tilde{E}^2_{|u=1} \\
\cong & & \cong \\
V \otimes V \otimes E & \mapsto & T^2_{\text{red}} V \otimes E
\end{array}
\]

\[
a_k \circledast a_l \circledast \gamma \mapsto a_k \circledast (x_l \otimes a_l(\gamma)) \mapsto x_k \otimes a_k a_l(\gamma) \mapsto x_k \otimes a_k a_k a_l s_1(\gamma)
\]

\[
(2.36)
\]

where \( k \in [n], \ l \in \{2, \ldots, n\} \).

There is a slightly tricky point here: the left-hand isomorphism of (2.36) comes from (2.35), but the right-hand isomorphism does not come from a similar application of Proposition 2.7.1. However, Proposition 2.7.1 also holds with the isomorphism \( g \circledast e \mapsto (g \circledast 1) \otimes gce \) replacing (2.28), where \( c \in G \) commutes with all of \( K \). In this
case we must choose $c = s_1$ (which commutes with $K = J_{n-2}'$) to make the diagram (2.36) commute.

2.7.5. There is a similar decomposition of $\hat{E}^2 := (\hat{\mathcal{H}}^+ \otimes \mathcal{H} (\hat{\mathcal{H}}^+ \otimes \mathcal{H} A\Gamma)_1)_1$ into a reduced and non-reduced part. Two applications of Proposition 2.7.8 yield $\hat{E}^2 = \mathcal{H} \otimes J \pi(\mathcal{H} \otimes J \pi E)$.

First, let us apply Theorem 2.7.6 to $\text{Res}_{J_n-1} \mathcal{H} \otimes J'_{n-1} \pi E$ analogously to the application in the previous subsection. In this case $I = J'_{n-1}$, $K = J_{n-1}$, and therefore $K W I = \{1, a_n\}$.

The $1$-subgraph is $\{\tilde{C}'_{a_n, \pi \gamma} : k < n, \pi \gamma \in \pi \Gamma\}$ and spans a cellular submodule of $\text{Res}_{J_n-1} \mathcal{H} \otimes J'_{n-1} \pi E$. This can be seen, for instance, by applying Proposition 2.2.2 with the order $\prec$ of §2.2.4 to obtain

$$A\{\tilde{C}'_{a_k, \pi \gamma} : k < n, \pi \gamma \in \pi \Gamma\} = A\{\tilde{T}_{a_k, \pi \gamma} : k < n, \pi \gamma \in \pi \Gamma\};$$

(2.37)

it is clear that this $A$-span of $\tilde{T}$’s is left stable under the action of $\mathcal{H}_{J_{n-1}}$. Now this submodule is isomorphic to $\mathcal{H}_{J_{n-1}} \otimes J_{n-1} \setminus s_1 \pi E$ (as a $W_{J_{n-1}}$-graph) by Theorem 2.7.6.

The $a_n$-subgraph is $\{\tilde{C}'_{a_n, \pi \gamma} : \pi \gamma \in \pi \Gamma\}$ and spans a cellular quotient since the only other $d$-subgraph spans a submodule. This quotient is isomorphic to $a_n \pi E$ as a $W_{J_{n-1}}$-graph. Moreover, $a_n \pi E$ is exactly $\text{Res}_{J_{n-1}} E$ as $L = K \cap a_n I a_n^{-1} = K = J_{n-1}$.

The following exact sequence summarizes what we have so far.

$$0 \longrightarrow \mathcal{H}_{J_{n-1}} \otimes J_{n-1} \setminus s_1 \pi E \longrightarrow \text{Res}_{J_{n-1}} \mathcal{H} \otimes J'_{n-1} \pi E \longrightarrow \text{Res}_{J_{n-1}} E \longrightarrow 0.$$ (2.38)
Applying $\pi$ to the $W_{J_n-1}$-graphs in this sequence to obtain $W_{J'_n-1}$-graphs (as explained before Theorem 2.7.6) and then tensoring with $\mathcal{H}$ yields

$$0 \to \mathcal{H} \otimes J_{n-1} \pi (\mathcal{H} \otimes J_{n-1})_{\pi E} \to \mathcal{H} \otimes J_{n-1} \pi (\mathcal{H} \otimes J_{n-1})_{\pi E} \to \mathcal{H} \otimes J_{n-1} \pi E \to 0$$

$$0 \to \mathcal{H} \otimes J_{n-2} \pi^2 E \to (\mathcal{H}^+ \otimes \mathcal{H}^+)_{\pi E} \to 0,$$

(2.39)

where $\text{Res}_{J_{n-2}E} \pi^2 E$ is the $W_{J'_n-2}$-graph obtained from $\text{Res}_{J_{n-2}} E$ by increasing descent set indices by 2. The leftmost isomorphism comes from the isomorphism of Coxeter group pairs $(W_{J_{n-1}\setminus s_1}, W_{J_{n-1}}) \cong (W_{J'_n\setminus s_1}, W_{J'_n})$ given by conjugation by $\pi$. The other two isomorphisms are applications of Proposition 2.7.8.

The submodule $\hat{F}^2 := \mathcal{H} \otimes J_{n-2} \text{Res}_{J_{n-2}} \pi^2 E$ (resp. the quotient $(\mathcal{H}^+ \otimes \mathcal{H}^+)_{\pi E}$) is the reduced (resp. non-reduced) part of $\hat{E}^2$. We have proved the following analogue of Proposition 2.7.10.

**Proposition 2.7.13.** The submodule and quotient of $\hat{E}^2$ given by (2.39) are cellular and the maps in (2.39) take canonical basis elements to canonical basis elements or to 0.

**Example 2.7.14.** The non-reduced part of $\hat{E}^2$ for $E = e^+$ is the top row of the $W$-graph in Figure 2.2. The cells comprising it are labeled “non-red” below the tree.

At $u = 1$, the decomposition (2.39) becomes

$$\hat{E}^2|_{u=1} \cong V \otimes V \otimes E \cong T^2_{\text{red}} V \otimes E \oplus V \otimes E$$

(2.40)
(with the left-hand isomorphism from Proposition 2.7.4), but the computation is different from that of §2.7.4. We omit the details.

2.8 Decomposing $V \otimes V \otimes E$ and the functor $Z^2$

In this section we study a $W$-graph version of the decomposition $V \otimes V \otimes E \cong S^2V \otimes E \oplus \Lambda^2V \otimes E$. Along the way, we come across a mysterious object, the sym-wedge functor $Z^2$. At $u = 1$, this is some kind of mixture of the functors $S^2_{\text{red}}V \otimes -$ and $\Lambda^2V \otimes -$, where $S^2_{\text{red}}V = \{x_i \otimes x_j + x_j \otimes x_i : i \neq j\} \subseteq S^2V \subseteq V \otimes V$.

2.8.1. Let $\Lambda$ be the $W_{S\backslash s_2}$-graph on $H_{S\backslash s_2} \otimes \Gamma$ obtained from Theorem 2.2.3. For any $W$-graph $\Upsilon$ and $s \in S$, define $\Upsilon^- = \{\gamma \in \Upsilon : s \in L(\gamma)\}$ and $\Upsilon^+ = \{\gamma \in \Upsilon : s \notin L(\gamma)\}$. In this case, $\Lambda^-_{s_1} = \{\tilde{C}_{s_1,\gamma} : \gamma \in \Gamma\}$, and $\Lambda^+_{s_1} = \{\tilde{C}'_{s_1,\gamma} : \gamma \in \Gamma\}$ as $L(w, \gamma) = L(w) \cup L(\gamma)$. Also note that $\tilde{C}_{1,\gamma} = C_1 \otimes \gamma$ and $\tilde{C}'_{s_1,\gamma} = C'_{s_1} \otimes \gamma$.

It is clear that in the case $\Gamma = \Gamma_{W_{j_{n-2}}}$, $\Lambda_{s_1}^-$ is a cellular submodule of $\Lambda \Lambda$. This is actually true in full generality as we will see shortly (Lemma 2.8.3). Now define the sym-wedge functor $Z^2$ by $Z^2 \Lambda = H \otimes_{S\backslash s_2} \Lambda \Lambda_{s_1}^-$, with a $W$-graph structure coming from Theorem 2.2.3.

**Theorem 2.8.1.** The $H$-module $Z^2 \Lambda$ is a cellular submodule of $\tilde{F}^2 := H \otimes_{J_{n-2}} \Lambda$.

**Proof.** By Lemma 2.8.3 (below), $\Lambda \Lambda_{s_1}^-$ is a cellular submodule of $\Lambda \Lambda$. Proposition 2.2.6 shows that $H \otimes_{S\backslash s_2} \Lambda \Lambda_{s_1}^-$ is a cellular submodule of $H \otimes_{S\backslash s_2} \Lambda \Lambda$, and $H \otimes_{J_{n-2}} \Gamma$ give the same $W$-graph structure on $\tilde{F}^2$ by Proposition 2.3.7. \qed
The sym-wedge functor was discovered by looking at examples. The preceding proof sort of explains why such a cellular submodule should exist, but it is still somewhat surprising it does not agree with $\text{S}^2_{\text{red}}V \otimes -$ at $u = 1$. We will determine what $Z^2 A\Gamma$ is at $u = 1$ in §2.8.2 and address its relation with $\text{S}^2_{\text{red}}V \otimes -$ and $\Lambda^2 V \otimes -$ in §2.8.5. It will be useful for us later to know the following additional structure possessed by $Z^2$.

**Proposition 2.8.2.** The rule $E \mapsto Z^2 E$ is a functor $Z^2 : \mathcal{H} \text{-Mod} \to \mathcal{H} \text{-Mod}$. Moreover, if $E = A\Gamma$ for some $W$-graph $\Gamma$, then taking cellular submodules or quotients of $E$ gives rise to cellular submodules and quotients of $Z^2 E$ in the same way induction does in Proposition 2.2.6.

**Proof.** As explained above, the proposed functor $Z^2$ is the composition

$$
\mathcal{H} \text{-Mod} \xrightarrow{\text{Res}_{n-2}} \mathcal{H}_{n-2} \text{-Mod} \xrightarrow{\mathcal{H}_{S^2} \otimes -} \mathcal{H}_{S^2} \text{-Mod} \xrightarrow{\zeta} \mathcal{H}_{S^2} \text{-Mod} \xrightarrow{\mathcal{H} \otimes -} \mathcal{H} \text{-Mod},
$$

where $\zeta(F)$ is the kernel of $F \xrightarrow{m_{C_{s_1}^-[2]}} F$ and $m_h$ is left multiplication by $h$ (by Lemma 2.8.3, $m_{C_{s_1}^-[2]}$ is an $\mathcal{H}_{S^2} \text{-mod}$ homomorphism and its kernel equals $A\Lambda_{s_1}^-$ in the case $F = A\Lambda$). Thus it suffices to show that $\zeta$ is a functor and respects cellular subquotients as claimed.

Let $F$ and $F^*$ be $W_{S^2}$-graphs and $f : F \to F^*$ be an $\mathcal{H}_{S^2} \text{-mod}$ homomorphism. As $f \circ m_{C_{s_1}^-[2]} = m_{C_{s_1}^-[2]} \circ f$, $f(\ker(m_{C_{s_1}^-[2]})) \subseteq \ker(m_{C_{s_1}^-[2]})$. Thus $f \mapsto f|_{\ker(m_{C_{s_1}^-[2]})}$ defines $\zeta$ on morphisms and this certainly respects composition of morphisms.
For the second statement, just observe that if $\Lambda$ is a $W_{S\setminus S_2}$-graph and $\Upsilon \subseteq \Lambda$ spans a cellular submodule, then $\zeta(\Lambda \Upsilon)$ is the intersection of the cellular submodules $\Lambda \Upsilon$ and $\Lambda \Lambda_{S_1}^-$, which is a cellular submodule of $\zeta(\Lambda \Lambda) = \Lambda \Lambda_{S_1}^-$. Similarly, if $\Upsilon^*$ is the vertex set $\Lambda \setminus \Upsilon$, then $\zeta(\Lambda \Upsilon^*) = \Lambda(\Upsilon^* \cap \Lambda_{S_1}^-)$, which is the cellular quotient $\Lambda \Lambda_{S_1}^-/\zeta(\Lambda \Upsilon)$ of $\Lambda \Lambda_{S_1}^-$. 

**Lemma 2.8.3.** For any $W$-graph $\Lambda$ and $s \in S$, the kernel of the map of abelian groups $m_{C'_s-[2]} : A\Lambda \to A\Lambda$ (where $m_h$ is left multiplication by $h$) is equal to $\Lambda \Lambda_s^-$. If $s$ commutes with $t$ for all $t \in S$ and $F$ is any $H$-module, then $m_{C'_s-[2]} : F \to F$ is an $H$-module homomorphism. Therefore, if $\Lambda$ is a $W_{S\setminus S_2}$-graph, then $\Lambda \Lambda_{S_1}^-$ is a cellular submodule of $A\Lambda$.

**Proof.** Certainly any $h \in \Lambda \Lambda_s^-$ is in the kernel of $m_{C'_s-[2]}$. To see that the kernel is no bigger, let $h = \sum_{\lambda \in \Lambda} c_{\lambda} \lambda$ ($c_{\lambda} \in A$) be an element of $A\Lambda$ satisfying $(C'_s-[2])h = 0$. We may assume that $c_{\lambda} = 0$ for $\lambda \in \Lambda_{S_1}^-$. Also, by multiplying the $c$’s by some power of $u$, we may assume that $c_{\lambda} \in A^-$ for all $\lambda$ and $c_{\lambda} \notin u^{-1}A^-\Lambda$ for at least one $\lambda$. Then computing mod $A^-\Lambda$, we have

$$0 = \sum_{\lambda \in \Lambda} c_{\lambda} \left( \sum_{\{\delta, s \in L(\delta)\}} \mu(\delta, \lambda)\delta \right) - [2] \sum_{\lambda \in \Lambda} c_{\lambda} \lambda \equiv -u \sum_{\lambda \in \Lambda} c_{\lambda} \lambda. \quad (2.42)$$

Therefore $c_{\lambda} \in u^{-1}A^-\Lambda$ for all $\lambda$, contradicting the earlier assumption.

The second statement is a special case of the fact that $m_h$ is an $H$-module homomorphism whenever $h$ is in the center of $H$. 

\[ \square \]
2.8.2. To better understand the functor $Z^2$, let us determine what it becomes at $u = 1$.

**Proposition 2.8.4.** The image of $Z^2 E|_{u=1}$ under the isomorphism $\tilde{F}^2|_{u=1} \simeq T^2_{\text{red}} V \otimes E$ of (2.36) is $S^2_{\text{red}} V \otimes E$ (resp. $\Lambda^2 V \otimes E$) if $\text{Res}_{W_{\{s_1\}}} E$ is a sum of copies of the trivial (resp. sign) representation.

**Proof.** Under the isomorphism $\tilde{F}^2|_{u=1} \simeq T^2_{\text{red}} V \otimes E$, the standard basis for $Z^2 E$ coming from realizing it as $\mathcal{H} \otimes_{S_{\{s_2\}} A \Lambda^+_{s_1}}$ (see the discussion before Theorem 2.8.1) satisfies

$$\left( T_{ak,l} \otimes_{S_{\{s_2\}}} C'_{s_1} \otimes_{J_{n-2}'} \gamma \right)|_{u=1} = (a_{k,l} + a_{l,k+1}) \otimes \gamma \leftarrow x_k \otimes x_l \otimes a_{k,l}s_1 \gamma + x_l \otimes x_k \otimes a_{k,l} \gamma \quad (k < l), \quad (2.43)$$

where (2.34) has been used freely. Therefore if $s_1$ acts trivially on $E$, then the rightmost expression in (2.43) becomes $(x_k \otimes x_l + x_l \otimes x_k) \otimes a_{k,l} \gamma$. If $s_1$ acts by -1 on $E$, then it becomes $(-x_k \otimes x_l + x_l \otimes x_k) \otimes a_{k,l} \gamma$. The proposition then follows, as $\mathbb{Z}\{a_{k,l} \gamma : \gamma \in \Gamma\} = E|_{u=1}$. \qed

2.8.3. A correct $W$-graph version of tensoring $S^2_{\text{red}} V$ with $E$ is $\mathcal{H} \otimes_{S_{\{s_2\}}} E$, and the projection $T^2_{\text{red}} V \otimes E \to S^2_{\text{red}} V \otimes E$ corresponds to

$$\tilde{F}^2 = \mathcal{H} \otimes_{J_{n-2}'} E = \mathcal{H} \otimes_{S_{\{s_2\}}} \mathcal{H}S_{\{s_2\}} \otimes_{J_{n-2}'} E \xrightarrow{\tilde{\beta}(E)} \mathcal{H} \otimes_{S_{\{s_2\}}} E, \quad (2.44)$$

where $\tilde{\beta}(E) = \mathcal{H} \otimes_{S_{\{s_2\}}} \beta(E)$. This is justified by the following calculation at $u = 1$. 
Proposition 2.8.5. The module $\mathcal{H} \otimes_{S \setminus s_2} E$ is a $u$-analogue of $S_{\text{red}}^2 V \otimes E$ (via the right vertical map of the following diagram, to be defined) in a way so that the diagram commutes.

\[
\begin{array}{c}
(\mathcal{H} \otimes j'_{n-2} E)_{u=1} \xrightarrow{\beta(E)} (\mathcal{H} \otimes S \setminus s_2 E)_{u=1} \\
\cong \downarrow \cong \\
T^2_{\text{red}} V \otimes E \rightarrow S^2_{\text{red}} V \otimes E
\end{array}
\]

(2.45)

Proof. Here we will think of $S_{\text{red}}^2 V$ as the subspace $\mathbb{Z}\{x_k x_l : k \neq l\}$ of $(\mathbb{Z}[x_1, \ldots, x_n])_2$, and the map $T^2_{\text{red}} V \otimes E \rightarrow S^2_{\text{red}} V \otimes E$ as the one sending $x_k \otimes x_l$ to $x_k x_l$. The right vertical map comes from an application of the modified Proposition 2.7.1 (in which $g \boxtimes e \mapsto (g \boxtimes 1) \otimes gce$ replaces (2.28), where $c \in G$ commutes with all of $K$). In this application, use $G = W, K = W S \setminus s_2, c = s_1$. We have $\mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbb{Z} \cong S^2_{\text{red}} V$ by $a_{k,l} \boxtimes 1 \mapsto x_k x_l$ for $k < l$. It is straightforward to check that this is a $\mathbb{Z}G$-module homomorphism; the most interesting case is $s_k a_{k,k+1} \boxtimes 1 = a_{k+1,k+1} \boxtimes 1 = a_{k,k+1} s_1 \boxtimes 1 = a_{k,k+1} \boxtimes 1$, which matches $s_k(x_k x_{k+1}) = x_k x_{k+1}$. It can be checked directly on the basis $\{a_{k,l} \boxtimes \gamma : k \in [n], l \in \{2, \ldots, n\}, \gamma \in \Gamma\}$ of $(\mathcal{H} \otimes j'_{n-2} E)_{u=1}$ that the diagram commutes. \hfill \Box

2.8.4. It is immediate from Proposition 2.7.4 that the right-hand vertical map in the diagram below is a $u$-analogue of the surjection $V \otimes V \otimes E \rightarrow S^2 V \otimes E$. Let us check that this is compatible with the projection $\tilde{\beta}(\pi^2 E) -$ the $u$-analogue of the projection $T^2_{\text{red}} V \otimes E \rightarrow S^2_{\text{red}} V \otimes E$. This amounts to checking that the following diagram commutes, where the top horizontal map is from (2.39) and the bottom
horizontal map we take to be the inclusion of the $\pi^2$-subgraph of $\text{Res}_H \hat{\mathcal{H}} \otimes \mathcal{H} E$.

$$\mathcal{H} \otimes J_{\mu-2} \mathcal{H} \otimes (\hat{\mathcal{H}}^+ \otimes \mathcal{H} E)_{11} \xrightarrow{\tilde{\beta}(\pi^2 E)} \mathcal{H} \otimes S_{\gamma} \otimes (\hat{\mathcal{H}}^+ \otimes \mathcal{H} E)_{11}$$

It is straightforward to check, given Theorem 2.7.6 and the derivation of (2.39), that standard basis elements behave as shown under the horizontal maps. This proves that the diagram commutes.

$$\tilde{T}_{a_k,l,\pi^2\gamma} \xrightarrow{T_{a_k,l,\pi^2\gamma}} \tilde{T}_{a_k,l,\pi^2\gamma} \quad \tilde{T}_{a_k,l,\pi^2\gamma} \xrightarrow{T_{a_k,l,\pi^2\gamma}} \tilde{T}_{a_k,l,\pi^2\gamma} \quad T_{a_{l-1},k} \otimes T_{s_1}(\pi^2\gamma) \xrightarrow{T_{a_{l-1},k} \pi^2 \otimes T_{s_{l-1}}\gamma}$$

The left-hand diagram is for $k < l$ and the right for $k \geq l > 1$.

This calculation will be used to show that the work we do in the next subsection for the $\mathcal{H} \otimes J$–version of tensoring with $V$ is also useful for the $(\hat{\mathcal{H}}^+ \otimes \mathcal{H}^-)_{11}$ version.

2.8.5. In this subsection we will partially determine the projection $\tilde{\beta}(E)$ on canonical basis elements. Despite the fact that $\mathcal{H} \otimes S_{\gamma} \otimes E$ is a $u$-analogue of $S^2_{\text{red}} V \otimes E$ and $Z^2 E$ is not, our study of $Z^2$ was not wasted. It will be helpful for determining what $\tilde{\beta}(E)$ does to canonical basis elements. This is not so easy to see directly, as it does not simply send canonical basis elements to canonical basis elements.

By Lemma 2.8.3, $A\Gamma^-_{s_1}$ is a cellular submodule of $\text{Res}_{S \setminus \gamma} A\Gamma$ with corresponding quotient $A\Gamma^+_{s_1}$, hence the exact sequence

$$0 \rightarrow A\Gamma^-_{s_1} \rightarrow \text{Res}_{S \setminus \gamma} A\Gamma \rightarrow A\Gamma^+_{s_1} \rightarrow 0.$$
Since $\tilde{E}^2, Z^2 A\Gamma, S_{\text{red}}^2 V A\Gamma$ only depend on $\text{Res}_{S_{\text{red}}} A\Gamma$, this sequence yields the three columns in the diagram below. The left column is exact by Proposition 2.8.2 and the other two are exact by exactness of induction. The left two squares commute because $\zeta$ (of the proof of Proposition 2.8.2) of a morphism just restricts its domain, and the right two squares commute because $\beta$ is a natural transformation.

![Diagram](image)

**Lemma 2.8.6.** Given $w \in W^{J_n-2}, \gamma \in \Gamma$, suppose that either $s_1 \notin R(w)$ or $s_1 \notin L(\gamma)$.

Then $\tilde{\beta}((A\Gamma))(\tilde{C}_{w,\gamma}'), \tilde{C}_{w, \gamma} \in H \otimes J_{n-2} A\Gamma$ lies in the lattice $L' := A^{-} H \otimes S_{\text{red}} \Gamma$.

**Proof.** First note that the standard basis for $H \otimes J_{n-2} A\Gamma$ coming from realizing $H \otimes S_{\text{red}} \otimes J_{n-2} A\Gamma$ satisfies

\[ \tilde{T}_{v, \tilde{C}_{s_1, \gamma}} = T_v \otimes S_{\text{red}} \subseteq \otimes J_{n-2} \gamma \quad \tilde{\beta}(A\Gamma) \quad T_v \otimes S_{\text{red}} \gamma, \text{ and} \]

\[ \tilde{T}_{v, \tilde{C}_{s_1, \gamma}} = T_v \otimes S_{\text{red}} C_{s_1} \otimes J_{n-2} \gamma \quad \tilde{\beta}(A\Gamma) \quad \begin{cases} \sum_{s_1 \in L(\delta)} \mu(\delta, \gamma) T_v \otimes S_{\text{red}} \delta & \text{if } s_1 \notin \Gamma, \\ \end{cases} \]

for $v \in W^{S_{\text{red}}}$. Then since the elements $T_v \otimes S_{\text{red}} \gamma$ are a standard basis for $H \otimes S_{\text{red}} A\Gamma,$
the lattice \( L = A^- H \otimes J_{s_{n-2}} \Gamma \) is sent to \( uL' \) by \( \tilde{\beta}(A \Gamma) \). Now for \( w \in W^{J_{s_{n-2}}} \), \( s_1 \notin R(w) \) implies \( w \in W^{S \setminus s_2} \). In this case,

\[
\tilde{C}'_{w,\gamma} \in \tilde{T}_w(\tilde{C}'_{s_1,\gamma}) + u^{-1} L \xrightarrow{\tilde{\beta}(A \Gamma)} T_w \otimes_{S \setminus s_2} \gamma + L' = L'.
\]

(2.51)

On the other hand if \( s_1 \in R(w) \), then \( w = vs_1 \) for \( v \in W^{S \setminus s_2} \), and in this case we are assuming \( s_1 \notin L(\gamma) \). Hence

\[
\tilde{C}'_{vs_1,\gamma} \in \tilde{T}_v(\tilde{C}'_{s_1,\gamma}) + u^{-1} L \xrightarrow{\tilde{\beta}(A \Gamma)} T_v \otimes_{S \setminus s_2} A^- \Gamma + L' = L'.
\]

(2.52)

For the remainder of the subsection set \( L^* = A^- H \otimes_{S \setminus s_2} \Gamma_{s_{n-2}} \).

**Theorem 2.8.7.** The arrows in (2.49) are compatible with the \( W \)-graph structures in the following sense.

(i) Vertical arrows take canonical basis elements to canonical basis elements or to 0.

(ii) The top non-zero row, on canonical basis elements, satisfies

\[
\tilde{C}'_{w,s_1,\gamma} \xrightarrow{\beta} [2] \tilde{C}'_{w,\gamma}, \text{ and } (w \in W^{S \setminus s_2}) \tag{2.53}
\]

\[
\tilde{C}'_{w,\gamma} \xrightarrow{\beta} 0 \mod L^*
\]

(iii) The bottom non-zero row, on canonical basis elements, satisfies

\[
\tilde{C}'_{w,s_1,\gamma} \xrightarrow{\beta} \tilde{C}'_{w,s_1,\gamma}, \text{ and } (w \in W^{S \setminus s_2}) \tag{2.54}
\]

\[
\tilde{C}'_{w,\gamma} \xrightarrow{\beta} \tilde{C}'_{w,\gamma}
\]
Proof. Statement (i) follows from Proposition 2.2.6 and Proposition 2.8.2.

The horizontal arrows on the left side of (2.49) are understood from Theorem 2.8.1; each is the inclusion of a cellular submodule.

To see (ii), first observe that Res$_{S \setminus s_2} \Gamma_{s_1}^-$ and $\Lambda_{s_1}^- \subseteq \mathcal{H}_{s_1} \otimes J_{n-2}^- A \Gamma_{s_1}^-$ (as in Theorem 2.8.1) are isomorphic as $W_{S \setminus s_2}$-graphs. This is clear from the remarks preceding Theorem 2.8.1 and from (2.9). An isomorphism, up to a global constant, between these two objects is given by

$$A \Lambda_{s_1}^- \xrightarrow{\beta(\Lambda_{s_1}^+)} A \Gamma_{s_1}^-; \tilde{C}_{s_1,\gamma} \mapsto [2] \gamma. \quad (2.55)$$

Therefore, tensoring $\beta(\Lambda_{s_1}^-)$ with $\mathcal{H}$ and applying the construction of Theorem 2.2.3 yields a map taking each canonical basis element to $[2]$ times a canonical basis element. This map is the composite of the maps in the top non-zero row of (2.49).

The second line of (2.53) follows from Lemma 2.8.6.

The proof of (iii) is similar to that of (ii). The $W_{S \setminus s_2}$-graphs Res$_{S \setminus s_2} \Gamma_{s_1}^+$ and $\Lambda_{s_1}^+ \subseteq \mathcal{H}_{s_1} \otimes J_{n-2}^+ A \Gamma_{s_1}^+$ are isomorphic via

$$A \Lambda_{s_1}^+ \xrightarrow{\beta(\Lambda_{s_1}^+)} A \Gamma_{s_1}^+; \tilde{C}_{1,\gamma} \mapsto \gamma. \quad (2.56)$$

Tensoring with $\mathcal{H}$ yields a map taking canonical basis elements to canonical basis elements, and this map is the bottom right horizontal map of (2.49).

To see the first line of (2.54), first observe that $\tilde{C}_{1,\gamma} = C_{s_1} \otimes \gamma \xrightarrow{\beta(\Lambda_{s_1}^+)} C_{s_1} \gamma = 0$, with the equality by definition of the quotient $A \Gamma_{s_1}^+$. Then use the fact that any $\tilde{C}_{w,\gamma}$ is in $A \{ T_x \otimes \tilde{C}_{s_1,\gamma} : x \in W_{S \setminus s_2}; \gamma \in \Gamma_{s_1}^+ \}$ (see Theorem 2.8.1 and the preceding
Theorem 2.8.8. The map $\tilde{\beta}(A\Gamma)$ (the middle right horizontal map of (2.49)), on canonical basis elements, satisfies

$$
\begin{align*}
\tilde{C}_{w,s_1,\gamma}' &\mapsto [2]\tilde{C}_{w,\gamma}' \quad \text{if } s_1 \in L(\gamma), \\
\tilde{C}_{w,\gamma}' &\mapsto 0 \mod L^* \quad \text{if } s_1 \in L(\gamma), \\
\tilde{C}_{w,s_1,\gamma}' &\mapsto 0 \mod L^* \quad \text{if } s_1 \notin L(\gamma), \\
\tilde{C}_{w,\gamma}' &\mapsto \tilde{C}_{w,\gamma}' \mod L^* \quad \text{if } s_1 \notin L(\gamma),
\end{align*}
$$

(2.57)

where $w$ is any element of $W_{S\setminus s_2}$ (and $L^* = A^{-}H \otimes S_{S\setminus s_2} \Gamma_{s_1}^{-}$).

Proof. The first and second line of (2.57) follow from Theorem 2.8.7 (ii) and the top right square of (2.49), as each vertical map in this square is the inclusion of a cellular submodule.

For the third line, apply Theorem 2.8.7 (iii) to show that $\tilde{C}_{w,s_1,\gamma}' \in H \otimes J_{n-2} \Gamma_{s_1}^{-}$, going down and then right, maps to $0 \in H \otimes S_{S\setminus s_2} \Gamma_{s_1}^{+}$. Therefore (going right) $\tilde{\beta}(A\Gamma)(\tilde{C}_{w,s_1,\gamma}') \in H \otimes S_{S\setminus s_2} \Gamma_{s_1}^{-} \subseteq H \otimes S_{S\setminus s_2} \Gamma_{s_1}$. Combining this with Lemma 2.8.6 yields the desired result. A similar argument proves the fourth line.

In a way made precise by the corollary below, the sets

$$Z^2\Gamma_{s_1}^{-} \cup (H \otimes J_{n-2} \Gamma_{s_1}^{+} \setminus Z^2\Gamma_{s_1}^{+}) \text{ and } Z^2\Gamma_{s_1}^{+} \cup (H \otimes J_{n-2} \Gamma_{s_1}^{-} \setminus Z^2\Gamma_{s_1}^{-})$$

(2.58)

are canonical bases for $S_{\text{red}}\mathcal{V} \otimes A\Gamma$ and $\Lambda^2 \mathcal{V} \otimes A\Gamma$, respectively, as $u \to 0$. We therefore call these subsets of $H \otimes J_{n-2} \Gamma$ combinatorial reduced sym and combinatorial wedge respectively.
Corollary 2.8.9. After adjoining $\frac{1}{|\mathcal{A}|}$ to $A$, there exists a $^T$-invariant basis \{ $c_{x,\gamma} : x \in W_{n-2}^J, \gamma \in \Gamma$ \} of $\mathcal{H} \otimes_{J_{n-2}} A\Gamma$ so that the transition matrix to the basis $\mathcal{H} \otimes_{J_{n-2}} \Gamma$ tends to the identity matrix as $u \to 0$, and so that under the map $\tilde{\beta}(A\Gamma)$

\[
\begin{align*}
    c_{w{s_1},\gamma} & \mapsto [2]C_{w,\gamma} & \text{if } s_1 \in L(\gamma), \\
    c_{w,\gamma} & \mapsto 0 & \text{if } s_1 \in L(\gamma), \\
    c_{w{s_1},\gamma} & \mapsto 0 & \text{if } s_1 \notin L(\gamma), \\
    c_{w,\gamma} & \mapsto \tilde{C}_{w,\gamma} & \text{if } s_1 \notin L(\gamma),
\end{align*}
\]

(2.59)

where $w$ is any element of $W_{S \setminus s_2}$.

Theorem 2.8.8 and Corollary 2.8.9 also apply with $\pi^2 A\Gamma$ replacing $A\Gamma$. There is a potential pitfall here as $\pi^2 A\Gamma$ is not the restriction of an $\mathcal{H}$-module to $\mathcal{H}_{J_{n-2}}$. However, it is an $\mathcal{H}_{S \setminus s_2}$-module, since $K_0 \cap \pi^2 K_0 \pi^{-2} = S \setminus s_2$, which is all that is needed to apply the results in this subsection. Also, by §2.8.4 the projection $\tilde{\beta}(\pi^2 E)$ specializes to the projection $T_{\text{red}}^2 V \otimes E \to S_{\text{red}}^2 V \otimes E$ at $u = 1$. Thus we can write $\mathcal{H} \otimes_{J_{n-2}} \pi^2 \Gamma$ as the disjoint union of

\[
\begin{align*}
    Z^2 \pi^2 \Gamma_{s_{n-1}}^{-1} \cup (\mathcal{H} \otimes_{J_{n-2}} \pi^2 \Gamma_{s_{n-1}}^{-1} \setminus Z^2 \pi^2 \Gamma_{s_{n-1}}^{-1}) & \quad \text{and} \\
    Z^2 \pi^2 \Gamma_{s_{n-1}}^+ \cup (\mathcal{H} \otimes_{J_{n-2}} \pi^2 \Gamma_{s_{n-1}}^+ \setminus Z^2 \pi^2 \Gamma_{s_{n-1}}^{-1})
\end{align*}
\]

(2.60)

which will also be called combinatorial reduced sym and combinatorial wedge.

Example 2.8.10. In the $W$-graph in Figure 2.1, combinatorial reduced sym is the lower triangular region consisting of the first $i$ entries of row $i$ for $i = 1, 2, 3$; combinatorial wedge is the upper triangular region consisting of the last $4 - i$ entries of
row $i$ for $i = 1, 2, 3$. For general $\Gamma$, the picture would be similar: the $W$-graph could be drawn in $n$ by $n$ chunks and combinatorial reduced sym would consist of lower triangular regions for $\gamma \in \Gamma_{x_1}^-$ and upper triangular regions for $\gamma \in \Gamma_{x_1}^+$. 

In the $W$-graph in Figure 2.2, combinatorial reduced sym is the lower triangular region consisting of the first $i - 1$ entries of row $i$ for $i = 2, 3, 4$; combinatorial wedge is the upper triangular region consisting of the last $5 - i$ entries of row $i$ for $i = 2, 3, 4$.

The labels “sym” and “wedge” below the trees mark the cells in combinatorial reduced sym and combinatorial wedge.

### 2.9 Combinatorial approximation of $V \otimes V \otimes E \rightarrow S^2V \otimes E$

For this section, let $\Gamma$ be a cell of $\Gamma_W$ labeled by a tableau $T^0$. We will describe the results of §2.7 and §2.8 in terms of cells and their tableau labels.

#### 2.9.1. For a tableau $P$, let $P_{r,c}$ be the square of $P$ in the $r$-th row and $c$-th column. Suppose that $P_{r_1,c_1}, \ldots, P_{r_l,c_l}$ are squares of $P$ such that $P_{r_i,c_i}$ is an outer corner of $P^{i-1} := P \backslash \{P_{r_1,c_1}, \ldots, P_{r_{i-1},c_{i-1}}\}$. Then referring to the sequence of tableaux $P, P^1, \ldots, P^l$, we say that $P_{r_1,c_1}, \ldots, P_{r_l,c_l}$ are removed from $P$ as a horizontal strip (resp. removed from $P$ as a vertical strip) if $c_1 > c_2 > \cdots > c_l$ (resp. $r_1 > r_2 > \cdots > r_l$). Equivalently, if $P^*$ is the skew tableau of squares $\{P_{r_1,c_1}, \ldots, P_{r_l,c_l}\}$ with
$l + 1 - i$ in $P_{r_1, c_1}$, then $\text{jdt}(P^*)$ is a single row (resp. column). Similarly, referring to the sequence of tableaux $P^l, \ldots, P^1, P$, we say that $P_{r_1, c_1}, \ldots, P_{r_l, c_l}$ are added to $P^l$ as a horizontal strip (resp. added to $P^l$ as a vertical strip) if $c_1 > c_2 > \cdots > c_l$ (resp. $r_1 > r_2 > \cdots > r_l$).

Recall the local rules for the RSK growth diagram (see, e.g., [38, 7.13]). Letting $\lambda, \mu, \nu$ be partitions with $\mu \subseteq \lambda, \nu$, we notate these local rules by

$$
\begin{align*}
\mathcal{G}(0; \lambda, \mu, \nu) &= \begin{cases} 
\lambda & \text{if } \lambda = \mu = \nu, \\
\lambda + \epsilon_{i+1} & \text{if } \lambda = \nu = \mu + \epsilon_i, \\
\lambda \cup \nu & \text{if } \lambda \neq \nu,
\end{cases} \\
\mathcal{G}(1; \lambda, \mu, \nu) &= \lambda + \epsilon_1, & \text{if } \lambda = \mu = \nu.
\end{align*}
$$

(2.61)

Here $\epsilon_i$ denotes the tuple with a 1 in its $i$-th coordinate and 0's elsewhere, and $\lambda \cup \nu$ denotes the partition whose $i$-th part is $\max(\lambda_i, \nu_i)$.

Let $a \rightarrow P$ (resp. $a \leftarrow P$) denote the column (resp. row) insertion of $a$ into $P$. For the next theorem we will use freely the descriptions of cells given in §2.4. The shorthand $P_\succ$ will be used for $P_\succ c = \text{jdt}(P^*)$, where $P^*$ is the skew subtableau of $P$ with entries $> c$ and $c$ is the smallest entry of $P$. In what follows we will use the somewhat redundant local sequences for cells that come from writing $\tilde{E}^1, \tilde{E}^2, \tilde{F}^2$ as $\mathcal{H} \otimes \mathcal{I}, \mathcal{J} \otimes \mathcal{J} \otimes \mathcal{A} \Gamma, \mathcal{H} \otimes \mathcal{J}, \mathcal{H} \otimes \mathcal{J}, \mathcal{H} \otimes \mathcal{J}, \mathcal{J} \otimes \mathcal{J} \otimes \mathcal{A} \Gamma, \mathcal{H} \otimes \mathcal{J}, \mathcal{J} \otimes \mathcal{J}, \mathcal{J} \otimes \mathcal{J} \otimes \mathcal{A} \Gamma, \mathcal{H} \otimes \mathcal{S}_{\cap \mathcal{A}}, \mathcal{S}_{\cap \mathcal{A}} \otimes \mathcal{J} \otimes \mathcal{J}, \mathcal{J} \otimes \mathcal{J} \otimes \mathcal{A} \Gamma$ respectively; these last two will be referred to as $\tilde{F}^2_j$ and $\tilde{F}^2_{S_{\cap \mathcal{A}}}$ respectively.

**Theorem 2.9.1.**
(i) The map $\mathcal{H} \otimes_J \alpha : \tilde{E}^1 \rightarrow \tilde{E}^2$ of (2.32) is given on cells by

$$(T^1, P, T^0) \mapsto (T^1, P, T^0),$$

where $P = 1 \rightarrow T^1$. In particular,

$$\text{sh}(P) = \mathcal{G}(1; \text{sh}(T^1), \text{sh}(T^1), \text{sh}(T^1)).$$

(ii) The inverse of the map $\tilde{E}^2 \rightarrow \tilde{E}_J^2$ of (2.32) is given on cells by

$$(T^2, P, T^0) \mapsto (T^2, P, T^0),$$

where

$$\text{sh}(P) = \mathcal{G}(0; \text{sh}(T^2), \text{sh}(T^2), \text{sh}(T^1));$$

$P$ is determined by its shape and $P > T^1$.

(iii) The isomorphism of $W$-graphs $\tilde{F}_J^2 \rightarrow \tilde{F}_{S \setminus s_2}^2$ of Proposition 2.3.7 is given on cells by

$$(T^2, P^1, T^0) \mapsto (T^2, (P^2, T^2), T^0),$$

where $P^1$ is the tableau $\begin{array}{|c|c|c|c|}
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}$ (resp. $\begin{array}{|c|c|c|c|}
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}$) if $T^0 \setminus T^1, T^1 \setminus T^2 > 2$ are removed from $T^0$ as a horizontal strip (resp. vertical strip), and $P^2$ is the tableau $\begin{array}{|c|c|c|c|}
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}$ (resp. $\begin{array}{|c|c|c|c|}
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}$) if $T^2 \setminus T^2 > 2, T^2 \setminus T^2 > 2$ are added to $T^2 > 2$ as a horizontal strip (resp. vertical strip).

(iv) The cells of $\tilde{F}_{S \setminus s_2}^2$ in combinatorial reduced sym (resp. combinatorial wedge) are those with local sequences $(T^2, (P^2, T^2), T^0)$ such that $P^2$ and $P^1$ have the same shape (resp. different shape).

Proof. For (i)–(iii), we will use $J = J_{n-1}$ instead of $J = J'_{n-1}$ and the comments in §2.4.4 to go back and forth between these conventions.
The map $\mathcal{H} \otimes J \alpha$ on canonical basis elements is given in stuffed notation by

$$(z_1, Jz_1, z_0) \mapsto (z_1, Jz_1, Jz_1n, Jz_1, z_0).$$  \hspace{1cm} (2.62)$$

Here the $z_i$ are thought of as words so that $Jz_1n$ is just the word $Jz_1$ with $n$ appended at the end. The map on cells is then $(T_1, T_1^1, T^0) \mapsto (T_1, T_1^1, P, T_1^1, T^0)$, where $P = T_1^1 \leftarrow n$. Statement (i) then follows by applying the Schützenberger involution.

For (ii), observe that the inverse of $\mathcal{H} \otimes J \tau$ of (2.32) is given in stuffed notation by

$$(z_2, Jz_2, z_0, Jn, z_0) \mapsto (z_2, Jz_2, z_1, z_0, z_0),$$

where $z_1 = Jz_2^*k$ ($Jz_2^*$ is obtained from $Jz_2$ by increasing all numbers $\geq k$ by 1) and $k$ is such that $z_0J = z_0^*Jn^{-2}k$ ($z_0^*Jn^{-2}$ is obtained from $z_0Jn^{-2}$ by increasing all numbers $\geq k$ by 1). Thus if $(T_2^2)^* := P(Jz_2^*)$ and $(T_2^2)^{-1} := P(J_1z_0^*)$, then

$$P := P(z_1) = (T_2^2)^* \leftarrow k,$$

and $T^1 := P(z_0J) = (T_2^2)^{-1} \leftarrow k$.  \hspace{1cm} (2.64)$$

Note that $k \neq n$ and $z_0Jn^{-2} = Jn^{-2}z_2$ imply $(T_2^2)^* \setminus (T_2^2)^{-1} \leftarrow k$ is a square containing an $n$. The element $k$ inserts in these tableau exactly the same way, except that the final step of $(T_2^2)^* \leftarrow k$ may bump the $n$ down one row; this case corresponds exactly to the case $\text{sh}((T_2^2)^*) = \text{sh}(T^1)$.

Statement (iii) is really two separate statements, one for a bijection of local sequences corresponding to $\text{Res}_{J_n^{-2}} \text{Res}_J A \Gamma \cong \text{Res}_{J_n^{-2}} \text{Res}_{S_2} A \Gamma$, and one for a bijection of local sequences corresponding to $\mathcal{H} \otimes \mathcal{B}_J \otimes \Upsilon \cong \mathcal{H} \otimes \mathcal{B}_{S_2} \otimes \Upsilon$ ($\Upsilon$ some cell of
\( \Gamma_{W_{J'_{n-2}}} \). The first bijection follows from [38, Lemma 7.11.2] (This includes the statement that if \( P \) is a tableau and \( j \leq k \), then the square \((P \leftarrow j) \leftarrow k)\backslash P \) lies strictly to the left of \(((P \leftarrow j) \leftarrow k)\backslash(P \leftarrow j)\). We also need that if \( j > k \), then the square \((P \leftarrow j)\backslash P \) lies weakly to the right of \(((P \leftarrow j) \leftarrow k)\backslash(P \leftarrow j)\), which is similar.) The second bijection is the definition of adding as a horizontal or vertical strip in the case that \( J = J_{n-1} \).

To see (iv), observe that the local labels of the cells of \( \text{Res}_{S_{\lessdot}2} \Gamma_{s_1}^+ \) (resp. \( \text{Res}_{S_{\lessdot}2} \Gamma_{s_1}^- \)) are of the form \((1,2, \mathbb{T}_{> 2})\) (resp. \((1,2, \mathbb{T}_{> 2})\)); the local labels of the cells of \( \Lambda_{s_1}^+ \) (resp. \( \Lambda_{s_1}^- \)) are of the form \((1,2, \mathbb{T}_{> 2})\) (resp. \((1,2, \mathbb{T}_{> 2})\)), where \( \Lambda = \mathcal{H}_{S_{\lessdot}2} \otimes J'_{n-2} \Gamma \).  

**Example 2.9.2.** Suppose \( T^0 = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 \\ \end{array} \). On the left is the local sequence of a cell of \( \tilde{E}^2 \) (reading from left to right, ignoring the bottom middle tableau) and the local sequence of the corresponding cell of \( \tilde{E}_{J}^2 \) (reading from left to right, ignoring the top middle tableau). The tableau are arranged this way to match an RSK growth diagram picture. Above the tableau are the coordinates of the stuffed notation for a canonical basis element in this cell.

On the right is the local sequence of the corresponding cell of \( \tilde{E}_{S_{\lessdot}2}^2 \).
2.9.2. For the $W$-graph version of tensoring with $V$ coming from the affine Hecke algebra, we have a similar theorem. Let $\mathcal{G}'(a; \lambda, \mu, \nu) = (\mathcal{G}(a; \lambda', \mu', \nu'))'$, where $'$ of a partition denotes its transpose. Let $\widehat{E}^1, \widehat{E}^2, \widehat{F}^2_j, \widehat{F}^2_{S\setminus s^2}$ be defined analogously to $\check{E}^1, \check{E}^2, \check{F}^2_j, \check{F}^2_{S\setminus s^2}$. More precisely, $\widehat{E}^1$ and $\widehat{E}^2$ will use three- and five-term local sequences as in Examples 2.4.6 and 2.9.4; $\check{F}^2_j$ refers to $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}_{J_n-2} \otimes \mathcal{H}_{J_{n-2}} \otimes \mathcal{A}$ with a six-term local sequence as in Example 2.9.4, and $\check{F}^2_{S\setminus s^2}$ refers to $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}_{S\setminus s^2} \otimes \mathcal{H}_{S_{s^2}} \otimes \mathcal{A}$ also with a six-term local sequence.

**Theorem 2.9.3.**

(i) The inverse of the map $\widehat{E}^2 \to \widehat{E}^1$ of (2.39) is given on cells by

$$(T^1, T^1_>, T^0) \mapsto (P^2, P^2_>, P^1, T^1_>, T^0),$$

where $P^1$ is determined by $\text{sh}(P^1) = \mathcal{G}'(1; \text{sh}(T^1_)), \text{sh}(T^1_>)$ and the entries in $P^2, P^2_>$ have the same relative order as those in $T^1, T^1_>$. 

(ii) The map $\widehat{F}^2_j \to \widehat{E}^2$ of (2.39) is given on cells by

$$(T^2, T^2_>, T^2_{>2}, T^1, T^0) \mapsto (P^2, P^2_>, P^1, T^1_>, T^0),$$

where $P^1$ is determined by $\text{sh}(P^1) = \mathcal{G}'(0; \text{sh}(T^2_>), \text{sh}(T^2_{>2}), \text{sh}(T^1))$ and the entries of $P^2, P^2_>$ have the same relative order as those in $T^2, T^2_>, T^1$.

(iii) The isomorphism of $W$-graphs $\widehat{F}^2_j \cong \widehat{F}^2_{S\setminus s^2}$ of Proposition 2.3.7 is given on cells
by

$$(T^2, T^2, T^2, \pi^{-2}T^2, T^1, T^0) \mapsto (T^2, (P^2, T^2), T^2, \pi^{-2}T^2, (\pi^{-2}T^2, P^1), T^0),$$

where $P^1$ is the tableau $\begin{array}{c} 1 \\ 2 \end{array}$ (resp. $\begin{array}{c} \pi^{-2} \\ T^2 \end{array}$) if $T^0\setminus T^1, T^1\setminus \pi^{-2}T^2$ are removed from $T^0$ as a horizontal strip (resp. vertical strip), and $P^2$ is the tableau $\begin{array}{c} \pi^{-2} \\ T^2 \end{array}$ (resp. $\begin{array}{c} \pi^{-2} \\ T^2 \end{array}$) if $T^2\setminus T^2, T^2\setminus T^2$ are added to $T^2$ as a horizontal strip (resp. vertical strip).

(iv) The cells of $\hat{F}_{S\setminus s_2}^2$ in combinatorial reduced sym (resp. combinatorial wedge) are those with local sequences

$$(T^2, (P^2, T^2), T^2, \pi^{-2}T^2, (\pi^{-2}T^2, P^1), T^0)$$

such that $P^2$ and $P^1$ have the same shape (resp. different shape).

Proof. Similar to that of Theorem 2.9.1, the main difference being for (ii): after applying the Schützenberger involution, the analogous statement to (2.64) is with column insertions instead of row insertions. $\square$

Example 2.9.4. The local sequence for a cell of $\hat{E}_2^2$ (top) and the local sequence for the corresponding cell of $\hat{F}_j^2$ (bottom).
Corollary 2.9.5. Theorem 2.9.1 gives a partition of the cells of $\tilde{E}^2$ into three parts: the non-reduced part corresponding to the image of (i), the inverse image of combinatorial reduced sym under (ii), and the inverse image of combinatorial wedge under (ii). Similarly, Theorem 2.9.3 gives a partition of the cells of $\hat{E}^2$ into three parts: the non-reduced part corresponding to the inverse image of (i), the image of combinatorial reduced sym under (ii), and the image of combinatorial wedge under (ii).

Remark 2.9.6. There is an obvious bijection between the cells of $\tilde{E}^2$ and $\hat{E}^2$ obtained by taking a local sequence $\Upsilon$ of a cell of $\tilde{E}^2$ to the cell of $\hat{E}^2$ with the same sequence of shapes as those of $\Upsilon$. Under this bijection, the cells of any of the three parts of $\tilde{E}^2$ coming from Corollary 2.9.5 do not match the corresponding parts of $\hat{E}^2$ in general. They do not, for example, in Figures 2.1 and 2.2.

2.10 Future work

There are some natural questions to ask about the inducing $W$-graphs construction that, as far as we know, remain unanswered. One question is whether the edge weights $\mu$ of the $W_J$-graph $\Gamma$ being nonnegative implies the same for the coefficients $\tilde{P}_{x,\delta,w,\gamma}$ of (2.9) or for the structure constants $h_{x,y,z}$, defined by $C'_x \tilde{C}'_y = \sum_z h_{x,y,z} \tilde{C}'_z$, $x \in W$, $y, z \in W^J \times \Gamma$. Our computations in the case $W = S_n$ are consistent with these positivity conjectures, but we have not investigated the inducing $W$-graphs construction outside this case. Presumably these should be provable in the special
case that \((W, S)\) is crystallographic, and \(\Gamma\) is the iterated induction of Hecke algebras of crystallographic Coxeter groups, by the same methods used to show the non-
negativity of the usual Kazhdan-Lusztig polynomials for such \(W\).

Another question concerns the partial order on the cells of \(\tilde{E}^{d-1} = \mathcal{H}_1 \otimes J_1 \ldots \otimes J_{d-1}\) \(\mathcal{H}_d\). For type \(A\), we have stated Conjecture 2.6.2. For general type, we might hope to extend Lusztig’s \(a\)-invariant to the induced \(W\)-graph setting. In particular, each cell of \(\tilde{E}^{d-1}\) is contained in a cellular subquotient isomorphic to \(\Gamma_{W_1}\) (Theorem 2.3.5), so inherits an \(a\)-invariant from this isomorphism; a natural question is whether \(z \leq_\Lambda z'\) and \(z, z'\) in different cells implies \(a(z) > a(z')\), where \(\Lambda\) is the \(W_1\)-graph structure on \(\tilde{E}^{d-1}\). In [10], Geck shows a slightly weaker version of this statement in the case \(\tilde{E}^{d-1} = \text{Res}_{J_1, \mathcal{H}_2}, d = 2\) and \(W_2\) crystallographic and bounded in the sense of [33, 1.1 (d)]. It seems likely that a similar proof will work for the general case, with all Coxeter groups crystallographic and bounded.

In the forthcoming paper [4], we look at the partial order on the cells of \(\text{Res}_{\mathcal{H}, \mathcal{H}^+} \otimes_{\mathcal{H}} e^+\). It appears that there are other important invariants besides the \(a\)-invariant and dominance order that put restrictions on this partial order.

We have put much effort into extending the results of §2.7-§2.9 to higher symmetric powers of \(V\) and have had only partial success. In a way, this is the subject of the forthcoming paper [4], however this focuses more on the extended affine Hecke algebra and less on iterated restriction and induction.
Chapter 3

Paper II: An insertion algorithm for catabolizability

Abstract

We give an insertion-like algorithm that takes a standard word \( w \) and outputs the catabolizability of the insertion tableau of \( w \). From this, we deduce a characterization of catabolizability as the statistic on words invariant under Knuth transformations, catabolism transformations, and non-zero (co)rotations, and satisfying some normalization condition. We also prove a Greene’s Theorem-like characterization of catabolizability, and a result about how cocyclage changes catabolizability, strengthening a similar result in [37].
3.1 Introduction

The ring of coinvariants $R_1^n = \mathbb{C}[y_1, \ldots, y_n]/(e_1, \ldots, e_n)$, thought of as a $\mathbb{C}S_n$-module with $S_n$ acting by permuting the variables, is a graded version of the regular representation. It has the Garsia-Procesi modules $R_\lambda$ as quotients (see [8]). Combining the work of Hotta-Springer and Lascoux (see [13],[27]) gives the Frobenius series

$$F_{R_\lambda}(t) = \sum_{\begin{array}{c} T \in \text{SYT} \\ \text{ctype}(T) \geq \lambda \end{array}} t^{\text{cocharge}(T)} s_{\text{sh}(T)},$$

(3.1)

where $\text{ctype}(T)$ is the catabolizability of $T$, defined in §3.2.2.

In [4] we exhibit a $q$-analogue $R_1^n$ of the ring of coinvariants that is endowed with a canonical basis and possesses $q$-analogues $R_\lambda$ of the $R_\lambda$ as cellular quotients. The elements of the canonical basis are in bijection with standard words (permutations of $1, \ldots, n$), the cells in bijection with SYT, and there is a natural grading on the cells that corresponds to cocharge under this bijection. Tableau of shape $\lambda$ do not appear in degree less than $\sum_i \left( \begin{array}{c} \lambda_i \\ 2 \end{array} \right)$ and there is a unique occurrence of a tableau of shape $\lambda$ in this degree; refer to this tableau and its corresponding cell as the Garnir tableau of shape $\lambda$. In our investigations of the $R_\lambda$, we found a way to go from any word $w$ to a word inserting to the Garnir tableau of shape $\text{ctype}(P(w))$ by a sequence of relations in the Kazhdan-Lusztig preorder.

Following this sequence of relations gives an algorithm for computing $\text{ctype}(P(w))$ for any standard word $w$. This leads to a characterization of catabolizability as being
the statistic on words invariant under Knuth transformations, non-zero (co)rotations (defined in §3.2.1), and a new catabolism transformation defined in §3.3.1 (and satisfying some normalization condition). This also allows us to prove a Greene’s Theorem-like characterization of catabolizability.

After reviewing the definitions of cocyclage and catabolism in §3.2, we present this algorithm and its corollaries in §3.3.

3.2 Cocyclage and catabolism

3.2.1. We recall the notions of cocyclage and catabolizability as defined in [28], [37], but restrict to the special case of standard tableaux and words. In what follows, $u, v, w$ will denote standard words.

The cocharge labeling $w^\text{cc}$ of a word $w$ is the (non-standard) word obtained from $w$ by reading the numbers of $w$ in increasing order; labeling the 1 of $w$ with a 0, and if the $i$ of $w$ is labeled by $k$, then labeling the $i + 1$ of $w$ with a $k$ (resp. $k + 1$) if the $i + 1$ in $w$ appears to the right (resp. left) of $i$. For example, for the word $w$, its cocharge labeling $w^\text{cc}$ appears below it.

\[
\begin{align*}
w &= 1 6 8 4 2 9 5 7 3 \\
w^\text{cc} &= 0 2 3 1 0 3 1 2 0.
\end{align*}
\]

The sum of the numbers in the cocharge labeling of $w$ is the cocharge of $w$ or $\text{cocharge}(w)$. 
For a word $w$ and number $a \neq 1$, $aw$ (resp. $wa$) is a corotation (resp. rotation) of $wa$ (resp. of $aw$). It is a non-zero corotation (resp. rotation) of $wa$ (resp. of $aw$) if, in the cocharge labeling of $wa$, $a$ is labeled with a number greater than 0. Similarly, define zero (co)rotations for the case $a$ is labeled with a 0.

(Co)rotations respect cocharge labeling in the following way: $v$ is a corotation of $w$ if and only if $v^{cc} = a + 1 \cdot y$, $w^{cc} = ya$.

Let $s_i$ be the simple reflection of $S_n$ that transposes $i$ and $i + 1$. Thinking of a standard word $w = w_1 \cdots w_n$ as being the map $w : [n] \to [n]$, $i \mapsto w_i$, we can act on $w$ on the right by $S_n$; then $ws_i$ is the word obtained from $w$ by swapping the numbers in positions $i$ and $i + 1$. We can also act on cocharge labelings with this same right action, however this does not always result in the cocharge labeling of some standard word. The following are easily seen to be equivalent.

\begin{align}
(i) & \quad w^{cc}s_i = (ws_i)^{cc}. \\
(ii) & \quad w^{cc} \text{ and } (ws_i)^{cc} \text{ have the same content.} \\
(iii) & \quad \text{cocharge}(w) = \text{cocharge}(ws_i). \\
(iv) & \quad |w_i - w_{i+1}| \neq 1.
\end{align}

If any (all) of these holds, then the transformation $w \rightsquigarrow ws_i$ is cocharge-preserving.

There is a cocyclage from the tableau $T$ to the tableau $T'$, written $T \overset{cc}{\rightarrow} T'$, if there exist words $u, v$ such that $v$ is the corotation of $u$ and $P(u) = T$ and $P(v) = T'$. The cocyclage poset is the poset on the set of SYT generated by the relation $\overset{cc}{\rightarrow}$.

Define the cocharge labeling $T^{cc}$ of a tableau $T$ to be $P(\text{rowword}(T)^{cc})$, and
cocharge\((T)\) to be the sum of the entries in \(T^{cc}\). Here rowword\((T)\) denotes the row reading word of \(T\). The tableau \(T^{cc}\) is also \(P(w^{cc})\) for any \(w\) inserting to \(T\). This follows from the fact that Knuth transformations do not change left descent sets. A cocyclage \(T \xrightarrow{cc} T'\) is a non-zero (resp. zero) cocyclage if any (equivalently, every) corotation inducing it is non-zero (resp. zero).

The following theorem is easy for the standard tableaux case. The cyclage poset is the poset dual to the cocyclage poset, i.e., the poset obtained by reversing all relations.

**Theorem 3.2.1** ([28]). The cyclage poset is graded, with rank function given by cocharge.

**3.2.2.** Let \(Z^r_\mu\) be the superstandard tableau of shape \(\mu\) with \(r\)'s in the first row, \(r+1\)'s in the second row, etc. Let \(Z^*_m\) be the single-row standard tableau with entries \(1, \ldots, m\). For a skew tableau \(T\) and index \(r\) (resp. index \(c\)), let \(H_r(T) = P(T_nT_s)\) (resp. \(V_c(T) = P(T_eT_w)\)), where \(T_n\) and \(T_s\) (resp. \(T_e\) and \(T_w\)) are the north and south (resp. east and west) subtableaux obtained by slicing \(T\) horizontally (resp. vertically) between its \(r\)-th and \((r+1)\)-th rows (resp. \(c\)-th and \((c+1)\)-th columns). Here we are thinking of tableaux as being drawn in English notation.

For \(\lambda \subseteq \text{sh}(T)\), let \(T_\lambda\) be the subtableau of \(T\) of shape \(\lambda\). If \((m) \subseteq \text{sh}(T)\), then define the \(m\)-catabolism (resp. \(m\)-column catabolism) of \(T\), notated \(\text{rcat}_m(T)\) (resp. \(\text{ccat}_m(T)\)), to be the tableau \(H_1(T - T_{(m)})\) (resp. \(V_m(T - T_{(m)})\)). For a partition \(\lambda \vdash -\)
\[ n := |T|, \lambda\text{-catabolizability (resp. } \lambda\text{-column catabolizability) is defined inductively as follows: } T \text{ is } \lambda\text{-}(column) \text{ catabolizable if } T_{(\lambda_1)} = Z_{\lambda_1} \text{ and the } \lambda_1\text{-}(column) \text{ catabolism of } T \text{ is } \hat{\lambda}\text{-}(column) \text{ catabolizable, where } \lambda = (\lambda_1, \hat{\lambda}); \text{ the empty tableau is } \emptyset\text{-}(column) \text{ catabolizable.}

The following is a consequence of Proposition 48 in [37] and the surrounding discussion.

Proposition 3.2.2. For any SYT \( T \), there is a unique maximal in dominance order partition \( \lambda \) such that \( T \) is \( \lambda\)-catabolizable.

The \( \lambda \) of this proposition is the catabolizability of \( T \), denoted \( \text{ctype}(T) \). Catabolizability is computed by performing the sequence of catabolisms to \( T \) in which \( \text{rcat}_m \) is applied with the largest \( m \) for which it is defined. Similarly, define the column catabolizability of \( T \) to be the partition obtained by performing the sequence of column catabolisms to \( T \) in which \( \text{ccat}_m \) is applied with the largest \( m \) for which it is defined. With the definitions of \( \lambda\text{-}(column) \) catabolizability above, the following proposition is quite tricky.

Proposition 3.2.3 ([37, Proposition 49]). A standard tableau is \( \lambda\)-catabolizable if and only if it is \( \lambda\)-column catabolizable.

We reprove a similar result in the next section using the catabolism insertion algorithm.
3.3 Catabolism insertion

3.3.1. We now describe the catabolism insertion algorithm, which takes a standard word $w$ as input and outputs a partition $F(w)$ that we will show to be equal to the catabolizability of the insertion tableau of $w$.

Let $\epsilon_i \in \mathbb{Z}^n$ be the standard basis vector with a 1 in its $i$-th coordinate and 0’s elsewhere.

Algorithm 3.3.1. Let $f$ be the function below, which takes a pair consisting of a (non-standard) word and a partition to another such pair. Let $x = ya$, $y$ a word and $a$ a number.

$$f(x, \nu) = \begin{cases} (y, \nu + \epsilon_{a+1}) & \text{if } \nu + \epsilon_{a+1} \text{ is a partition,} \\ (a + 1 y, \nu) & \text{otherwise.} \end{cases}$$ (3.4)

Given the input standard word $w$, first determine the cocharge labeling $z$ of $w$.

Next, apply $f$ to $(z, \emptyset)$ repeatedly, obtaining the sequence of pairs $f^{(i)}(z, \emptyset)$, stopping when the word of the pair is empty. Output the partition of this final pair, and denote this output $F(w)$.

The transition from $(x, \nu)$ to $f(x, \nu)$ is a step of the algorithm. We say that $a$ is presented to $\nu$, and in the top case of (3.4), $a$ is inserted into $\nu$, while in the bottom case, $a$ is corotated. The step of the algorithm in the top case is an insertion, and in the bottom a corotation.
It is convenient to think of the $\nu$ in the algorithm as the tableau $Z^0_\nu$, as illustrated by the following example.

**Example 3.3.2.** The word $w = 1 \ 6 \ 8 \ 4 \ 2 \ 9 \ 5 \ 7 \ 3$ has cocharge labeling $z = 0 \ 2 \ 3 \ 1 \ 0 \ 3 \ 1 \ 2 \ 0$. The sequence of words and partitions produced by the algorithm is

<table>
<thead>
<tr>
<th>$i$</th>
<th>$f^{(i)}(z, \emptyset)$</th>
<th>$i$</th>
<th>$f^{(i)}(z, \emptyset)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>023103120 0</td>
<td>7</td>
<td>44302 0</td>
</tr>
<tr>
<td>1</td>
<td>02310312 0</td>
<td>8</td>
<td>4430 0</td>
</tr>
<tr>
<td>2</td>
<td>30231031 0</td>
<td>9</td>
<td>443 0</td>
</tr>
<tr>
<td>3</td>
<td>3023103 0</td>
<td>10</td>
<td>44 0</td>
</tr>
<tr>
<td>4</td>
<td>4302310 0</td>
<td>11</td>
<td>4 0</td>
</tr>
<tr>
<td>5</td>
<td>430231 0</td>
<td>12</td>
<td>5 0</td>
</tr>
<tr>
<td>6</td>
<td>43023 0</td>
<td>13</td>
<td>0 0</td>
</tr>
</tbody>
</table>

Letting $z$ be the cocharge labeling of $w$, a *catabolism transformation* of $w$ (or of $z$) is an operation taking $w$ to $ws_i$ (or $zs_i$) if $|z_i - z_{i+1}| > 1$:

$$
\cdots z_i z_{i+1} \cdots \leftrightarrow \cdots z_{i+1} z_i \cdots .
$$

(3.5)

It is easy to see from (3.3) (iv) that a catabolism transformation is cocharge-preserving.
Some properties of Algorithm 3.3.1 are more easily seen from the following variant, which is clearly equivalent to it.

**Algorithm 3.3.3.** A step of Algorithm 3.3.1 from \((ya, \nu)\) to \(f(ya, \nu)\) is rephrased as follows. Instead of keeping track of a partition \(\nu\), keep track of the corresponding superstandard tableau \(Z^0_\nu\). Replace presenting \(a\) to \(\nu\) with column-inserting \(a\) into \(Z^0_\nu\).

In the insertion case, this produces the same result as in Algorithm 3.3.1.

The corotation case is broken into three parts, the first of which is this column-insertion; let \(T\) be the tableau resulting from this insertion. The tableau \(T\) contains \(Z^0_\nu\) and \(T - Z^0_\nu\) is a single square containing an \(a\). The corresponding number of \(\text{rowword}(T)\) is at least two more than all the numbers to the right of it. The second part performs the sequence of catabolism transformations taking \(\text{rowword}(T)\) to \(\text{rowword}(Z^0_\nu)\; a\). The third part then corotates

\[
y \; \text{rowword}(Z^0_\nu) \; a
\]

to obtain

\[
a + 1 \; y \; \text{rowword}(Z^0_\nu),
\]

and this word is output as the pair \((a + 1 \; y, Z^0_\nu)\).

**Theorem 3.3.4.** Algorithm 3.3.1, with input a standard word \(u\) of length \(n\), satisfies:

(i) after every step, the word-partition pair \((x, \nu)\) is such that \(x \; \text{rowword}(Z^0_\nu)\) is the cocharge labeling of some standard word of length \(n\),

(ii) terminates (in at most \(n + \binom{n}{2}\) steps),
(iii) \( F(u) = F(v) \) if \( v \) is a non-zero corotation of \( u \),

(iv) \( F(u) = F(v) \) if \( u \leadsto v \) is a catabolism transformation,

(v) \( F(u) = F(v) \) if \( u \leadsto v \) is a Knuth transformation,

(vi) \( F(u) = \text{ctype}(P(u)) \).

Proof. Statement (i) is easy to see from Algorithm 3.3.3. It realizes each step as a sequence of Knuth transformations, catabolism transformations, or non-zero (co)rotations on the word \( x \ \rowword(Z^0_\nu) \), all of which could be applied by converting back to a standard word, performing the operation, and then taking the cocharge labeling.

The algorithm terminates in at most \( n + \binom{n}{2} \) steps because for any step from \((x, \nu)\) to \((x^*, \nu^*) := f(x, \nu)\), either

\[
|x^*| = |x| - 1 \quad \text{or} \quad \text{cocharge}(x^* \ \rowword(Z^0_{\nu^*})) = \text{cocharge}(x \ \rowword(Z^0_{\nu})) + 1;
\]

cocharge never exceeds \( \binom{n}{2} \).

Statement (iii) is clear.

For (iv), suppose \( u^{cc} = \cdots ab \cdots \leadsto v^{cc} = \cdots ba \cdots \) is a catabolism transformation. Whether \( a \) is corotated or inserted does not depend on what happens to \( b \) and similarly whether \( b \) is corotated or inserted does not depend on what happens to \( a \). Therefore, after \( a \) and \( b \) are presented, the resulting pair \((x, \nu)\) is the same for the algorithm applied to \( u \) and applied to \( v \), unless \( a \) and \( b \) are both corotated. In this case, the
result follows by induction since we can apply the same argument to the catabolism transformation \( \cdots a + 1 \ b + 1 \cdots \mapsto \cdots b + 1 \ a + 1 \cdots. \)

Given (iv), to show (v) it suffices to show that

\[
u^{cc} = \cdots bac \cdots \mapsto v^{cc} = \cdots bca \cdots \tag{3.6}
\]

implies \( F(u) = F(v) \) only in the case \( b = c, \ a = b - 1 \) (we must also check \( u^{cc} = \cdots acb \cdots \mapsto v^{cc} = \cdots cab \cdots \), where \( b = a \) and \( c = b + 1 \), but this is similar). To check this there are four cases depending on whether \( \nu_a - \nu_{a+1}, \ \nu_b - \nu_{b+1} \) are 0 or not (define \( \nu_0 = \infty \)), where \( \nu \) is the partition that \( c \) is presented to for the algorithm on \( u \). If they are both 0, then the result follows by induction from the Knuth transformation \( \cdots b + 1 \ a + 1 \ c + 1 \cdots \mapsto \cdots b + 1 \ c + 1 \ a + 1 \cdots. \) Otherwise, the algorithm run on \( u \) or \( v \) produces the same pair after \( a, b, \) and \( c \) are presented.

Given (v), at any stage in the algorithm we are free to replace \( x \) with something Knuth equivalent to it without changing the final output. Also, at any stage we may run the algorithm on anything Knuth equivalent to \( x \) rowword\((Z^0_\nu)\) and get the same output. Thus for (vi), run the algorithm on rowword\((P(u))\). The first steps of the algorithm corotate the numbers in the first row of \( P(u)^{cc} \) that are not 0. The next \( m \) steps are insertions of 0’s, where \( m \) is the number of 0’s in \( P(u)^{cc} \), also the largest integer such that \( P(u)(m) = Z^*_m \). Letting \((x, \nu)\) be the pair at this stage, we have \( P(x) = rcat_m(P(u)) \) and \( \nu = m \). The result follows from

\[
F(u) = (m, F(v)) = (m, ctype(P(v))) = ctype(P(u)), \tag{3.7}
\]
where \(v\) is obtained from \(x\) by subtracting a 1 from all numbers and then taking the standard word of this cocharge labeling. The leftmost equality is because the word \(x\) contains at most \(m\) 1’s, and the middle equality is the inductive statement \(F(v) = \text{ctype}(P(v)).\)

From now on, we write \(\text{ctype}(u)\) for \(\text{ctype}(P(u))\). From the theorem and remarks in the proof of (i), we obtain the following.

**Corollary 3.3.5.** Catabolizability is characterized as the statistic on standard words that is invariant under non-zero (co)rotations, catabolism transformations, and Knuth transformations, and satisfies \(\text{ctype}(\text{rowword}(Z^*_\lambda)) = \lambda\).

A similar proof to that of (vi) gives

**Corollary 3.3.6.** The catabolizability of a tableau \(T\) equals the column catabolizability of \(T\).

3.3.2. Corollary 3.3.5 allows us to characterize catabolizability in a similar way to the Greene’s Theorem interpretation of the shape of the insertion tableau of a word (see [7, Lemma A1.1.7]). This is reminiscent of the combinatorial description of two-sided cells in the affine Weyl group of type \(A\) (see, for instance, [41]), and also of the usual way of computing cocharge of semistandard words.

Let \(w\) be a standard word and \(z\) its cocharge labeling. Define \(\tilde{w} : \mathbb{Z}_{\leq n} \to \mathbb{Z}_{\geq 0}\) by \(i \mapsto z_{i+kn} + k\), where \(k\) is the unique integer so that \(i + kn \in [n]\). Also let \(\tilde{z}\) refer to
this same map. Let \( \hat{\cdot} : \mathbb{Z}_{\leq n} \to \mathbb{Z}/n\mathbb{Z} \) be the map sending an integer to its congruence class mod \( n \).

For \( k \in [n] \), a \( k \)-bounded chain of \( \tilde{w} \) is a sequence \( j = (j_{k'}, j_{k'-1}, \ldots, j_0) \) satisfying

(i) \( k' \leq k \),

(ii) \( j_{k'} < j_{k'-1} < \cdots < j_0 \),

(iii) \( \tilde{w}(j_i) = i \), for all \( i \in [0, k'] \),

(iv) \( \hat{j}_i \neq \hat{j}_{i'} \), for all \( i, i' \in [0, k'] \) with \( i \neq i' \).

The length of the chain \( j = (j_{k'}, j_{k'-1}, \ldots, j_0) \) is \( k' + 1 \). The underlying set of \( j \) is \( j^* := \{j_{k'}, j_{k'-1}, \ldots, j_0\} \).

A \( k \)-bounded chain family of \( \tilde{w} \) is a set \( \mathcal{A} = \{A_1, \ldots, A_l\} \) of \( k \)-bounded chains of \( \tilde{w} \) such that the subsets \( A_i \) of \( \mathbb{Z}/n\mathbb{Z} \) are disjoint. The support of \( \mathcal{A} \) is \( \bigcup_i A_i^* \subseteq \mathbb{Z}_{\leq n} \) and the size of a \( k \)-bounded chain family is the cardinality of its support. The maximum size of a \( k \)-bounded chain family of \( \tilde{w} \) is denoted \( I_k(\tilde{w}) \).

Let \( s_d : \mathbb{Z}_{\leq n} \to \mathbb{Z}_{\leq n} \) for \( d \in [n-1] \) be the affine versions of the simple reflections of \( S_n \): \( s_d \) transposes \( d + kn \) and \( d + 1 + kn \) for all \( k \leq 0 \). Certainly \( \tilde{u}s_d = \tilde{u}s_d \) if \( u \sim u s_d \) is cocharge-preserving. The action of \( S_n \) on chains is given by \( s_d(j) = (s_d(j_{k'}), s_d(j_{k'-1}), \ldots, s_d(j_0)) \).

Now assume that standard words \( u \) and \( v \) differ from each other by a simple transposition, \( v = u s_d \), so that the transformation \( u \sim v \) is cocharge-preserving.

**Lemma 3.3.7.** With \( u, v \) as above, suppose \( u_d^c \neq u_{d+1}^c + 1 \). Then, if \( j \) is a \( k \)-bounded
chain of $\tilde{u}$, then $s_d(j)$ is a $k$-bounded chain of $\tilde{v}$.

Proof. First observe

$$\tilde{v}(s_d(j)) = \tilde{u}(s_d(s_d(j))) = \tilde{u}(j). \quad (3.8)$$

The only way $s_d(j)$ is not a $k$-bounded chain of $\tilde{v}$ is if $j_i = j_{i-1} - 1 \equiv d \mod n$ for some $i \in [0, k]$, which is excluded by the assumption $u_d^{cc} \neq u_{d+1}^{cc} + 1$. \qed

Theorem 3.3.8. With the notation above,

$$\sum_{i=1}^{k+1} \text{c-type}(w)_i = I_k(\tilde{w}). \quad (3.9)$$

Proof. By Corollary 3.3.5, it suffices to check that $I_k(\tilde{w})$ is $\sum_{i=1}^{k+1} \lambda_i$ for rowword($Z^0_\lambda$), is invariant under non-zero (co)rotations, catabolism transformations, and Knuth transformations.

For $w = \text{rowword}(Z^0_\lambda)$, $I_k(\tilde{w}) = \sum_{i=1}^{k+1} \lambda_i$. There holds

$$|\{i : \tilde{w}(i) \leq k\}| = \sum_{i=1}^{k+1} \lambda_i$$

so that $I_k(\tilde{w})$ cannot possibly exceed this number. It is easy to exhibit a $k$-bounded chain family of $\tilde{w}$ of size $\sum_{i=1}^{k+1} \lambda_i$.

Non-zero (co)rotations. If $v$ is a non-zero corotation of $u$, then $\tilde{v}(i) = \tilde{u}(i - 1)$ and $\tilde{u}(n) \neq 0$. Thus the support of a $k$-bounded chain family of $\tilde{u}$ cannot contain $n$ and the notions of a $k$-bounded chain family for $\tilde{u}$ and $\tilde{v}$ differ only by shifting indices by 1.
Catabolism transformations. This follows from the special case of Lemma 3.3.7 in which $|u_{cc}^{c} - u_{cc}^{c+1}| \neq 1$.

Knuth transformations. We may assume that $u \rightsquigarrow v$ is a Knuth transformation with $v = us_d$ and $u_{cc}^{c} = a + 1$, $u_{cc}^{c+1} = a$, $u_{cc}^{c+2} = a$.

$$u = \cdots a + 1 a a \cdots \rightsquigarrow v = \cdots a a + 1 a \cdots$$  \hspace{1cm} (3.10)

By Lemma 3.3.7 and its proof, any $k$-bounded chain family of $\tilde{v}$ yields one of the same size for $\tilde{u}$, and any $k$-bounded chain family $A$ of $\tilde{u}$ yields one of the same size for $\tilde{v}$ provided any $j \in A$ does not satisfy $j_i = j_{i-1} - 1 \equiv d \mod n$ for some $i \in [0, k]$. If this is the case, then one checks that $\{s_d s_{d+1}(j) : j \in A\}$ is a $k$-bounded chain family of $\tilde{v}$, clearly of the same size as $A$.

This proof differs in an important way with that of the Greene’s Theorem interpretation of insertion tableau [7, Lemma A1.1.7]. None of the alterations of chains in the proof change the length of chains. This allows us to conclude the stronger statement:

**Theorem 3.3.9.** If $\text{ctype}(w) = \lambda$ then $\tilde{w}$ has an $\ell(\lambda)$-bounded chain family consisting of chains of lengths $\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda'}$.

Applying Lemma 3.3.7 and Theorem 3.3.8, we obtain

**Corollary 3.3.10.** If $u$ is a standard word with $u_{cc}^{c} > u_{cc}^{c+1}$ and the transformation $u \rightsquigarrow us_d$ is cocharge-preserving, then $\text{ctype}(u) \supseteq \text{ctype}(us_d)$. 
The next corollary is a strengthening of [37, Lemma 51].

**Corollary 3.3.11.** If $v$ is a non-zero corotation of $u$, then $\text{ctype}(v) = \text{ctype}(u)$. If $v$ is a zero corotation of $u$, then $\text{ctype}(v) \triangleleft \text{ctype}(u)$. 
Chapter 4

Paper III: A factorization theorem for affine Kazhdan-Lusztig basis elements

Abstract

The lowest two-sided cell of the extended affine Weyl group $W_e$ is the set \{\(w \in W_e : w = x \cdot w_0 \cdot z\), for some \(x, z \in W_e\}\}, denoted \(W(\nu)\). We prove that for any \(w \in W(\nu)\), the canonical basis element \(C'_w\) can be expressed as \(\frac{1}{n!} \chi_{\lambda}(Y) C'_{v_1 w_0} C'_{w_0 v_2}\), where \(\chi_{\lambda}(Y)\) is the character of the irreducible representation of highest weight \(\lambda\) in the Bernstein generators, and \(v_1\) and \(v_2^{-1}\) are what we call primitive elements. Primitive elements are naturally in bijection with elements of the finite Weyl group \(W_f \subseteq W_e\), thus
this theorem gives an expression for any $C'_w$, $w \in W(\nu)$ in terms of only finitely many canonical basis elements. After completing this paper, we realized that this result was first proved by Xi in [41]. The proof given here is significantly different and somewhat longer than Xi’s, however our proof has the advantage of being mostly self-contained, while Xi’s makes use of results of Lusztig from [30] and Cells in affine Weyl groups I-IV and the positivity of Kazhdan-Lusztig coefficients.

4.1 Introduction

This work came about from a desire to better understand the polynomial ring $\mathbb{C}[y_1, \ldots, y_n]$ as an $\mathbb{C}S_n$-module in a way compatible with the multiplicative structure of the polynomial ring. The hope was that a quantization of the polynomial ring and its $S_n$ action would rigidify the structure and make a combinatorial description more transparent. This hope has largely been realized by the type $A$ extended affine Hecke algebra and its canonical basis and, indeed, this is the subject of the forthcoming paper [4].

While the use of crystal bases of quantum groups to do tableau combinatorics is well established and used prolifically, the connection between combinatorics and canonical bases of Hecke algebras is less developed. This may be because such combinatorics involves computing the weights $\mu$ of $W$-graph edges, which is difficult, or finding a way to determine cells that avoids such computation. The main theorem of this paper simplifies the description of some canonical basis elements, the combina-
torial implications of which will be discussed in [4]. We show that the canonical basis elements $C'_w$ of the extended affine Hecke algebra $H(W)$, for $w$ in the two-sided cell $W_{(\nu)}$, can be expressed in terms of a finite subset of the canonical basis and symmetric polynomials in the Bernstein basis. After completing this paper, we realized that this result was first proved by Xi in [41]. This paper gives a different proof that is somewhat longer but relies on less machinery than Xi’s.

This theorem allows us to construct a $q$-analogue of the ring of coinvariants $\mathbb{C}[y_1, \ldots, y_n]/(e_1, \ldots, e_n)$ with a canonical basis (see [4]), but this application does not use the full strength of this theorem. It may also be possible to use the theorem to construct a $q$-analogue of the ring of coinvariants in other types. In addition, this theorem may shed some light on computing Kazhdan-Lusztig polynomials, although this is not our main focus.

The remainder of this paper is organized as follows. In §4.2 we introduce the extended affine Weyl group and its Hecke algebra along with the Bernstein presentation and canonical bases. In §4.3 we define the primitive elements of an extended affine Weyl group associated to a simply connected reductive algebraic group $G$ over $\mathbb{C}$, which are in bijection with elements of the associated finite Weyl group $W_f$. Finally, §4.4 is devoted to a proof of our main result (Corollary 4.4.2), which expresses a canonical basis element $C'_w$, $w \in W_{(\nu)}$ in terms of the $C'_x$ for $x$ primitive.
4.2 Weyl groups and Hecke algebras

Here we introduce Weyl groups and Hecke algebras in full generality and then specialize to the affine case. We also recall the important presentation of the extended affine Hecke algebra due to Bernstein and state a crucial theorem of Lusztig expressing certain canonical basis element in terms of the Bernstein generators. This material is explained more fully in [14], and [20] gives a good exposition of the more basic notions.

4.2.1. A root system $(X, (\alpha_i), (\alpha_i^\vee))$ consists of a finite-rank free abelian group $X$, its dual $X^\vee := \text{Hom}(X, \mathbb{Z})$, simple roots $\alpha_1, \ldots, \alpha_n \in X$, and simple coroots $\alpha_1^\vee, \ldots, \alpha_n^\vee \in X^\vee$ such that the $n \times n$ matrix with $(i, j)$-th entry $\langle \alpha_j, \alpha_i^\vee \rangle$ is a generalized Cartan matrix. Assume that the root system is non-degenerate, i.e. the simple roots are linearly independent.

Let $W$ be the Weyl group of this root system and $S = \{s_1, \ldots, s_n\}$ the set of simple reflections. The group $W$ is the subgroup of automorphisms of the lattice $X$ (and of $X^\vee$) generated by the reflections $s_i$. Let $R, R_+, R_-, Q$ be the roots, positive roots, negative roots, and root lattice.

The dominant weights $X_+$ and the dominant regular weights $X_{++}$ are the cones
in $X$ given by

\[
X_+ = \{ \lambda \in X : \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i \},
\]

\[
X_{++} = \{ \lambda \in X : \langle \lambda, \alpha_i^\vee \rangle \geq 1 \text{ for all } i \}.
\] (4.1)

The pair $(W, S)$ is a Coxeter group with length function $\ell$ and Bruhat order $\leq$. The length $\ell(w)$ of $w$ is the minimal $\ell$ such that $w = s_1 \ldots s_l$ for some $s_i \in S$, also equal to $|R_\cap w(R_+)|$. If $\ell(uv) = \ell(u) + \ell(v)$, then $uv = u \cdot v$ is a reduced factorization.

The notation $L(w), R(w)$ will denote the left and right descent sets of $w$.

It is often convenient to use the geometry and topology of the real vector space $X^\vee_\mathbb{R} := X^\vee \otimes \mathbb{Z} \mathbb{R}$. This space contains the root hyperplanes $H_\alpha = \{ x \in X^\vee_\mathbb{R} : \langle \alpha, x \rangle = 0 \}$. The connected components of $X^\vee_\mathbb{R} - \bigcup_\alpha H_\alpha$ are Weyl chambers and the dominant Weyl chamber is $C_0 = \{ x \in X^\vee_\mathbb{R} : \langle \alpha, x \rangle > 0 \text{ for all } \alpha \in R_+ \}$. Its closure is a fundamental domain for the action of $W$ on $X^\vee_\mathbb{R}$.

Certain relations in the Bruhat order may be understood in several ways. The following are equivalent:

(i) $s_\alpha w < w$.

(ii) one of $\alpha$ and $w^{-1}(\alpha)$ is in $R_+$ and the other is in $R_-$.

(iii) $C_0$ and $w(C_0)$ are on opposite sides of $H_\alpha$.

For the equivalence of (i) and (ii), see [20, §5.7], while the equivalence of (ii) and (iii) follows from the identity $\langle \alpha, w(C_0) \rangle = \langle w^{-1}(\alpha), C_0 \rangle$, where for a set $Z \subseteq X^\vee_\mathbb{R}$, $\langle Z, \alpha \rangle$ is defined to be the set $\{ \langle z, \alpha \rangle : z \in Z \}$. 

\[115\]
4.2.2. For any $J \subseteq S$, the parabolic subgroup $W_J$ is the subgroup of $W$ generated by $J$. Each left (resp. right) coset of $wW_J$ (resp. $W_Jw$) contains an unique element of minimal length called a minimal coset representative. The set of all such elements is denoted $W^J$ (resp. $^JW$). For any $w \in W$, define $w^J, {}^Jw$ by

$$w = w^J \cdot {}^Jw, \ w^J \in W^J, \ {}^Jw \in W_J.$$ \hspace{1cm} (4.3)

Similarly, define $w_J, {}^Jw$ by

$$w = w_J \cdot {}^Jw, \ w_J \in W_J, \ {}^Jw \in ^JW.$$ \hspace{1cm} (4.4)

4.2.3. Any finite Weyl group $W_f$ contains a unique longest element $w_0$. The action of $w_0$ on $R$ satisfies $w_0(\alpha_i) = -\alpha_{d(i)}$ for some automorphism $d$ of the Dynkin diagram. Left (or right) multiplication by $w_0$ induces a Bruhat order-reversing involution on $W_f$ and therefore takes elements of length $l$ to elements of length $\ell(w_0) - l$. In particular, it takes the elements of length $\ell(w_0) - 1$ to the simple reflections. Conjugation by $w_0$ leaves stable the set of simple reflections $S$ and acts on $S$ by the automorphism $d$.

4.2.4. Let $(Y, \alpha'_i, \alpha''_i), i \in [n]$ be the finite root system specifying a reductive algebraic group $G$ over $\mathbb{C}$. Denote the Weyl group, simple reflections, roots, and root lattice by $W_f, S_f, R_f, Q_f$. The extended affine Weyl group is the semidirect product

$$W_e := Y \rtimes W_f.$$ 

Elements of $Y \subseteq W_e$ will be denoted by the multiplicative notation $y^\lambda, \lambda \in Y$. 


The group \(W_e\) is also equal to \(\Pi \ltimes W_a\), where \(W_a\) is the Weyl group of an affine root system we will now construct and \(\Pi\) is an abelian group. Let \(X = Y^\vee \oplus \mathbb{Z}\) and \(\delta\) be a generator of \(\mathbb{Z}\). The pairing of \(X\) and \(X^\vee\) is obtained by extending the pairing of \(Y\) and \(Y^\vee\) together with \(\langle \delta, Y \rangle = 0\). Let \(\phi'\) be the dominant short root of \((Y, \alpha'_i, \alpha'_i^\vee)\) and \(\theta = \phi'^\vee\) the highest coroot. For \(i \neq 0\) put \(\alpha_i = \alpha_i'^\vee\) and \(\alpha_i^\vee = \alpha'_i\); put \(\alpha_0 = \delta - \theta\) and \(\alpha_0^\vee = -\phi'\). Then \((X, \alpha_i, \alpha_i^\vee), i \in [0, n]\) is an affine root system. Let \(W_a\) denote its Weyl group and use the notation of §4.2.1 for its roots, root lattice, etc. The roots \(R\) may be expressed in terms of the coroots \(R_i^\vee\) of the system \((Y, \alpha'_i, \alpha'_i^\vee)\) as \(R = R_i^\vee + \mathbb{Z}\delta\), and the positive roots \(R_+\) as \(R_+ = (R_i^\vee + \mathbb{Z}_{>0}\delta) \cup R_i^\vee\).

The abelian group \(Q'_f\) is realized as a subgroup of \(W_a\) acting on \(X\) and \(X^\vee\) by translations: for \(\beta' = \alpha'_i \in R'_f \subseteq Q'_f\) (\(i \in [n]\)), define \(y^{\beta'} = s_{\alpha_i^\vee} s_{\alpha_i'^\vee}\). Then
\[
y^{\beta'}(x) = x - \langle x, \beta' \rangle \delta, \quad x \in X, \tag{4.5}
\]
\[
y^{\beta'}(x^\vee) = x^\vee + \langle \delta, x^\vee \rangle \beta', \quad x^\vee \in X^\vee, \tag{4.6}
\]
and for any \(\beta' \in Q'_f\), these equations define an action of \(Q'_f\) on \(X\) and \(X^\vee\). This action of \(Q'_f\) by translations extends to an action of \(Y\), which realizes \(W_e\) as a subgroup of the automorphisms of \(X\) and \(X^\vee\). The inclusion \(W_a \hookrightarrow W_e\) is given on simple reflections by \(s_i \mapsto s_i\) for \(i \neq 0\) and \(s_0 \mapsto y^{\phi'} s_{\phi'}\). The subgroup \(W_a\) is normal in \(W_e\) with quotient \(W_e/W_a \cong Y/Q'_f\), denoted \(\Pi\). And, as was our goal, we have \(W_e = \Pi \ltimes W_a\).

Let \(H = \{x \in X_R^\vee : \langle \delta, x \rangle = 1\}\) be the level 1 plane. It follows from (4.6) that the action of \(W_e\) on \(X^\vee\) restricts to an action of \(W_e\) on \(H\). The space \(H\) contains the affine hyperplanes \(h_\alpha := H_\alpha \cap H, \alpha \in R\). The connected components of \(H - \bigcup_{\alpha \in R} h_\alpha\) are
alcoves, and the basic alcove is $A_0 = H \cap C_0$. Its closure is a fundamental domain for the action of $W_o$ on $H$, and its stabilizer for the action of $W_e$ is $\Pi$. We also distinguish the finite dominant Weyl chamber $C_{f0} = \{ x \in H : \langle \alpha, x \rangle > 0 \text{ for all } \alpha \in R_f^\vee \}$. 

A basic fact we will make frequent use of is that any element $w \in W_e$ can be written uniquely as a product 

$$w = u \cdot y^\beta v, \quad (4.7)$$

such that $u, v \in W_f$, $\beta \in Y$, and $y^\beta v$ is minimal in its right coset $W_f w$. This last condition implies $u \cdot (y^\beta v)$ is a reduced factorization and $\beta \in Y_+$. In terms of the alcove picture, $y^\beta v$ takes $A_0$ to $y^\beta v(A_0) \subseteq C_{f0}$ and then $u$ moves this alcove into $u(C_{f0})$. Note that $\beta \in Y_{++}$ forces $y^\beta v$ to be minimal in $W_f y^\beta v$ for any $v \in W_f$.

The group $W_e$ is an extended Coxeter group. The length function and partial order on $W$ extend to $W_e$: $\ell(\pi v) = \ell(v)$, and $\pi v \leq \pi' v'$ if and only if $\pi = \pi'$ and $v \leq v'$, where $\pi, \pi' \in \Pi$, $v, v' \in W$. The definitions of left and right descent sets and reduced factorization carry over identically.

4.2.5. We will give examples in type $A$ throughout the paper, and now fix notation for this special case. See [4, 42] for a more extensive treatment of this case.

For $G = GL_n$, the lattices $Y$ and $Y^\vee$ are equal to $\mathbb{Z}^n$ and $\alpha'_i = \epsilon_i - \epsilon_{i+1}$, $\alpha_i^\vee = \epsilon_i^\vee - \epsilon_{i+1}^\vee$, where $\epsilon_i$ and $\epsilon_i^\vee$ are the standard basis vectors of $Y$ and $Y^\vee$. The finite Weyl group $W_f$ is $S_n$ and the subgroup $\Pi$ of $W_e$ is $\mathbb{Z}$. The element $\pi = y_1 s_1 s_2 \ldots s_{n-1} \in \Pi$ is a generator of $\Pi$. This satisfies the relation $\pi s_i = s_{i+1} \pi$, where the subscripts of
the $s_i$ are taken mod $n$.

For $G = SL_n$, the lattice $Y$ is the quotient of the weight lattice for $GL_n$ by $\mathbb{Z}\varepsilon$, where $\varepsilon = \epsilon_1 + \cdots + \epsilon_n$ and the simple roots are the images of those for $GL_n$. The dual lattice $Y^\vee$ is the coroot lattice of $GL_n$, and the coroots are the same as those for $GL_n$. The finite Weyl group $W_f$ is the same as for $GL_n$ and the subgroup $\Pi = \langle \pi \rangle$ is that for $GL_n$ with the additional relation $\pi^n = id$.

Another description of $W_e$ for $GL_n$, due to Lusztig, identifies it with the group of permutations $w : \mathbb{Z} \to \mathbb{Z}$ satisfying $w(i + n) = w(i) + n$ and $\sum_{i=1}^n (w(i) - i) \equiv 0 \mod n$. The identification takes $s_i$ to the permutation transposing $i + kn$ and $i + 1 + kn$ for all $k \in \mathbb{Z}$, and takes $\pi$ to the permutation $k \mapsto k + 1$ for all $k \in \mathbb{Z}$. We take the convention of specifying the permutation of an element $w \in W_e$ by the word

$$w(1) w(2) \cdots w(n).$$

We refer to this as the word of $w$, also written as $w_1 w_2 \cdots w_n$; this is understood to be part of an infinite word so that $w_i = i - \hat{\iota} + \hat{w}_i$, where $\hat{\iota}$ denotes the element of $[n]$ congruent to $i \mod n$. For example, if $n = 4$ and $w = \pi^2 s_2 s_0 s_1$, then the word of $w$ is $5 2 4 7$.

The extended affine Weyl group for $SL_n$ may be obtained from this permutation group by quotienting by the subgroup generated by $\pi^n = n + 1 n + 2 \cdots 2n$.

4.2.6. Let $A = \mathbb{Z}[u, u^{-1}]$ be the ring of Laurent polynomials in the indeterminate $u$ and $A^- = \mathbb{Z}[u^{-1}]$. The Hecke algebra $\mathcal{H}(W)$ of a (extended) Coxeter
group \( (W, S) \) is the free \( A \)-module with basis \( \{ T_w : w \in W \} \) and relations generated by

\[
T_u T_v = T_{uv} \quad \text{if} \ uv = u \cdot v \text{ is a reduced factorization}
\]

(4.8)

\[
(T_s - u)(T_s + u^{-1}) = 0 \quad \text{if} \ s \in S.
\]

The bar-involution, \( \overline{\cdot} \), of \( \mathcal{H} \) is the additive map from \( \mathcal{H} \) to itself extending the involution \( \overline{\cdot} \colon A \to A \) given by \( q \mapsto q^{-1} \) and satisfying \( \overline{T_w} = T_w^{-1} \). Define the lattice

\[
\mathcal{L} = A^{-\langle T_w : w \in W \rangle}.
\]

**Theorem 4.2.1** (Kazhdan-Lusztig [22]). For each \( w \in W \), there is a unique element \( C'_w \in \mathcal{H}(W) \) such that \( \overline{C'_w} = C'_w \) and \( C'_w \) is congruent to \( T_w \mod u^{-1}\mathcal{L} \).

The set \( \{ C'_w : w \in W \} \) is an \( A \)-basis for \( \mathcal{H}(W) \) called the canonical basis or Kazhdan-Lusztig basis.

The coefficients of the \( C'_w \)'s in terms of the \( T \)'s are the Kazhdan-Lusztig polynomials \( P'_{x,w} \):

\[
C'_w = \sum_{x \in W} P'_{x,w} T_x.
\]

(4.9)

(Our \( P'_{x,w} \) are equal to \( q^{(\ell(x)-\ell(w))/2} P_{x,w} \), where \( P_{x,w} \) are the polynomials defined in [22].)

4.2.7. The extended affine Hecke algebra \( \widehat{\mathcal{H}} \) is the Hecke algebra \( \mathcal{H}(W_e) \). Just as the extended affine Weyl group \( W_e \) can be realized both as \( \Pi \ltimes W_a \) and \( W_f \ltimes Y \), the extended affine Hecke algebra can be realized in two analogous ways:
The algebra $\hat{H}$ contains the Hecke algebra $H(W_a)$ and is isomorphic to the twisted group algebra $\Pi \cdot H(W_a)$ generated by $\Pi$ and $H(W_a)$ with relations generated by

$$\pi T_w = T_{\pi w \pi^{-1}}$$

for $\pi \in \Pi$, $w \in W_a$.

There is also a presentation of $\hat{H}$ due to Bernstein. For any $\lambda \in Y$ there exist $\mu, \nu \in Y_+$ such that $\lambda = \mu - \nu$. Define

$$Y^\lambda := T_{y^\mu} (T_{y^\nu})^{-1},$$

which is independent of the choice of $\mu$ and $\nu$. The algebra $\hat{H}$ is the free $A$-module with basis $\{Y^\lambda T_w : w \in W_f, \lambda \in Y\}$ and relations generated by

$$T_i Y^\lambda = Y^\lambda T_i$$

if $\langle \lambda, \alpha_i^{\vee} \rangle = 0$,

$$T_i^{-1} Y^\lambda T_i^{-1} = Y^{s_i(\lambda)}$$

if $\langle \lambda, \alpha_i^{\vee} \rangle = 1$,

$$(T_i - u)(T_i + u^{-1}) = 0$$

for all $i \in [n]$, where $T_i := T_{s_i}$.

4.2.8. Given $\lambda \in Y_+$, let $\chi_\lambda(Y) = \sum_{\mu} d_{\mu,\lambda} Y^\mu$, where $d_{\mu,\lambda}$ is the dimension of the $\mu$-weight space of the irreducible representation of Lie($G$) of highest weight $\lambda$.

**Theorem 4.2.2** (Lusztig [31, Proposition 8.6]). For $\lambda \in Y_+$, the canonical basis element $C'_{w_0 Y_\lambda}$ can be expressed in terms of the Bernstein generators as

$$C'_{w_0 Y_\lambda} = \chi_\lambda(Y) C'_{w_0} = C'_{w_0} \chi_\lambda(Y).$$
4.3 Primitive elements

4.3.1. The primitive elements of $W_e$ that we are about to define are most natural in the case $G$ is simply connected, so let us assume this from now on. This is equivalent to the assumption that fundamental weights $\varpi_i, i \in [n]$ exist and are $\mathbb{Z}$-basis for $Y$. (The weight $\varpi_i$ is defined by $\langle \varpi_i, \alpha_j' \rangle$ equals 1 if $i = j$ and 0 otherwise.)

We give three descriptions of primitive elements and show that they are equivalent in Proposition 4.3.1. The first description is a geometric one from [30]. A box is a connected component of $H - \bigcup_{i \in [n], k \in \mathbb{Z}} h_{\alpha_i + k\delta}$. We denote by $B_0$ the box containing $A_0$. It is bounded by the hyperplanes $h_{\alpha_i}$ and $h_{\alpha_i - \delta}$ for $i \in [n]$.

The action of $Y$ on $H$ by translations gives the action $y^\lambda(h_\alpha) = h_{\alpha - (\alpha, \lambda)\delta}$ of $Y$ on hyperplanes. This further gives an action of $Y$ on boxes. Put $\lambda = \sum_{i=1}^n c_i \varpi_i$, $c_i \in \mathbb{Z}$, and define $B_\lambda = y^\lambda(B_0)$. It is the box that is the bounded component of

$$H - \bigcup_i h_{\alpha_i - c_i \delta} - \bigcup_i h_{\alpha_i - (c_i + 1)\delta}.$$ 

Thus our assumption that the fundamental weights are a basis for $Y$ implies that $Y$ acts simply transitively on boxes. Additionally, the $B_\lambda$ for $\lambda \in Y^+$ are the connected components of $C_{t_0} - \bigcup_{i \in [n], k \in \mathbb{Z}} h_{\alpha_i + k\delta}$.

Set $\rho = \sum_{i=1}^n \varpi_i$. One checks that $w_0(B_0) = B_{-\rho}$.

Given any $v \in W_f$, let $J = R(v)$. The element $v^{-1}$ takes the basic alcove $A_0$ to some alcove $v^{-1}(A_0)$ whose closure contains the origin. There is a unique minimal
\[ \lambda \in Y_+ \text{ such that } y^{-\lambda}v^{-1}(A_0) \subseteq w_0(C_0). \] Minimality implies that \( y^{-\lambda}v^{-1}(A_0) \subseteq \mathbf{B}_{-\rho}. \) It is a consequence of the next proposition that this \( \lambda \) is given by

\[ \lambda = \sum_{s_i \in S_f \setminus J} \varpi_i. \]

Now define \( w \) to be

\[ v \cdot y^\lambda = y^{v(\lambda)}v, \]

which is maximal in its left coset \( wW_f. \) For example, for \( G = SL_5 \) (see §4.2.5), if \( v = 5 2 3 1 4, \) then \( J = \{s_1, s_3\}, \lambda = (2, 2, 1, 1, 0), \) and \( v(\lambda) = (1, 2, 1, 0, 2). \)

**Proposition 4.3.1.** The following are equivalent for an element \( w \in W_e: \)

(i) \( w^{-1}(A_0) \subseteq \mathbf{B}_0. \)

(ii) \( w(\alpha_i) \in (R_- + \delta) \cap R_+ = (R_f^{\vee} + \delta) \cup R_f^{\vee} \) for \( i \in [n]. \)

(iii) \( w = vy^\lambda w_0 \) such that \( v \in W_f, \) and \( \lambda = \sum_{s_i \in S_f \setminus J} \varpi_i, \) where \( J = R(v) \) as above.

**Proof.** The equivalence of (i) and (ii) is proved by observing that each of the statements below is equivalent to the next. The equivalence of (a) and (b) follows from (4.2).

(a) \( A_0 \) and \( w^{-1}(A_0) \) are on the same side of \( h_{\alpha_i + k\delta} \) for all \( i \in [n], k \in \mathbb{Z}. \)

(b) \( (w(\alpha_i + k\delta) \in R_+ \text{ and } \alpha_i + k\delta \in R_+) \) or \( (w(\alpha_i + k\delta) \in R_- \text{ and } \alpha_i + k\delta \in R_-) \) for all \( i \in [n], k \in \mathbb{Z}. \)
(c) \((w(\alpha_i) + k\delta \in R_+ \text{ and } k \geq 0) \) or \((w(\alpha_i) + k\delta \in R_- \text{ and } k < 0)\) for all \(i \in [n], k \in \mathbb{Z}\).

(d) \(w(\alpha_i) \in R_+ \text{ and } w(\alpha_i) - \delta \in R_- \) for all \(i \in [n]\).

(e) \(w(\alpha_i) \in (R_- + \delta) \cap R_+ \) for all \(i \in [n]\).

To see (iii) implies (ii), for any \(i \in [n]\), compute

\[
vy^\lambda w_0(\alpha_i) = vy^\lambda(-\alpha_j) = v(-\alpha_j - \langle \lambda, -\alpha_j \rangle \delta) = v(-\alpha_j) + \begin{cases} 
\delta & \text{if } j \not\in J, \\
0 & \text{if } j \in J 
\end{cases},
\]

where \(j = d(i)\) (with \(d\) as in §4.2.3 so that \(w_0(\alpha_i) = -\alpha_j\)). The condition \(v(-\alpha_j) \in R_{f+}^{\vee}\) is equivalent to \(j \in R(v) = J\), hence \(vy^\lambda w_0(\alpha_i) \in (R_{f+}^{\vee} + \delta) \cup R_{f+}^{\vee}\).

Now assume \(w\) satisfies (ii). Put \(J = \{s_j : w_0(-\alpha_j) \in R_{f+}^{\vee}\}\) and define \(\lambda := \sum_{s_i \in S_f \setminus J} \varpi_i\). Then, define \(v := w_0 y^{-\lambda}\) and compute

\[
w_0 y^{-\lambda}(-\alpha_j) = w_0(-\alpha_j - \langle -\lambda, -\alpha_j \rangle \delta) = w_0(-\alpha_j) - \begin{cases} 
\delta & \text{if } j \not\in J, \\
0 & \text{if } j \in J 
\end{cases}
\]

where, as above, \(i = d(j)\). By the assumption (ii) and definition of \(J\), \(v(-\alpha_j) \in R_{f}^{\vee}\) for \(j \in [n]\). Writing \(v = uy^\mu, u \in W_f, \mu \in Y\), and using that the fundamental weights form a basis for \(Y\), we may conclude that \(\mu = 0\), i.e., \(v \in W_f\). Also, \(R(v) = \{s_j : v(-\alpha_j) \in R_{f+}^{\vee}\} = J\), so \(w = vy^\lambda w_0\) with the conditions in (iii) satisfied.

\[\square\]

**Definition 4.3.2.** A \(w \in W_e\) satisfying any (all) of the preceding conditions is called primitive.

**Proposition 4.3.3.** For \(G = SL_n\), \(x \in W_e\) is primitive if and only if \(1 \leq x_{i+1} - x_i \leq n\) for \(i = 1, \ldots, n - 1\), where \(x_1 x_2 \ldots x_{n-1} x_n\) is the word of \(x\) (see §4.2.5).
Proof. The word of \( x \) and \( x \) as an automorphism of \( X \) are related by

\[
x(\epsilon_i^\vee) = \epsilon_i^\vee + \left( \frac{x_i - \hat{x}_i}{n} \right) \delta.
\]

Define a function \( \tau : \mathbb{R} \rightarrow \mathbb{Z} \) by \( \alpha \mapsto \langle \alpha, \rho + n\Lambda^\vee \rangle \), where \( \Lambda^\vee \) is the generator of \( \mathbb{Z} \) in \( X^\vee = Y \oplus \mathbb{Z} \) satisfying \( \langle \delta, \Lambda^\vee \rangle = 1 \). This function takes \( \alpha_i \) to 1 for \( i \in [n] \). The inverse image of \([n]\) under this map is \((R^\vee_{f-} + \delta) \cup R^\vee_{f+}\). Then

\[
\tau(x(\epsilon_i^\vee)) = n - \hat{x}_i + \frac{x_i - \hat{x}_i}{n} n = n - x_i,
\]

so \( \tau(x(\alpha_i)) = x_{i+1} - x_i \) is in \([n]\) if and only if \( x(\alpha_i) \in (R^\vee_{f-} + \delta) \cup R^\vee_{f+} \). \( \square \)

Example 4.3.4. For \( G = SL_4 \), the primitive elements of \( W_e \), expressed as products of simple reflections (top lines) and words (bottom lines), are

\[
id
1 2 3 4
\]

\[
\pi s_1 s_0
1 2 4 7 1 3 4 6 2 3 4 5
\]

\[
\pi^2 s_1 s_3 s_0
1 3 6 8 1 3 5 7 2 3 5 8 2 4 5 7 3 4 5 6
\]

\[
\pi^3 s_2 s_1 s_3 s_0
1 4 7 10 2 4 7 9 3 4 6 9 2 5 7 8 3 5 6 8 4 5 6 7
\]

\[
\pi^4 s_1 s_3 s_0
3 5 8 10 2 6 8 11 3 6 8 9 4 5 7 10 4 6 7 9
\]

\[
\pi^5 s_2 s_1 s_3 s_0
3 6 9 12 4 6 9 11 4 7 9 10
\]

\[
\pi^6 s_2 s_1 s_3 s_0
4 7 10 13
\]
4.3.2. Let us now establish some properties of primitive elements. The first two are basic and the third is more substantial.

**Proposition 4.3.5.** For any $\pi \in \Pi$, $w \in W_e$ is primitive if and only if $\pi w$ is.

*Proof.* Given any $i \in [n]$, $w$ primitive implies $w(\alpha_i) \in R_+$ and $w(\alpha_i - \delta) \in R_-$. Since $\pi(R_+) = R_+$ (and $\pi(R_-) = R_-$), $\pi w(\alpha_i) \in R_+$ and $\pi w(\alpha_i - \delta) \in R_-$. In other words, $\pi w(\alpha_i) \in (R_- + \delta) \cap R_+$, so $\pi w$ is primitive. The same argument with $\pi^{-1}$ in place of $\pi$ and $\pi w$ in place of $w$ gives the other direction. 

**Proposition 4.3.6.** If $x \cdot y$ is a reduced factorization of a primitive element, then $y$ is primitive.

*Proof.* For $i \in [n]$, $y(\alpha_i) \in R_+$ because if not, $y(\alpha_i) \in R_-$ and $x \cdot y$ reduced implies by Lemma 4.4.3 that $xy(\alpha_i) \in R_-$, contradiction. Also, $x \cdot y$ reduced, $\alpha_i - \delta \in R_-$, and $xy(\alpha_i - \delta) \in R_-$ implies $y(\alpha_i - \delta) \in R_-$. Thus $y(\alpha_i) \in (R_- + \delta) \cap R_+$ shows $y$ is primitive.

The lowest two-sided cell of $W_e$ is the set $W_{(\nu)} = \{ w \in W_e : w = x \cdot w_0 \cdot z, \text{ for some } x, z \in W_e \}$ (see [41, 42]).

**Proposition 4.3.7.** For any $w \in W_{(\nu)}$, there exists a unique expression for $w$ of the form

$$w = v_1 \cdot w_0 y^\lambda \cdot v_2$$

where $v_1, v_2^{-1}$ are primitive and $\lambda \in Y_+$. 
Proof. Write \( w = x \cdot w_0 \cdot z \). By (4.7), \( z(A_0) \subseteq C_{f_0} \) so \( z(A_0) \subseteq B_{\eta} \) for some \( \eta \in Y_+ \). Thus by Proposition 4.3.1 and the discussion preceding it, \( z = y^\eta v_2 \) for some primitive element \( v_2^{-1} \). Similarly, \( x = v_1 y^{-\mu} \) for \( v_1 \) primitive and \( \mu \in Y_+ \). Next, we can write

\[
y^{-\mu} w_0 y^\eta = w_0 y^{w_0(-\mu)+\eta}.
\]

Setting \( \lambda = w_0(-\mu) + \eta \) and noting that \( w_0(-\mu) \in Y_+ \) yields the desired expression \( w = v_1 w_0 y^\lambda v_2 \), with \( v_1, v_2^{-1} \) primitive. To see that this factorization is reduced, use that \( v_1 \cdot w_0 \) and \( w_0 \cdot y^\lambda v_2 \) are reduced and Proposition 4.4.5 (see §4.4.1 below) to conclude that \( v_1 \cdot w_0 y^\lambda v_2 \) is reduced. Similarly, by rewriting \( w_0 y^\lambda = y^{w_0(\lambda)} w_0 \) we may conclude \( v_1 y^{w_0(\lambda)} w_0 \cdot v_2 \) is reduced.

For uniqueness, suppose that \( v_1 \cdot w_0 y^\lambda \cdot v_2 = w = v_1' \cdot w_0 y^{\lambda'} \cdot v_2' \) for \( v_1, v_1', v_2^{-1}, v_2'^{-1} \) primitive and \( \lambda, \lambda' \in Y_+ \). Put \( v_1 w_0 = u_1 y^\mu \), \( v_1' w_0 = u_1' y^{\mu'} \) with \( u_1, u_1' \in W_f \), \( \mu, \mu' \in Y_+ \) as in Proposition 4.3.1 (iii). Then \( w(A_0) = u_1 y^{\mu + \lambda} v_2(A_0) \subseteq u_1(C_{f_0}) \) and also \( u_1'(A_0) = u_1' y^{\mu' + \lambda'} v_2'(A_0) \subseteq u_1'(C_{f_0}) \). Thus \( u_1 = u_1' \), implying \( v_1 = v_1' \). Similarly, \( v_2 = v_2' \) and then \( \lambda = \lambda' \) follows easily.

\[ \square \]

### 4.4 The factorization theorem

**4.4.1.** This section is devoted to a proof of our main theorem, which we now state. Suppose \( v \in W_e \) such that \( v \cdot w_0 \) is reduced. It is well-known that

\[
C'_{vw_0} = \sum_x \overrightarrow{P}_{x,v} T_x C'_{w_0},
\]

where the sum is over \( x \leq v \) such that \( x \cdot w_0 \) is reduced and \( \overrightarrow{P}_{x,v} := P_{xw_0,vw_0} \) (see [1] for the more general construction of which this is a special case). Define...
\[ \vec{C}_v' = \sum_x \vec{P}_{x,v} T_x, \] with the sum over the same \( x \) as above. Similarly, for \( v \) such that \( w_0 \cdot v \) is reduced, define \( \vec{P}_{x,v} = P'_{w_0 x, w_0 v} \) and \( \vec{C}_v' = \sum_x \vec{P}_{x,v} T_x. \)

**Theorem 4.4.1.** Let \( v_1, v_2 \in W_e \) such that \( v_1 \cdot w_0 \) and \( w_0 \cdot v_2 \) are reduced factorizations. If \( v_1 \) is primitive, then \( C'_{v_1 w_0 v_2} = \vec{C}_v' \chi_\lambda(Y) \vec{C}_v' \). 

This theorem together with Theorem 4.2.2 and Proposition 4.3.7 have the following powerful corollary for any \( w \) in the lowest two-sided cell of \( W_e \). This is phrased a little differently from the result stated in the abstract, but is equivalent to it.

**Corollary 4.4.2.** For \( w \in W_\nu \) and with \( w = v_1 \cdot w_0 y^\lambda \cdot v_2 \) as in Proposition 4.3.7, we have the factorization

\[ C'_w = \chi_\lambda(Y) \vec{C}_v' \vec{C}_w' \vec{C}_v'. \] (4.11)

We will need the two general lemmas about Coxeter groups that follow. Their proofs are straightforward and are given in the appendix.

**Lemma 4.4.3.** For any \( x, y \in W_e \), \( x \cdot y \) is a reduced factorization if and only if \( x(y(R_+) \cap R_-) \subseteq R_- \).

The next lemma holds for any Weyl group, but it is stated in the less general setting in which it will be applied.

**Lemma 4.4.4.** Suppose \( a, y \in W_f \) and \( \alpha \in R_f^{\nu-} \).

(i) If \( s_\alpha a < a \), then \( s_\alpha ay < ay \iff a^{-1}s_\alpha ay > y \).

(ii) If \( s_\alpha a > a \), then \( s_\alpha ay > ay \iff a^{-1}s_\alpha ay > y \).
The next proposition is the crux of the proof of Theorem 4.4.1 and is the only place the primitive assumption is used directly.

**Proposition 4.4.5.** Let \( x, z \in W_e \) such that \( x \cdot w_0 \) and \( w_0 \cdot z \) are reduced factorizations. Then \( x \cdot w_0 \cdot z \) is a reduced factorization. Furthermore, if \( x \) is primitive, then \( x \cdot y \cdot z \) is a reduced factorization for every \( y \in W_f \) with \( \ell(y) = \ell(w_0) - 1 \).

**Proof.** Given \( \alpha \in R_+ \), suppose \( z(\alpha) \in R_- \). We will show that \( xw_0z(\alpha) \) remains in \( R_- \), which by Lemma 4.4.3 shows \( xw_0 \cdot z \) is a reduced factorization. Write \( z(\alpha) = \alpha' - k\delta \), for some \( \alpha' \in R_f^{\vee} \) and \( k \geq 0 \). In fact, \( \alpha' \in R_f^{\vee} \) since otherwise \( z(\alpha + k\delta) \in R_f^{\vee}_- \subseteq R_- \) and \( w_0z(\alpha + k\delta) \in R_f^{\vee}_+ \subseteq R_+ \), contradicting \( w_0 \cdot z \) reduced by Lemma 4.4.3. Now

\[
xw_0z(\alpha) = xw_0(\alpha' - k\delta) = x(w_0(\alpha')) - k\delta.
\]

The root \( x(w_0(\alpha')) \) is in \( R_- \) because \( x \cdot w_0 \) is reduced and \( \alpha' \in R_+ \) while \( w_0(\alpha') \in R_- \). Therefore \( x(w_0(\alpha')) - k\delta \) is in \( R_- \), as desired.

Now suppose \( x \) is primitive and \( y \in W_f \) with \( \ell(y) = \ell(w_0) - 1 \). By §3.4.2.3, we have \( y = w_0s_i \) for some \( i \in [n] \). Certainly \( x \cdot y \) is reduced since a reduced factorization for \( xy \) in terms of simple reflections can be obtained from one for \( xw_0 \) that ends in \( s_i \), by deleting that last \( s_i \). Proceed as in the first part of the proof by considering any \( \alpha \in R_+ \) such that \( z(\alpha) \in R_- \), and showing that \( xyz(\alpha) \) remains in \( R_- \). As above, \( z(\alpha) = \alpha' - k\delta \in R_- \) with \( \alpha' \in R_f^{\vee}_+ \), and hence \( k \geq 1 \). We may assume \( \alpha' = \alpha_i \) because if not, the same argument used above works. Then

\[
xyz(\alpha) = x(y(\alpha')) - k\delta = x(\alpha_j) - k\delta,
\]
where $j \in [n]$ so that $\alpha_j = w_0(-\alpha_i)$ (see §4.2.3). Now $x$ primitive implies $x(\alpha_j) \in (R_f^\vee + \delta) \cup R_f^\vee$ and thus $xyz(\alpha) = x(\alpha_j) - k\delta \in R_-$, as $k \geq 1$.

\[4.4.2.\] Here we prove two technical lemmas whose significance may not become clear until seeing their application in the main thread of the proof in §4.4.4.

Given $x \in W_a$ and an element $\tilde{z} = uy^\beta v$ factored as in (4.7), we will say $\tilde{z}$ is large with respect to $x$ if any (all) of the following equivalent conditions is satisfied:

(i) $\langle \beta, \alpha_i \rangle >> \ell(x) + \ell(w_0)$ for $i \in [n]$.

(ii) $\tilde{z}(A_0)$ is far from the affine hyperplanes $h_\alpha$ (i.e. $\langle \alpha, \tilde{z}(A_0) \rangle >> \ell(x) + \ell(w_0)$) for all $\alpha \in R_f^\vee$.

The next lemma is the only place the largeness assumption is used directly.

Before stating the lemma, define the homomorphism

$$\Psi : W_a \cong W_f \times Q'_f \twoheadrightarrow W_f$$

(4.13)

to be the projection onto the first factor. Geometrically, this map can be understood by the action of $W_a$ on $X^\vee$. This action leaves stable the level 0 plane \{ $x \in X^\vee_\mathbb{R}$ : $\langle \delta, x \rangle = 0$ \} $\cong Y$, and $\Psi$ is given by restricting the action of $W_a$ to this plane. On simple reflections, $\Psi$ is given by

$$\Psi(s_i) = \begin{cases} 
  s_i & \text{if } i \in [n], \\
  s_{\phi'} & \text{if } i = 0.
\end{cases}$$

(4.14)

\textbf{Lemma 4.4.6.} If $x \in W_a$, $\tilde{z} = uy^\beta v$ as in (4.7), and $\tilde{z}$ large with respect to $x$, then
(i) if \( x \bar{z} = u'y^\beta v \) is the unique factorization of (4.7), then \( u' = \Psi(x)u \in W_f \), i.e., \( (x \bar{z})_{s_f} = \Psi(x) \bar{z}_{s_f} = \Psi(x)u \),

(ii) \( s_0 \bar{z} < \bar{z} \) if and only if \( \Psi(s_0) \bar{z} > \bar{z} \),

(iii) if \( u = id \), then \( s_0 x \bar{z} < x \bar{z} \iff \Psi(s_0 x) > \Psi(x) \). Similarly, if \( u = id \) and \( i \in [n] \), then \( s_i x \bar{z} < x \bar{z} \iff \Psi(s_i x) < \Psi(x) \).

\[ \text{Proof.} \] Part (i) of the lemma follows easily from the case \( x = s_0 \). In that case,

\[ s_0 \bar{z} = y^{\phi'} s_{\phi'} u y^{\beta} v = s_{\phi'} u (u^{-1} s_{\phi'} y^{\phi'} s_{\phi'} u) y^{\beta} v = \Psi(s_0) u y^{u^{-1} s_{\phi'} (\phi') + \beta} v. \]

Define \( \beta' = u^{-1} s_{\phi'} (\phi') + \beta \). Since \( \langle \beta, \alpha_i \rangle >> 0 \), the same holds for \( \beta' \) because \( \langle u^{-1} s_{\phi'} (\phi'), \alpha_i \rangle \) is bounded by a constant depending only on \( W_f \). As mentioned in §4.2.4, \( \beta' \in Y_{++} \) implies that \( \Psi(s_0) u y^{\beta'} v \) is the desired factorization for \( s_0 \bar{z} \). That \( \bar{z} \) is large with respect to \( x \) ensures that we can multiply \( \bar{z} \) on the left by \( \ell(x) \) simple reflections to obtain \( u'y^{\beta'} v \) and still have \( \beta' \in Y_{++} \).

For statement (ii) we have the equivalences

\[ s_0 \bar{z} < \bar{z} \iff \quad \text{A}_0 \text{ and } \bar{z}(A_0) \text{ are on opposite sides of } h_{\alpha_0} \iff \]

\[ \text{A}_0 \text{ and } \bar{z}(A_0) \text{ are on the same side of } h_\theta \iff \]

\[ \Psi(s_0) \bar{z} > \bar{z} \]

where the first and the third equivalences follow from (4.2) and the second from (4.12).
By applying statement (ii) to \( x \tilde{z} \), we have \( s_0x\tilde{z} < x\tilde{z} \) if and only if \( \Psi(s_0)x\tilde{z} > x\tilde{z} \). Next use that the factorizations \( x\tilde{z} = u' \cdot y^{\beta'} v \) and \( \Psi(s_0)x\tilde{z} = \Psi(s_0)u' \cdot y^{\beta'} v \) are reduced to conclude the left-hand equality of

\[
\ell(\Psi(s_0)x\tilde{z}) - \ell(x\tilde{z}) = \ell(\Psi(s_0)u') - \ell(u') = \ell(\Psi(s_0)x) - \ell(\Psi(x)).
\]

The right-hand equality is the substitution \( u' = \Psi(x)u = \Psi(x) \), which uses statement (i) and the assumption \( u = id \). Therefore, \( \Psi(s_0)x\tilde{z} > x\tilde{z} \) if and only if \( \Psi(s_0x) > \Psi(x) \) giving the first part of statement (iii). The statement for \( i \in [n] \) has the same proof except with the statement (ii) reference replaced by the triviality \( s_i x \tilde{z} < x \tilde{z} \iff \Psi(s_i)x\tilde{z} < x\tilde{z} \).

**Example 4.4.7.** Lemma 4.4.6 (iii) is fairly intuitive in type \( A \). For example, for \( G = SL_5 \), let \( \tilde{z}^{-1} = -27 -13 4 16 35 \) (with the convention of §4.2.5), and \( x^{-1} \) be any of the six possibilities shown below.

<table>
<thead>
<tr>
<th>( x^{-1} )</th>
<th>( \tilde{z}^{-1}x^{-1} )</th>
<th>( \Psi(x^{-1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( id )</td>
<td>( -27 ) ( -13 ) 4 16 35</td>
<td>1 2 3 4 5</td>
</tr>
<tr>
<td>( s_0 )</td>
<td>30 ( -13 ) 4 16 ( -22 ) &lt;</td>
<td>5 2 3 4 1 &gt;</td>
</tr>
<tr>
<td>( s_0s_4 )</td>
<td>30 ( -13 ) 4 ( -22 ) 16 &lt;</td>
<td>5 2 3 1 4 &lt;</td>
</tr>
<tr>
<td>( s_0s_4s_1 )</td>
<td>( -13 ) 30 4 ( -22 ) 16 &lt;</td>
<td>2 5 3 1 4 &lt;</td>
</tr>
<tr>
<td>( s_0s_4s_1s_0 )</td>
<td>11 30 4 ( -22 ) ( -8 ) &lt;</td>
<td>4 5 3 1 2 &gt;</td>
</tr>
<tr>
<td>( s_0s_4s_1s_0s_1 )</td>
<td>30 11 4 ( -22 ) ( -8 ) &gt;</td>
<td>5 4 3 1 2 &gt;</td>
</tr>
</tbody>
</table>

The third (resp. fifth) column compares the length of an element \( \tilde{z}^{-1}x^{-1} \) (resp. \( \Psi(x^{-1}) \)) to the element immediately above it. To make these length comparisons,
we use the fact that \( ws_i > w \) if and only if \( w_i < w_{i+1} \), where \( w_1 \ldots w_n \) is the word of \( w \) (see [42]). Lemma 4.4.6 (iii) says that the inequalities in the third and fifth columns will match exactly when the value for \( x^{-1} \) differs from the value above it by right-multiplication by some \( s \in S_f \).

Let \( x \in W_a, \tilde{z} \in W_e \) such that \( x \cdot w_0 \) and \( w_0 \cdot \tilde{z} \) are reduced. Choose reduced factorizations \( x = s_{i_1}s_{i_2} \ldots s_{i_l}, \tilde{z} = s_{j_1}s_{j_2} \ldots s_{j_k} \). Fix \( y \in W_f \) and note that \( x \cdot y \) and \( y \cdot z \) are also reduced factorizations. Suppose \( s_{i_1} \in L(s_{i_2} \ldots s_{i_l}y\tilde{z}) \). Then by the Strong Exchange Condition (see, e.g., [20, §5.8]) there is an \( r \in [k] \) such that

\[
xy\tilde{z} = s_{i_1}s_{i_2} \ldots s_{i_l}ys_{j_1}s_{j_2} \ldots s_{j_k} = s_{i_2} \ldots s_{i_l}ys_{j_1}s_{j_2} \ldots \hat{s}_{j_r} \ldots s_{j_k}.
\]

The Strong Exchange Condition only says that if \( s \in L(w) \) (in this case, \( w = s_{i_1}xy\tilde{z} \), \( s = s_{i_1} \)), then in any expression for \( w \) as a product of simple reflections, \( sw \) can be obtained by omitting one. In this case however, \( x \cdot y \) reduced implies the omitted reflection must occur in the expression for \( \tilde{z} \). Define \( \tilde{z}' \) by \( s_{j_1}s_{j_2} \ldots \hat{s}_{j_r} \ldots s_{j_k} = u \cdot \tilde{z}' \), where \( u \in W_f \), \( \tilde{z}' \) minimal in \( W_f\tilde{z}' \). Also put \( x' = s_{i_2} \ldots s_{i_l}y' = yu \). We then have

\[
xyz = s_{i_1}s_{i_2} \ldots s_{i_l}y\tilde{z} = s_{i_2} \ldots s_{i_l}y'\tilde{z}' = x'y'\tilde{z}' .
\] (4.17)

We now can state a tricky lemma.

**Lemma 4.4.8.** With the notation above, if \( \tilde{z} \) (and therefore \( \tilde{z}' \)) are large with respect to \( x \), then \( y' > y \).
Example 4.4.9. It may be helpful to follow the proof with an example. Let $\tilde{z}^{-1} = -27 -13 4 16 35$, $y^1 = 4 3 1 2 5$, and $x^1$ be the six possibilities shown below.

<table>
<thead>
<tr>
<th>$x^{-1}$</th>
<th>$z^{-1}y^{-1}x^{-1}$</th>
<th>$\Psi(y^{-1}x^{-1})$</th>
<th>$y'^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>id</td>
<td>16 4 -27 -13 35</td>
<td>4 3 1 2 5</td>
<td></td>
</tr>
<tr>
<td>$s_0$</td>
<td>30 4 -27 -13 21</td>
<td>5 3 1 2 4</td>
<td></td>
</tr>
<tr>
<td>$s_0s_4$</td>
<td>30 4 -27 21 -13</td>
<td>5 3 1 4 2</td>
<td></td>
</tr>
<tr>
<td>$s_0s_4s_1$</td>
<td>4 30 -27 21 -13</td>
<td>3 5 1 4 2</td>
<td></td>
</tr>
<tr>
<td>$s_0s_4s_1s_0$</td>
<td>-18 30 -27 21 9</td>
<td>2 5 1 4 3</td>
<td></td>
</tr>
<tr>
<td>$s_0s_4s_1s_0s_2$</td>
<td>-18 -27 30 21 9</td>
<td>2 1 5 4 3</td>
<td></td>
</tr>
</tbody>
</table>

The third column indicates whether this value for $z^{-1}y^{-1}x^{-1}$ is less than or greater than the value immediately above it. The values for $y'^{-1}$ may be computed by $y'^{-1} = \Psi(y^{-1}x^{-1}x')$ (see the proof below). According to Lemma 4.4.8, we must have $5 3 1 2 4 > 4 3 1 2 5$ as $s_0 \in L(y\tilde{z})$, $4 5 1 2 3 > 4 3 1 2 5$ as $s_1 \in L(s_4s_0y\tilde{z})$, and $4 3 5 2 1 > 4 3 1 2 5$ as $s_2 \in L(s_0s_4s_0s_0y\tilde{z})$.

Proof of Lemma 4.4.8. Since $w_0 \cdot \tilde{z}$ and $w_0 \cdot \tilde{z}'$ are reduced, $\Psi(xy) = \Psi(x'y')$ by Lemma 4.4.6(i) applied to the left- (resp. right-) hand side of (4.17) with $xy$ (resp. $x'y'$) for $x$ of the lemma. Put $a = \Psi(x') \in W_f$. Then

$$y' = a^{-1}\Psi(s_{i_1})ay. \quad (4.18)$$

First suppose $i_1 \neq 0$. Then $s_{i_1} \in L(x'y\tilde{z})$ implies $s_{i_1} = \Psi(s_{i_1}) \in L(ay)$ by Lemma 4.4.6(iii). On the other hand, $x \cdot w_0 \cdot \tilde{z}$ and $(s_{i_1}x) \cdot w_0 \cdot \tilde{z}$ are reduced so $s_{i_1} \notin L(x'w_0\tilde{z})$
implies $\Psi(s_{i_1}) \notin L(aw_0)$ again by Lemma 4.4.6(iii). By §4.2.3 this implies $s_{i_1} \in L(a)$ and applying Lemma 4.4.4(i) with $\alpha = \alpha_{i_1}$ yields the desired result.

If $i_1 = 0$, then $s_{i_1} \in L(x'y\tilde{z})$ implies $\Psi(s_{i_1})a > a$ by Lemma 4.4.6(iii). On the other hand, $x \cdot w_0 \cdot \tilde{z}$ and $(s_{i_1}x) \cdot w_0 \cdot \tilde{z}$ are reduced so $s_{i_1} \notin L(x'w_0\tilde{z})$ implies $\Psi(s_{i_1})aw_0 < aw_0$ again by Lemma 4.4.6(iii). By §4.2.3 this implies $\Psi(s_{i_1})a > a$ and applying Lemma 4.4.4(ii) with $\alpha = \theta$ yields the desired result.

4.4.3. We need a basic lemma about multiplying $T$'s before giving the proof of Theorem 4.4.1. For $w_1, w_2 \in W_e$, define the structure coefficients $f_{w_1, w_2, w_3} \in A$ by

$$T_{w_1}T_{w_2} = \sum_{w_3 \in W_e} f_{w_1, w_2, w_3}T_{w_3}. \quad (4.19)$$

Let $\xi$ be the element $u - u^{-1} \in A$.

Lemma 4.4.10. The coefficients $f_{w_1, w_2, w_3}$ are polynomials in $\xi$ with non-negative integer coefficients.

Proof. Write $w_1 = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_l}$ as a reduced product of simple reflections. For any subset $A = \{j_1, j_2, \ldots, j_{|A|}\} \subseteq [l]$ ($j_1 < j_2 < \cdots < j_{|A|}$), define $x_A = s_{i_{j_1}} \ldots s_{i_{j_{|A|}}}$ Also define the indicator function $I(P)$ of a statement $P$ to take the value 1 if $P$ is true and 0 if $P$ is false. By a direct calculation, the product $T_{w_1}T_{w_2} = T_{i_1}T_{i_2} \ldots T_{i_l}T_{w_2}$ is equal to

$$\sum_{A \subseteq [l]} \xi^{[-|A|]}T_{x_A w_2} \prod_{j \notin A} I(s_{i_j} \in L(x_{\{k \in A: k > j\}}w_2)), \quad (4.20)$$

which implies the desired result.  \(\square\)
We will also use the important observation

For any $f$ in $A$ that is a polynomial in $\xi$ with non-negative
integer coefficients, there holds $\deg_\xi(f) = \deg_u(f)$.

(4.21)

4.4.4.

Proof of Theorem 4.4.1. It is convenient to assume $v_1 \in W_a$, and this is possible
because we can always write $v_1 = \pi v'_1$ with $\pi \in \Pi$ and $v'_1 \in W_a$. Then the theorem
for $v'_1$ gives it for $v_1$ as $C'_{v_1 w_0 v_2} = \pi C'_{v'_1 w_0 v_2} = \pi \overrightarrow{C'}_{v'_1} \overrightarrow{C'}_{w_0} \overrightarrow{C'}_{v_2} = \overrightarrow{C'}_{v_1} \overrightarrow{C'}_{w_0} \overrightarrow{C'}_{v_2}$. This uses
that $v'_1$ is primitive (Proposition 4.3.5).

Begin by expanding out the product of canonical bases as follows:

$$
\overrightarrow{C'}_{v_1} \overrightarrow{C'}_{w_0} \overrightarrow{C'}_{v_2} = \left( \sum \overrightarrow{P}_{x,v_1} T_x \right) \left( \sum_{y \in W_f} (q^{1/2})^{\ell(y)-\ell(w_0)} T_y \right) \left( \sum \overrightarrow{P}_{z,v_2} T_z \right). \tag{4.22}
$$

The first sum is over $x \leq v_1$ such that $x \cdot w_0$ is reduced, and the third sum is over
$z \leq v_2$ such that $w_0 \cdot z$ is reduced. The canonical basis element $C'_{v_1 w_0 v_2}$ is characterized
by being bar invariant and equivalent to $T_{v_1 w_0 v_2} \mod u^{-1}\mathcal{L}$. Therefore, to prove the
theorem it suffices to show that the only term in the expansion of (4.22) not in $u^{-1}\mathcal{L}$ is
$\overrightarrow{P}_{v_1,v_1} T_{v_1} T_{w_0} \overrightarrow{P}_{v_2,v_2} T_{v_2} = T_{v_1 w_0 v_2}$ (which are equal by Proposition 4.4.5). The proof
takes four steps (A)–(D). In (A) and (B) it is shown that a term in the expansion of
(4.22) is in $u^{-1}\mathcal{L}$ provided $z$ is large. The intuition is that it is only easier to get a
large power of $u$ if $z$ is large. This is made precise by (C) and (D), which reduce the
general case to the case $z$ is large.
(A) Write $v_1 = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_l}$ as a reduced product of simple reflections. Then for any $k \in \{0, \ldots, l\}$ and $y \in W_f$ with $y \neq w_0$,

$$(q^{1/2})^{\ell(y) - \ell(w_0)} T_{s_{i_1}} \ldots s_{i_k} T_{s_{i_{k+1}}} \ldots s_{i_l} y \bar{z} \in u^{-1} \mathcal{L}$$

provided $\bar{z}$ is large with respect to $v_1$. In particular,

$$(q^{1/2})^{\ell(y) - \ell(w_0)} T_{v_1} T_y T_{\tilde{z}} \in u^{-1} \mathcal{L}.$$ 

We will prove the main statement of (A) by induction on $\ell(w_0) - \ell(y) + k$. The case $\ell(w_0) - \ell(y) = 1$ and $k$ arbitrary holds because

$$T_{s_{i_1}} \ldots s_{i_k} T_{s_{i_{k+1}}} \ldots s_{i_l} y \bar{z} = T_{s_{i_1}} \ldots s_{i_k} T_{s_{i_{k+1}}} \ldots s_{i_l} y \bar{z} = T_{v_1} T_y \bar{z} = T_{v_1} y \bar{z}$$

by Propositions 4.4.5 and 4.3.6. The result is trivial for $k = 0$. Now assume $k > 0$. If $s_{i_k} \notin L(s_{i_{k+1}} \ldots s_{i_l} y \bar{z})$, then

$$T_{s_{i_1}} \ldots s_{i_k} T_{s_{i_{k+1}}} \ldots s_{i_l} y \bar{z} = T_{s_{i_1}} \ldots s_{i_{k-1}} T_{s_{i_k}} \ldots s_{i_l} y \bar{z}$$

and we are done by induction.

Now suppose $s_{i_k} \in L(s_{i_{k+1}} \ldots s_{i_l} y \bar{z})$. By Lemma 4.4.8,

$$s_{i_k} \ldots s_{i_l} y \bar{z} = s_{i_{k+1}} \ldots s_{i_l} y' \bar{z'},$$

where $\ell(y') > \ell(y)$, $y' \in W_f$, $\bar{z'}$ large with respect to $v_1$, and $w_0 \cdot \bar{z'}$ reduced. Now compute

$$T_{s_{i_k}} T_{s_{i_{k+1}}} \ldots s_{i_l} y \bar{z} = \xi T_{s_{i_{k+1}}} \ldots s_{i_l} y \bar{z} + T_{s_{i_k}} \ldots s_{i_l} y \bar{z}$$
\[ = \xi T_{s_1 k} s_{k+1} \ldots s_i y y' z' + T_{s_1 k} \ldots s_i y y' z'. \]

Multiplying on the left by \((q^{1/2})^{\ell(y) - \ell(w_0)} T_{s_1 k} \ldots s_i y y' z'\), we obtain
\[ (q^{1/2})^{\ell(y) - \ell(w_0)} T_{s_1 k} \ldots s_i y y' z' = \]
\[ (q^{1/2})^{\ell(y) - \ell(w_0)} \xi T_{s_1 k} \ldots s_i y y' z' + (q^{1/2})^{\ell(y) - \ell(w_0)} T_{s_1 k} \ldots s_i y y' z'. \]

The first term is in \(u^{-1} L\) by induction since \((u)^{\ell(y') - \ell(w_0)}\) has at least as high a power of \(u\) as \((q^{1/2})^{\ell(y) - \ell(w_0)}\xi\) (tracing this induction back to the base case involves at most \(\ell(w_0)\) changes to \(\tilde{z}\), so the largeness assumption remains valid). The second term is in \(u^{-1} L\) by the inductive statement with \(k\) decreased by 1.

**B** The product \((q^{1/2})^{\ell(y) - \ell(w_0)} \overrightarrow{P}_{x, v_1} T_x T_y T \tilde{z} \in u^{-1} L\) for \(x < v_1, x \cdot w_0\) reduced, \(y \in W_f\), and \(\tilde{z}\) large with respect to \(x\).

The proof is the same as that for (A) except that the base case is for \(y = w_0\), which holds since \(\overrightarrow{P}_{x, v_1}\) is a polynomial in \(u^{-1}\) with no constant term and \(x \cdot w_0 \cdot \tilde{z}\) is reduced by Proposition 4.4.5.

**C** Given \(x, z \in W_e\) such that \(w_0 \cdot z\) is reduced, there exists a \(v \in W_e\) so that \(y^\lambda := zv\) is large with respect to \(x\) and \(z \cdot v\) and \(w_0 \cdot y^\lambda\) are reduced factorizations.

Choose any \(\lambda\) such that \(\langle \lambda, \alpha_i \rangle >> \ell(z) + \ell(x) + \ell(w_0)\) for \(i \in [n]\) and put \(v := z^{-1} y^\lambda\). We will use Lemma 4.4.3 to show that \(z \cdot v\) is reduced. It is convenient to instead show \(v^{-1} \cdot z^{-1}\) is reduced. Suppose \(\alpha + k \delta \in R_+\) with \(\alpha \in R^\vee_f\) and that \(z^{-1}(\alpha + k \delta) \in R_-\). Then \(z^{-1}(\alpha) \in R_-\), and because the product \(w_0 \cdot z\) is reduced, we must have \(\alpha \in R^\vee_{f-}\) (a similar fact was shown in Proposition 4.4.5). Now
\[ v^{-1}z^{-1}(\alpha + k\delta) = \alpha + (k + \langle \lambda, \alpha \rangle)\delta. \] The integer \( k + \langle \lambda, \alpha \rangle << 0 \) because \( z^{-1}(\alpha + k\delta) \in R_- \) implies \( k \) is bounded by a constant times \( \ell(z) \). Therefore \( v^{-1}z^{-1}(\alpha + k\delta) \in R_- \), as desired.

(D) Any term \( \overrightarrow{P}_{x,v_1}(q^{1/2})^{\ell(y)-\ell(w_0)}\overrightarrow{P}_{z,v_2}T_xT_yT_z \) from the expansion of (4.22) is in \( u^{-1}\mathcal{L} \).

Choose \( \tilde{z} \) large with respect to \( x \), as was shown to exist in (C), such that there exists \( v \) so that \( \tilde{z} = z \cdot v \) is reduced. Then compute

\[ T_xT_yT_z = T_xT_yT_v = \left( \sum_{a \in W_e} f_{x,yz,a}T_a \right) T_v = \sum_{b \in W_e} \left( \sum_{a \in W_e} f_{x,yz,a}f_{a,v,b} \right) T_b. \] \hspace{1cm} (4.23)

By (4.21), the highest power of \( u \) occurring in \( T_xT_yT_z \) is \( \max_a(\deg_\xi(f_{x,yz,a})) \). Let \( a' \in W_e \) be an element with \( f_{x,yz,a'} \) realizing this maximum degree and \( b' \in W_e \) an element with \( f_{a',v,b} \) nonzero. Then since the \( f \)'s are polynomials in \( \xi \) with non-negative coefficients (Lemma 4.4.10),

\[ \deg_\xi(f_{x,yz,a'}) \leq \deg_\xi \left( \sum_{a \in W_e} f_{x,yz,a}f_{a,v,b'} \right). \] \hspace{1cm} (4.24)

Moreover, again by (4.21), the right-hand side of this inequality is the \( u \)-degree of the coefficient of \( T_{b'} \) in \( T_xT_yT_z \). Thus \( \overrightarrow{P}_{x,v_1}(q^{1/2})^{\ell(y)-\ell(w_0)}T_xT_yT_z \) in \( u^{-1}\mathcal{L} \) (by (A) and (B)) implies the same for \( \overrightarrow{P}_{x,v_1}(q^{1/2})^{\ell(y)-\ell(w_0)}\overrightarrow{P}_{z,v_2}T_xT_yT_z \), as desired. \hfill \Box

### 4.5 Concluding remarks

We have tried to find an analog of Theorem 4.4.1 for \( w \) not in \( W_\nu \), or just in the finite Weyl group setting replacing \( w_0 \) with the longest element \( w_{0,J} \) of some
parabolic subgroup $J$. One can ask, for instance, for which $v_1, v_2 \in W_f$ the identity $C_{v_1w_0jv_2}' = \overrightarrow{C}_{v_1}C_{w_0}'\overrightarrow{C}_{v_2}'$ holds. We concluded after a cursory investigation that this holds so rarely that it wouldn't be of much use. We could certainly have overlooked something, but it's more likely that a nice extension of this result requires that a factorization like the above holds but only after quotienting by some submodule spanned by a subset of the canonical basis.

4.6 Appendix

Lemma. For any $x, y \in W_e$, $x \cdot y$ is a reduced factorization if and only if $x(y(R_+) \cap R_-) \subseteq R_-$. 

Proof. Given $\alpha \in R$, there are eight possibilities for the signs of $\alpha$, $y(\alpha)$, and $xy(\alpha)$. Let

$$N_{\epsilon_1\epsilon_2\epsilon_3} = |\{\alpha \in R : xy(\alpha) \in R_{\epsilon_1}, y(\alpha) \in R_{\epsilon_2}, \alpha \in R_{\epsilon_3}\}|$$

where $\epsilon_i \in \{+, -\}$. With this notation, $x(y(R_+) \cap R_-) \cap R_+$ has cardinality $N_{++-}$, so the condition $x(y(R_+) \cap R_-) \subseteq R_-$ is equivalent to $N_{++-} = 0$. We have

$$\ell(xy) = N_{--} + N_{++},$$
$$\ell(y) = N_{--} + N_{+--},$$
$$\ell(x) = N_{--} + N_{+--}.$$ 

A root $\alpha$ contributes to $N_{--}$ if and only if $-\alpha$ contributes to $N_{++-}$. Hence

$$\ell(x) + \ell(y) = N_{--} + N_{++} + N_{+++} + N_{--} = \ell(xy) + 2N_{++}.$$
Therefore $x \cdot y$ is a reduced factorization if and only if $N_{+-} = 0$. □

**Lemma.** Suppose $a, y \in W_f$ and $\alpha \in R_f^\vee$. 

(i) If $s_\alpha a < a$, then $s_\alpha ay < ay \iff a^{-1} s_\alpha ay > y$.

(ii) If $s_\alpha a > a$, then $s_\alpha ay > ay \iff a^{-1} s_\alpha ay > y$.

**Proof.** For (i), we have $s_\alpha a < a$ implies $a^{-1}(\alpha) \in R_f^\vee$ by (4.2). The element $a^{-1} s_\alpha a$ is a reflection $s_\beta$ where $\beta = -a^{-1}(\alpha) \in R_f^\vee$. Now compute

$$y^{-1}(\beta) = y^{-1}(-a^{-1}(\alpha)) = -(ay)^{-1}(\alpha).$$

Hence

$$s_\alpha ay < ay \iff -(ay)^{-1}(\alpha) \in R_f^\vee \iff y^{-1}(\beta) \in R_f^\vee \iff s_\beta y > y,$$

where the first and last equivalence again use (4.2). The proof of (ii) is the same except with $\beta$ defined to be $a^{-1}(\alpha) \in R_f^\vee$ instead of $-a^{-1}(\alpha)$. □
Chapter 5

Paper IV: Cyclage, catabolism, and the affine Hecke algebra

Abstract

We identify a subalgebra $\hat{H}^+_n$ of the extended affine Hecke algebra $\hat{H}_n$ of type $A$. The subalgebra $\hat{H}^+_n$ is a $u$-analogue of the monoid algebra of $S_n \ltimes \mathbb{Z}_{\geq 0}^n$ and inherits a canonical basis from that of $\hat{H}_n$. We show that its left cells are naturally labeled by tableaux filled with positive integer entries having distinct residues mod $n$, which we term positive affine tableaux (PAT).

We then exhibit a cellular quotient $\mathcal{B}_1^n$ of $\hat{H}^+_n$ that is a $u$-analogue of the ring of coinvariants $\mathbb{C}[y_1, \ldots, y_n]/(e_1, \ldots, e_n)$ with left cells labeled by PAT that are essentially standard Young tableaux with cocharge labels. Multiplying canonical basis
elements by a certain element $\pi \in \hat{H}_n^+$ corresponds to rotations of words, and on cells corresponds to cocyclage. We further show that $R_1^n$ has cellular quotients $R_\lambda$ that are $u$-analogues of the Garsia-Procesi modules $R_\lambda$ with left cells labeled by (a PAT version of) the $\lambda$-catabolizable tableaux.

We conjecture that the $k$-atoms of Lascoux, Lapointe, and Morse [25] and the $R$-catabolizable tableaux of Shimozono and Weyman [37] have cellular counterparts in $\hat{H}^+$. We extend the idea of atom copies of [25] to positive affine tableaux and give descriptions, mostly conjectural, of these copies in terms of catabolizability.

5.1 Introduction

It is well-known that the ring of coinvariants $R_{1^n} = \mathbb{C}[y_1, \ldots, y_n]/(e_1, \ldots, e_n)$, thought of as a $\mathbb{C}S_n$-module with $S_n$ acting by permuting the variables, is a graded version of the regular representation. However, how a decomposition of this module into irreducibles is compatible with multiplication by the $y_i$ remains a mystery.

A precise question one can ask along these lines goes as follows. Let $E \subseteq R_d$ be an $S_n$-irreducible, where $R_d$ is the $d$-th graded part of the polynomial ring $R = \mathbb{C}[y_1, \ldots, y_n]$. Suppose that the isotypic component of $R_d$ containing $E$ is $E$ itself. Then define $I \subseteq R$ to be the sum of all homogeneous ideals $J \subseteq R$ that are left stable under the $S_n$-action and satisfy $J \cap E = 0$. The quotient $R/I$ contains $E$ as the unique $S_n$-irreducible of top degree $d$. It is natural to ask
What is the graded character of $R/I$?

The most familiar examples of such quotients are the Garsia-Procesi modules $R_\lambda$ (see [8]), which correspond to the case that $E$ is of shape $\lambda$ and $d = n(\lambda) = \sum_i (i-1)\lambda_i$; refer to this representation $E \subseteq R_{n(\lambda)}$ as the **Garnir representation of shape** $\lambda$ or, more briefly, $G_\lambda$. Combining the work of Hotta-Springer (see [13]) and Lascoux [27] (see also [37]) gives the Frobenius series

$$\mathcal{F}_{R_\lambda}(t) = \sum_{\substack{T \in \text{SYT} \:\text{ctype}(T) \geq \lambda}} t^{\text{cocharge}(T)} s_{\text{sh}(T)},$$

where $\text{ctype}(T)$ is the catabolizability of $T$ (see §5.5.4).

Though this interpretation of the character of $R_\lambda$ has been known for some time, the only proofs were difficult and indirect. One of the goals of this research, towards which we have been partially successful, was to give a more transparent explanation of the appearance of catabolism in the combinatorics of the coinvariants.

More recent work suggests that there are other combinatorial mysteries hiding in the ring of coinvariants. We strongly suspect that modules with graded characters corresponding to the $k$-atoms of Lascoux, Lapointe, and Morse [25] and a generalization of $k$-atoms due to Li-Chung Chen [5] sit inside the coinvariants as subquotients. It is also natural to conjecture that the generalization of catabolism due to Shimozono and Weyman [37] gives a combinatorial description of certain subquotients of the coinvariants which are graded versions of induction products of $S_n$ irreducibles.

This paper describes an approach to these problems using the theory of canonical
bases, which has so far been quite successful, and we are hopeful will help solve some of the difficult conjectures in this area. After reviewing the necessary background on Weyl groups and Hecke algebras (§5.2) and canonical bases and cells (§5.3), we introduce the central algebraic object of our work, a subalgebra \( \hat{H}^+ \) of the extended affine Hecke algebra which is a \( u \)-analogue of the monoid algebra of \( S_n \ltimes \mathbb{Z}_{\geq 0}^n \). In §5.4, we establish some basic properties of this subalgebra and describe its left cells. It turns out that these cells are naturally labeled by tableaux filled with positive integer entries having distinct residues mod \( n \), which we term positive affine tableaux (PAT). Our investigations have thoroughly convinced us that these are excellent combinatorial objects for describing graded \( S_n \)-modules.

After some preparatory combinatorics and formalism in §5.5, we go on to show in §5.6 that \( \hat{H}^+ \) has a cellular quotient \( R_1^n \) that is a \( u \)-analogue of \( R_1^n \). The module \( R_1^n \) has a canonical basis labeled by affine words that are essentially standard words with cocharge labels, with left cells labeled by PAT that are essentially standard tableaux with cocharge labels. Multiplying canonical basis elements by a certain element \( \pi \in \hat{H}^+ \) corresponds to rotations of words, and on left cells corresponds to cocyclage.

In this cellular picture of the coinvariants, \( G_\lambda \) corresponds to a left cell of \( R_1^n \) labeled by a PAT of shape \( \lambda \), termed the Garnir tableau of shape \( \lambda \), again denoted \( G_\lambda \). In §5.8, we identify a \( u \)-analogues \( R_\lambda \) of the \( R_\lambda \) and give several equivalent descriptions of these objects. Most importantly, we show that \( R_\lambda \) is cellular and its
left cells are labeled by (a PAT version of) the $\lambda$-catabolizable tableaux. The proof uses several ingredients:

- The positivity of the structure coefficients of the canonical basis of $\hat{H}^+$,

- Identifying certain canonical basis elements of $\hat{H}^+$ as elementary symmetric functions in subsets of Bernstein generators $Y_1, \ldots, Y_n$ (Theorem 5.8.5),

- The $u = 1$ results of Garsia-Procesi and Bergeron-Garsia.

Given these ingredients, the proof is quite easy. One of the hopes of this approach was to give a proof of equation (5.1) not relying on the $u = 1$ results. Though we have not yet achieved this goal, the cellular picture provided by $\hat{H}^+$ seems to give an extremely good way of connecting representation theory with difficult combinatorics, both intuitively and conjecturally.

The final goal of this paper, the subject of §5.9, is to describe our progress towards connecting more elaborate combinatorics with other cellular subquotients of $\hat{H}^+$. We conjecture that the $R$-catabolizable tableaux of Shimozono and Weyman, the $k$-atoms of Lascoux, Lapointe, and Morse, and Chen’s atoms all have cellular counterparts in $\hat{H}^+$. In fact, there appear to be many isomorphic copies of such atoms in $\hat{H}^+$, generalizing the notion of atom copies in [25]. We give combinatorial descriptions, mostly conjectural, of some of these atom copies. Though we believe these copies to be isomorphic as cellular subquotients, they come with genuinely different combinatorics, just as the cocyclage poset on semistandard tableaux is not
obviously isomorphic to a subposet of the cocyclage poset on standard tableaux. We believe that a critical problem towards understanding $k$-atoms and catabolizability is to produce a combinatorial structure less rigid than tableaux that makes it obvious that these copies are isomorphic.

### 5.2 Hecke algebras

Following [14] (see also [33]), we introduce Weyl groups and Hecke algebras in full generality. In §5.4 and on, we work only in type A.

#### 5.2.1

Let $(W, S)$ be a Coxeter group and $\Pi$ an abelian group acting on $(W, S)$ by automorphisms. The extended Coxeter group associated to this data is the pair $(W_e, S)$, where $W_e$ is the semidirect product $\Pi \rtimes W$. The length function $\ell$ and partial order $\leq$ on $W$ extend to $W_e$: $\ell(\pi v) = \ell(v)$, and $\pi v \leq \pi'v'$ if and only if $\pi = \pi'$ and $v \leq v'$, where $\pi, \pi', v, v' \in \Pi, v, v' \in W$.

If $\ell(uv) = \ell(u) + \ell(v)$, then $uv = u \cdot v$ is a reduced factorization. The notation $L(w) = \{s \in S : sw < w\}, R(w) = \{s \in S : ws < w\}$ will be used for the left and right descent sets of $w$.

Although it is possible to allow parabolic subgroups to be extended Coxeter groups, we define a parabolic subgroup of $W_e$ to be an ordinary parabolic subgroup of $W$ to simplify the discussion (this is the only case we will need).

For any $J \subseteq S$, the parabolic subgroup $W_{eJ} = W_J$ is the subgroup of $W_e$ generated
by $J$. Each left (resp. right) coset of $wW_eJ$ (resp. $W_eJw$) contains an unique element of minimal length called a minimal coset representative. The set of all such elements is denoted $W_e^J$ (resp. $^JW_e$). For any $w \in W_e$, define $w^J, {}^Jw$ by

$$w = w^J \cdot {}^Jw, \ w^J \in W_e^J, \ {}^Jw \in W_eJ.$$  
(5.2)

Similarly, define $w_J, {}^Jw$ by

$$w = w_J \cdot {}^Jw, \ w_J \in W_eJ, \ {}^Jw \in {}^JW_e.$$  
(5.3)

5.2.2. Let $(Y, \alpha'_i, \alpha_i^\vee)$, $i \in [n-1]$ be the root system specifying a reductive algebraic group $G$ over $\mathbb{C}$. Write $Y^\vee$ for the dual lattice $\text{Hom}(Y, \mathbb{Z})$ and $\langle \ , \ \rangle$ for the pairing. Let $W_f$ be the Weyl group of this root system and $S = \{s_1, \ldots, s_{n-1}\}$ the set of simple reflections. The group $W_f$ is the subgroup of automorphisms of the lattice $Y$ generated by the reflections $s_i$. Let $R'_f$ be the set of roots and $Q'_f$ the root lattice.

The extended affine Weyl group is the semidirect product

$$W_e := Y \rtimes W_f.$$ 

Elements of $Y \subseteq W_e$ will be denoted by the multiplicative notation $y^\lambda, \lambda \in Y$.

The group $W_e$ is also equal to $\Pi \rtimes W_a$, where $W_a$ is the Weyl group of an affine root system we will now construct and $\Pi$ is an abelian group. Let $X = Y^\vee \oplus \mathbb{Z}$ and $\delta$ be a generator of $\mathbb{Z}$. The pairing of $X$ and $X^\vee$ is obtained by extending the pairing of $Y$ and $Y^\vee$ together with $\langle \delta, Y \rangle = 0$. Let $\phi'$ be the dominant short root of $(Y, \alpha'_i, \alpha_i^\vee)$ and $\theta = \phi'^\vee$ the highest coroot. For $i \neq 0$ put $\alpha_i = \alpha_i^\vee$ and $\alpha_i^\vee = \alpha_i^\vee$; put $\alpha_0 = \delta - \theta$
and \( \alpha'_0 = -\phi' \). Then \((X, \alpha_i, \alpha'_i), i \in [0, n - 1]\) is an affine root system with Weyl group \( W_a \).

The abelian group \( Q'_f \) is realized as a subgroup of \( W_a \) acting on \( X \) and \( X^\vee \) by translations. This action extends to an action of \( Y \), which realizes \( W_e \) as a subgroup of the automorphisms of \( X \) and \( X^\vee \). The inclusion \( W_a \hookrightarrow W_e \) is given on simple reflections by \( s_i \mapsto s_i \) for \( i \neq 0 \) and \( s_0 \mapsto y^{\phi'} s_0' \). The subgroup \( W_a \) is normal in \( W_e \) with quotient \( W_e/W_a \cong Y/Q'_f \), denoted \( \Pi \). And, as was our goal, we have \( W_e = \Pi \rtimes W_a \).

The set of dominant weights \( Y_+ \) is the cone in \( Y \) given by

\[
Y_+ = \{ \lambda \in Y : \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i \}. \tag{5.4}
\]

Let \( K = \{ s_0, s_1, \ldots, s_{n-1} \} \) be the set of simple reflections of \( W_a \). The pairs \((W_f, S)\) and \((W_a, K)\) are Coxeter groups, and \((W_e, K)\) is an extended Coxeter group. The parabolic subgroup \( W_eS \) is equal to \( W_f \).

5.2.3. Let \( A = \mathbb{Z}[u, u^{-1}] \) be the ring of Laurent polynomials in the indeterminate \( u \) and \( A^- \) be the subring \( \mathbb{Z}[u^{-1}] \). The Hecke algebra \( \mathcal{H}(W) \) of an (extended) Coxeter group \((W, S)\) is the free \( A \)-module with basis \( \{ T_w : w \in W \} \) and relations generated by

\[
T_u T_v = T_{uv} \quad \text{if } uv = u \cdot v \text{ is a reduced factorization} \tag{5.5}
\]

\[
(T_s - u)(T_s + u^{-1}) = 0 \quad \text{if } s \in S.
\]

For each \( J \subseteq S \), \( \mathcal{H}(W)_J \) denotes the subalgebra of \( \mathcal{H}(W) \) with \( A \)-basis \( \{ T_w : w \in W_J \} \), which is also the Hecke algebra of \( W_J \).
5.2.4. The extended affine Hecke algebra $\hat{\mathcal{H}}$ is the Hecke algebra $\mathcal{H}(W_e)$. Just as the extended affine Weyl group $W_e$ can be realized both as $\Pi \ltimes W_a$ and $Y \ltimes W_f$, the extended affine Hecke algebra can be realized in two analogous ways:

The algebra $\hat{\mathcal{H}}$ contains the Hecke algebra $\mathcal{H}(W_a)$ and is isomorphic to the twisted group algebra $\Pi \cdot \mathcal{H}(W_a)$ generated by $\Pi$ and $\mathcal{H}(W_a)$ with relations generated by

$$\pi T_w = T_{\pi w \pi^{-1}} \pi$$

for $\pi \in \Pi$, $w \in W_a$.

There is also a presentation of $\hat{\mathcal{H}}$ due to Bernstein. For any $\lambda \in Y$ there exist $\mu, \nu \in Y_+$ such that $\lambda = \mu - \nu$. Define

$$Y^\lambda := Y^\mu (T^\nu)^{-1},$$

which is independent of the choice of $\mu$ and $\nu$. The algebra $\hat{\mathcal{H}}$ is the free $A$-module with basis $\{Y^\lambda T_w : w \in W_f, \lambda \in Y\}$ and relations generated by

$$T_i Y^\lambda = Y^\lambda T_i \quad \text{if } \langle \lambda, \alpha_i^\vee \rangle = 0,$$

$$T_i^{-1} Y^\lambda T_i^{-1} = Y^{s_i(\lambda)} \quad \text{if } \langle \lambda, \alpha_i^\vee \rangle = 1,$$

$$(T_i - u)(T_i + u^{-1}) = 0$$

for all $i \in [n-1]$, where $T_i := T_{s_i}$. From this, one may deduce the more general commutation relation for $\lambda \in Y$:

$$T_i Y^\lambda - Y^{s_i(\lambda)} T_i = \frac{(u - u^{-1})(Y^{\alpha_i^\vee})}{Y^{\alpha_i^\vee} - 1} (Y^\lambda - Y^{s_i(\lambda)}), \quad i \in [n-1].$$

(5.6)
Be aware that, in the language of [14], we are using the right affine Hecke algebra, so this equation differs slightly from its counterpart [14, (19)] for the left.

We will make use of the following three important bases of \( \hat{H} \); the last one, the canonical basis, will be defined in the next section.

(i) The standard basis \( \{T_w : w \in W_e\} \),

(ii) The Bernstein basis \( \{Y^{\lambda}T_w : \lambda \in Y, w \in W_f\} \),

(iii) The canonical basis \( \{C'_w : w \in W_e\} \).

We remark that \( \{T_wY^{\lambda} : \lambda \in Y, w \in W_f\} \) is also a basis of \( \hat{H} \) and that the results we state using the basis (ii) have counterparts using this basis, but we will not state them explicitly.

### 5.3 Canonical bases and cells

#### 5.3.1.

The bar-involution, \( \overline{\cdot} \), of \( \mathcal{H}(W) \) is the additive map from \( \mathcal{H}(W) \) to itself extending the involution \( \overline{\cdot} : A \rightarrow A \) given by \( \overline{u} = u^{-1} \) and satisfying \( \overline{T_w} = T_{w^{-1}} \).

Observe that \( \overline{T_s} = T_s^{-1} = T_s + u^{-1} - u \) for \( s \in S \). Some simple \( \overline{\cdot} \)-invariant elements of \( \mathcal{H}(W) \) are \( C'_{id} := T_{id} \) and \( C'_s := T_s + u^{-1} = T_{s^{-1}} + u, s \in S \). The \( \overline{\cdot} \)-invariant \( u \)-integers are \( [k] := \frac{u^k - u^{-k}}{u - u^{-1}} \in A \).

#### 5.3.2.

In [22], Kazhdan and Lusztig introduce \( W \)-graphs as a combinatorial structure for describing an \( \mathcal{H}(W) \)-module with a special basis. A \( W \)-graph consists of a vertex
set $\Gamma$, an edge weight $\mu(\delta, \gamma) \in \mathbb{Z}$ for each ordered pair $(\delta, \gamma) \in \Gamma \times \Gamma$, and a descent set $L(\gamma) \subseteq S$ for each $\gamma \in \Gamma$. These are subject to the condition that $A\Gamma$ has a left $\mathcal{H}(W)$-module structure given by

$$C_s'\gamma = \begin{cases} [2]_\gamma & \text{if } s \in L(\gamma), \\ \sum_{\{\delta \in \Gamma : s \in L(\delta)\}} \mu(\delta, \gamma)\delta & \text{if } s \notin L(\gamma). \end{cases} \quad (5.7)$$

We will use the same name for a $W$-graph and its vertex set. If an $\mathcal{H}(W)$-module $E$ has an $A$-basis $\Gamma$ that satisfies (5.7) for some choice of descent sets, then we say that $\Gamma$ gives $E$ a $W$-graph structure, or $\Gamma$ is a $W$-graph on $E$.

It is convenient to define two $W$-graphs $\Gamma, \Gamma'$ to be isomorphic if they give rise to isomorphic $\mathcal{H}(W)$-modules with basis. That is, $\Gamma \cong \Gamma'$ if there is a bijection $\alpha : \Gamma \to \Gamma'$ of vertex sets such that $L(\alpha(\gamma)) = L(\gamma)$ and $\mu(\alpha(\delta), \alpha(\gamma)) = \mu(\delta, \gamma)$ whenever $L(\delta) \not\subseteq L(\gamma)$.

Define the lattice

$$\mathcal{L} = A^{-1}\{T_w : w \in W\}.$$ 

**Theorem 5.3.1** (Kazhdan-Lusztig [22]). For each $w \in W$, there is a unique element $C_w' \in \mathcal{H}(W)$ such that $\overline{C_w'} = C_w'$ and $C_w'$ is congruent to $T_w \mod w^{-1}\mathcal{L}$. There exist integers $\mu(x, w)$, $x, w \in W$ so that $\{C_w' : w \in W\}$ gives $\mathcal{H}(W)$ a $W$-graph structure.

The $A$-basis $\{C_w' : w \in W\}$ of $\mathcal{H}(W)$ is the canonical basis or Kazhdan-Lusztig basis. The corresponding $W$-graph is denoted $\Gamma_W$.

The coefficients of the $C_w'$s in terms of the $T$'s are the Kazhdan-Lusztig polynomials
\( P'_{x,w} \),

\[
C'_w = \sum_{x \in W} P'_{x,w} T_x.
\]  

(5.8)

(Our \( P'_{x,w} \) are equal to \( q^{(\ell(x)-\ell(w))/2} P_{x,w} \), where \( P_{x,w} \) are the polynomials defined in [22].) The \( W \)-graph \( \Gamma_W \) may be described in terms of Kazhdan-Lusztig polynomials as follows: the edge-weight \( \mu(x,w) \) is equal to the coefficient of \( u^{-1} \) in \( P'_{x,w} \) (resp. \( P'_{w,x} \)) if \( x \leq w \) (resp. \( w \leq x \)).

**Remark 5.3.2.** Not all of the integers \( \mu(x,w) \) matter for the \( \mathcal{H}(W) \)-module structure on \( A\Gamma_W \), i.e., different choices of certain edge-weights would lead to isomorphic \( W \)-graphs. However, the convention above in which \( \mu(w,x) = \mu(x,w) \) is sometimes convenient and we maintain this throughout the paper.

**5.3.3.** Let \( \Gamma \) be a \( W \)-graph and put \( E = A\Gamma \). The preorder \( \leq_{\Gamma} \) (also denoted \( \leq_{E} \)) on the vertex set \( \Gamma \) is generated by the relations/edges

\[
The preorder \( \leq_{\Gamma} \) (also denoted \( \leq_{E} \)) on the vertex set \( \Gamma \) is generated by the relations/edges

\[
\delta \leftarrow_{\Gamma} \gamma
\]

if there is an \( h \in \mathcal{H}(W) \) such that \( \delta \) appears with non-zero coefficient in the expansion of \( h\gamma \) in the basis \( \Gamma \).

Equivalence classes of \( \leq_{\Gamma} \) are the **left cells** of \( \Gamma \). Sometimes we will speak of the left cells of \( E \) or the preorder on \( E \) to mean that of \( \Gamma \), when the \( W \)-graph \( \Gamma \) is clear from context. A **cellular submodule** of \( E \) is a submodule of \( E \) that is spanned by a subset of \( \Gamma \) (and is necessarily a union of left cells). A **cellular quotient** of \( E \) is a quotient of \( E \) by a cellular submodule, and a **cellular subquotient** of \( E \) is a cellular submodule of a cellular quotient. We will abuse notation and sometimes refer to a
cellular subquotient by its corresponding union of cells.

5.3.4. The preorder \( \leq_E \) induces a partial order on the cells of \( E \). This seems to be quite difficult to compute completely; it is not even known for the \( S_n \)-graph \( \Gamma_{S_n} \). We will see some partial results along these lines throughout the paper. We can state one such result now, which originated in the work of Barbasch and Vogan on primitive ideals, and is proven in the generality stated here by Roichman [35] (see also [1, §3.3]).

**Proposition 5.3.3.** Let \( J \subseteq S \) and \( E = \text{Res}_{\mathcal{H}(W_J)} \mathcal{H}(W) \). Then for any \( x \in J W \),

\[
A\{C'_{vx} : v \in W_J\} \xrightarrow{\sim} \mathcal{H}(W_J), C'_{vx} \mapsto C'_v
\]  

(5.10)

is an isomorphism of \( \mathcal{H}(W_J) \)-modules with basis (equivalently, the corresponding map of \( W_J \)-graphs is an isomorphism). In particular, any left cell of \( E \) is isomorphic to one occurring in \( \mathcal{H}(W_J) \).

Despite the difficulty of computing \( \leq_E \), there are two kinds of easy edges that will be of interest to us.

If \( \Gamma \) is a cellular subquotient of the \( W \)-graph \( \Gamma_W \), then

\[
C'_{sw} \leq_{\Gamma} C'_{w}, \quad \text{if } sw > w, s \in S.
\]  

(5.11)

We will refer to such edges as *ascent-edges* and the corresponding edges between cells as *ascent-induced edges* (that is, for left cells \( T_1, T_2 \) of \( \Gamma \), \( T_1 \leq_{\Gamma} T_2 \) is an ascent-induced edge if there exist \( \gamma_1 \in T_1, \gamma_2 \in T_2 \) such that \( \gamma_1 \leq_{\Gamma} \gamma_2 \) is an ascent-edge).
If $\Gamma$ is a cellular subquotient of the $W_e$-graph $\Gamma_{W_e}$, then
\[
C'_{\pi w} \leq_{\Gamma} C'_{w}, \text{ for any } \pi \in \Pi, w \in W_e. \tag{5.12}
\]

We refer to such edges as corotation-edges and the corresponding edges between cells as cocyclage-edges. We will soon see that cocyclage-edges are a generalization of cocyclage for standard Young tableaux.

## 5.4 Type $A$ and the positive part of $\mathcal{H}$

Here we introduce a subalgebra $\mathcal{H}^+$ of $\mathcal{H}$ that plays a crucial role in our goal of relating subquotients of $R$ to tableau combinatorics. We also introduce the set of affine tableaux (AT) and positive affine tableaux (PAT), which label left cells of $\text{Res}_{\mathcal{H}} \mathcal{H}$ and $\mathcal{H}^+$.

### 5.4.1. From now on, specialize to the case $G = GL_n$.

The groups $W_f, W_a, W_e$, roots $R'_f$, root lattice $Q'_f$, etc. are now understood to be those of type A. Let $\mathcal{H}, \overline{\mathcal{H}}, \mathcal{H}$ denote the Hecke algebras of $W_f, W_a, W_e$, sometimes decorated with a subscript $n$ to emphasize that they correspond to type $A_{n-1}$ or $\mathbb{A}_n$.

The lattices $Y$ and $Y^\vee$ are equal to $\mathbb{Z}^n$ and $\alpha'_i = \epsilon_i - \epsilon_{i+1}$, $\alpha''_i = \epsilon'_i - \epsilon''_{i+1}$, where $\epsilon_i$ and $\epsilon'_i$ are the standard basis vectors of $Y$ and $Y^\vee$. The finite Weyl group $W_f$ is $S_n$ and the subgroup $\Pi$ of $W_e$ is $\mathbb{Z}$. The element $\pi = y_1s_1s_2\ldots s_{n-1} \in \Pi$ is a generator of $\Pi$. This satisfies the relation $\pi s_i = s_{i+1} \pi$, where, here and from now on, the subscripts of the $s_i$ are taken mod $n$. 
Here is a table that summarizes the algebras defined so far and some to be defined shortly.

<table>
<thead>
<tr>
<th>Group, monoid, etc.</th>
<th>Group algebra over $\mathbb{C}$</th>
<th>$u$-analogue</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_n = W_f$</td>
<td>$\mathbb{C}S_n$</td>
<td>$\mathcal{H}_n$</td>
</tr>
<tr>
<td>$\tilde{S}_n = W_a \cong Q \rtimes W_f$</td>
<td>$\mathbb{C}\tilde{S}_n$</td>
<td>$\tilde{\mathcal{H}}_n$</td>
</tr>
<tr>
<td>$\check{S}_n = W_e \cong Y \rtimes W_f$</td>
<td>$\mathbb{C}[y_1^{\pm1}, \ldots, y_n^{\pm1}] \rtimes S_n := \mathbb{C}(Y \rtimes W_f)$</td>
<td>$\check{\mathcal{H}}_n$</td>
</tr>
<tr>
<td>$\check{S}_n^+ = W_e^+ \cong Y^+ \rtimes W_f$</td>
<td>$\mathbb{C}[y_1, \ldots, y_n] \rtimes S_n := \mathbb{C}(Y^+ \rtimes W_f)$</td>
<td>$\check{\mathcal{H}}^+_n$</td>
</tr>
<tr>
<td>$Y^+$</td>
<td>$R = \mathbb{C}[y_1, \ldots, y_n]$</td>
<td>$\mathcal{R}$</td>
</tr>
<tr>
<td>$D^S$</td>
<td>$R_1^n = \mathbb{C}[y_1, \ldots, y_n]/(e_1, \ldots, e_n)$</td>
<td>$\mathcal{R}_1^n$</td>
</tr>
</tbody>
</table>

**5.4.2.** Another description of $W_e$, due to Lusztig, identifies it with the group of permutations $w : \mathbb{Z} \to \mathbb{Z}$ satisfying $w(i + n) = w(i) + n$ and $\sum_{i=1}^n (w(i) - i) \equiv 0 \text{ mod } n$. The identification takes $s_i$ to the permutation transposing $i + kn$ and $i + 1 + kn$ for all $k \in \mathbb{Z}$, and takes $\pi$ to the permutation $k \mapsto k + 1$ for all $k \in \mathbb{Z}$. We take the convention of specifying the permutation of an element $w \in W_e$ by the word

$$n + 1 - w^{-1}(1) \ n + 1 - w^{-1}(2) \ldots \ n + 1 - w^{-1}(n).$$

We refer to this as the *inverted window word*, *affine word*, or simply *word* of $w$, and, when there is no confusion, the word of $w$ will be written as $w_1w_2\cdots w_n$; this is understood to be part of an infinite word so that $w_i = \hat{i} - i + w_i$, where $\hat{i}$ denotes the element of $[n]$ congruent to $i$ mod $n$. For example, if $n = 4$ and $w = \pi^2 s_2 s_0 s_1$, then the word of $w$ is $8352$. 
The following formulas relate multiplication of elements of $W_e$ with manipulations on words. We adopt the convention of writing $a.b$ in place of $na + b$ ($a, b \in \mathbb{Z}$). In examples with actual numbers, $a$ and $b$ will always be single-digit numbers and we will omit the dot.

<table>
<thead>
<tr>
<th>Element of $W_e$</th>
<th>inverted window word</th>
</tr>
</thead>
<tbody>
<tr>
<td>$id$</td>
<td>$n \cdot n - 1 \ldots 2 \cdot 1$ (5.13)</td>
</tr>
<tr>
<td>$w$</td>
<td>$x_1 \cdot x_2 \cdots x_n$ (5.14)</td>
</tr>
<tr>
<td>$s_iw$</td>
<td>$x_1 \cdot x_2 \cdots x_{i+1} \cdot x_i \cdots x_n$ $i \in [n-1]$ (5.15)</td>
</tr>
<tr>
<td>$s_0w$</td>
<td>$1 \cdot x_n \cdot x_2 \cdots x_{n-1} \cdot (-1) \cdot x_1$ (5.16)</td>
</tr>
<tr>
<td>$ws_{n-i}$</td>
<td>$x_1 \cdots x_j + 1 \cdots x_k - 1 \cdots x_n$ $x_j \equiv i, x_k \equiv i + 1, i \in [n]$ (5.17)</td>
</tr>
<tr>
<td>$y^\lambda w$</td>
<td>$\lambda_1 \cdot x_1 \cdots \lambda_2 \cdot x_2 \cdots \lambda_n \cdot x_n$ (5.18)</td>
</tr>
<tr>
<td>$\pi w$</td>
<td>$1 \cdot x_n \cdot x_1 \cdots x_{n-1}$ (5.19)</td>
</tr>
<tr>
<td>$w\pi$</td>
<td>$x_1 + 1 \cdot x_2 + 1 \cdots x_n + 1$ (5.20)</td>
</tr>
</tbody>
</table>

Here are some basic facts we will need about words of $W_e$. See [42] for a thorough treatment.

**Proposition 5.4.1.** For $w \in W_e$ and $s_i \in S$, $s_iw > w$ if and only if $w_i > w_{i+1}$.

Similarly, $ws_{n-i} > w$ if and only if $j > k$, where $j$ and $k$ are such that $w_j = i, w_k = i + 1$.

**Proposition 5.4.2.** For $w \in W_e$, the length of $w$ may be expressed in terms of its word by

$$l(w) = \sum_{1 \leq i < j \leq n} \left\lfloor \frac{|w_i - w_j|}{n} \right\rfloor,$$

(5.21)
where $\lfloor x \rfloor$ is the greatest integer less than $x$.

**Proposition 5.4.3.** Given $w \in W_e$, let $x_1x_2\cdots x_n$ be the result of replacing the numbers of the word $w_1w_2\cdots w_n$ of $w$ by the numbers $1, \ldots, n$ so that relative order is preserved. Then $x$ is the word of $w_S$ (the notation $w_S$ is defined in §5.2.1).

**Proof.** Left-multiply $w$ by a sequence $s_i, s_{i_2}, \ldots, s_{i_l}$, $i_j \in [n-1]$ until the resulting element $w'$ has word $w'_1w'_2\cdots w'_n$ such that $w'_1 > w'_2 > \cdots > w'_n$ and $\{w'_1, w'_2, \ldots, w'_n\} = \{w_1, w_2, \ldots, w_n\}$. This may be done so that each left-multiplication decreases length by 1. The same sequence of left-multiplications transforms $x_1x_2\cdots x_n$ into $id = n \ n-1 \ \cdots \ 2 \ 1$. By Proposition 5.4.1, $L(w') \subseteq \{s_0\}$. Therefore, $sw = w'$ and $w_S = s_is_{i_2}\cdots s_{i_l}$, and $s_{i_1}\cdots s_{i_l}$ has word $x_1\cdots x_n$. \hfill $\Box$

Let $w^j$ be the subword of $w$ in the alphabet $[jn+1, (j+1)n]$ and $(w^j)^*$ denote the result of subtracting $jn$ from all the numbers in $w^j$.

**Proposition 5.4.4.** For $w \in W_e$, the word of $sw$ is given by $w^0(w^1)^*(w^2)^*\ldots$. Equivalently, $sw$ is given by $w_{j_1}w_{j_2}\cdots w_{j_n}$, where $j_1 < j_2 < \cdots < j_n$ are such that $w_{j_i} \in [n]$.

**Proof.** The proof is essentially the same as that of Proposition 5.4.3, but right-multiplications on words are harder to deal with. By looking at the word of $w$ on the subword $w_{j_1}w_{j_2}\cdots w_{j_n}$ and using Proposition 5.4.1, we can see that the subword on the indices $j_1, j_2, \ldots, j_n$ can be transformed into $n \ n-1 \ \cdots \ 1$ by a sequence of right-multiplications by $s_i \in S$ that decrease length by 1. Then again by Proposition 5.4.1, the resulting word $w'$ satisfies $R(w') \subseteq \{s_0\}$, so $w' = w^S$. Therefore, the sequence of
right-multiplications gives a factorization of $sw$ into a product of simple reflections, from which the result follows.  

5.4.3. There is an automorphism $\Delta$ of $W_e$ given on generators by $s_i \mapsto s_{n-i}, \pi \mapsto \pi^{-1}$.

**Definition 5.4.5.** Let $\Psi : W_e \to W_e$ be the anti-automorphism defined by $\Psi(w) = \Delta(w^{-1}) = (\Delta(w))^{-1}$. This restricts to an anti-automorphism $\Psi : W_e^+ \to W_e^+$. Finally, also denote by $\Psi$ the maps $\hat{\mathcal{H}} \to \hat{\mathcal{H}}$ and $\hat{\mathcal{H}}^+ \to \hat{\mathcal{H}}^+$ given by $T_w \mapsto T_{\Psi(w)}$.

The word of $\Psi(w)$ is given by $x_1 \cdots x_n$ where $x_i$ is determined by $w_{x_i} = i$.

5.4.4. The subset $Y^+ := \mathbb{Z}_{\geq 0}^n$ of the weight lattice $Y$ is left stable under the action of the Weyl group $W_f$. Thus $Y^+ \rtimes W_f$ is a submonoid of $W_e$. Note that this is only true in type $A$.

**Proposition 5.4.6.** The following subsets of $W_e$ are equal.

1. $Y^+ \rtimes W_f$,
2. The submonoid of $W_e$ generated by $\pi$ and $W_f$,
3. $\{w \in W_e : w_i > 0 \text{ for all } i \in [n]\}$.

**Proof.** We will show $(1) \subseteq (2) \subseteq (3) \subseteq (1)$. As $y_i = s_{i-1} s_{i-2} \cdots s_1 y_1 s_1 s_2 \cdots s_{i-1}$ and $y_1 = \pi s_{n-1} s_{n-2} \cdots s_1$, $(1) \subseteq (2)$. The inclusion to $(2) \subseteq (3)$ is clear from (5.15) and (5.19).

The word of any $w \in W_e$ can be written uniquely as

$$\lambda_1 x_1 \lambda_2 x_2 \cdots \lambda_n x_n$$
with \( x_i \in [n] \) and \( \lambda \in Y \). Then by (5.18) \( w = y^\lambda v \) and \( v \) has word \( x_1 x_2 \cdots x_n \). Therefore \( v \in W_f \). Then since \( w_i > 0 \) implies \( \lambda_i \geq 0 \), we have (3) \( \subseteq \) (1).

**Definition 5.4.7.** The *positive part* of \( W_e \), denoted \( W_e^+ \), is any (all) of the subsets in Proposition 5.4.6.

Let \((Y^+)_d\) (resp. \((Y^+)_{\geq d}\)) denote the set \( \{\lambda \in Y^+ : |\lambda| = d\} \) (resp. \( \{\lambda \in Y^+ : |\lambda| \geq d\} \)). The proof of Proposition 5.4.6 also shows that the *degree d part* \((W_e^+)_d\) of \( W_e^+ \) has the corresponding descriptions

\[
\begin{align*}
(i') \quad & (Y^+)_d \rtimes W_f, \\
(ii') \quad & \{w \in W_e^+ : w = \pi^d v, v \in W_a\}, \\
(iii') \quad & \{w \in W_e^+ : \sum_{i=1}^n (w_i - i) = dn\}. 
\end{align*}
\]

**Lemma 5.4.8.** Any \( w \in W_e^+ \) has a reduced expression of the form \( w = v_1 \cdot \pi \cdot v_2 \cdot \pi \cdot \cdots v_d \cdot \pi \cdot v_{d+1} \), where \( v_i \in W_f \).

**Proof.** Use the description (3) of Proposition 5.4.6. By Proposition 5.4.1, one checks that any word of \( w \in W_e \) with \( w_i > 0 \) of the form (3) can be brought to the identity by a sequence of left-multiplications by \( \pi^{-1} \) and left-multiplications by \( s_i \in S \) that decrease length by 1. This yields a desired reduced expression for \( w \).

**Proposition 5.4.9.** The following subsets of \( \hat{H} \) are equal.

\[
\begin{align*}
(i) \quad & A\{Y^\lambda T_w : \lambda \in Y^+, w \in W_f\}, \\
(ii) \quad & A\{T_w : w \in W_e^+\}, \\
(iii) \quad & A\{C'_w : w \in W_e^+\},
\end{align*}
\]
(iv) the subalgebra of \( \hat{\mathcal{H}} \) generated by \( \pi \) and \( \mathcal{H} \).

Proof. As \( Y_i = T_{i-1} T_{i-2} \cdots T_1 Y_1 T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1} \) and \( Y_1 = \pi T_{n-1} T_{n-2} \cdots T_1 \), (i) \( \subseteq \) (iv). Then since \( \pi \in (i) \) and \( \mathcal{H} \subseteq (i) \), (iv) \( \subseteq \) (i) follows if we can show that (i) is a subalgebra. This can be seen from the relations (5.6) since \( \frac{Y^\lambda - Y_{\alpha_i}(\lambda)}{Y^\alpha_i - 1} \) is a polynomial in the \( Y_i \) whenever \( \lambda \in \mathbb{Z}_{\geq 0}^n \).

The inclusion (ii) \( \subseteq \) (iv) follows from Lemma 5.4.8. Again, showing that (ii) is a subalgebra will prove (iv) \( \subseteq \) (ii). Given \( w_1, w_2 \in W_e^+ \),

\[
T_{w_1} T_{w_2} = \sum_{v_1 \leq w_1, v_2 \leq w_2} c_{v_1, v_2} T_{v_1} T_{v_2}, \quad c_{v_1, v_2} \in A. \tag{5.23}
\]

By Lemma 5.4.8, \( w \in W_e^+ \) implies \( v \in W_e^+ \) for any \( v \leq w \). Also \( v_1, v_2 \in W_e^+ \) implies \( v_1 v_2 \in W_e^+ \) as \( W_e^+ \) is a monoid. Thus the right-hand side of (5.23) is in (ii). The equality (iv) = (iii) is similar to (iv) = (ii).

\( \square \)

**Definition 5.4.10.** Let \( \hat{\mathcal{H}}^+ \subseteq \hat{\mathcal{H}} \) denote any (all) of these subsets.

The *degree d part* of \( \hat{\mathcal{H}}^+ \), \( (\hat{\mathcal{H}}^+)_d \), has the corresponding descriptions:

\[
(i') \quad A\{Y^\lambda T_w : \lambda \in (Y^+)_d, w \in W_f\},
\]

\[
(ii') \quad A\{T_w : w \in (W_e^+)_d\}, \tag{5.24}
\]

\[
(iii') \quad A\{C'_w : w \in (W_e^+)_d\}.
\]

Also define \( (\hat{\mathcal{H}}^+)_d = \oplus_{i \geq d} (\hat{\mathcal{H}}^+)_d \) and \( (\hat{\mathcal{H}}^+)_{\leq d} = \oplus_{i \leq d} (\hat{\mathcal{H}}^+)_d \). The decomposition \( \hat{\mathcal{H}}^+ = (\hat{\mathcal{H}}^+)_0 \oplus (\hat{\mathcal{H}}^+)_1 \oplus \ldots \) makes \( \hat{\mathcal{H}}^+ \) into a graded \( A \)-algebra. The descriptions (i), (ii), (iii) of Proposition 5.4.9 give three \( A \)-bases for \( \hat{\mathcal{H}}^+ \) consisting of homogeneous elements.
Just as we write $H(W)$ for the Hecke algebra of an extended Coxeter group $W$, generalizing the notion of a Hecke algebra of a Coxeter group, we further extend this to saying that $\hat{H}^+$ is the Hecke algebra of the monoid $W^+_e$.

5.4.5. The left cells of $\text{Res}_H \hat{H}, \hat{H}^+$ can be determined by Proposition 5.3.3. These results are stated as the two corollaries below. Keep in mind our convention from §5.4.2 for the word of $w$.

The work of Kazhdan and Lusztig [22] shows that the left cells of $H$ are in bijection with the set of SYT and the left cell containing $C'_w$ corresponds to the insertion tableau of $w$ under this bijection. The left cell containing those $C'_w$ such that $w$ has insertion tableau $P$ is the left cell labeled by $P$, denoted $\Gamma_P$. A combinatorial discussion of left cells in type $A$ is given in [1, §4].

**Definition 5.4.11.** An affine tableau (AT) of size $n$ is a semistandard Young tableau filled with integer entries that have distinct residues mod $n$. A positive affine tableau (PAT) of size $n$ is a semistandard Young tableau filled with positive integer entries that have distinct residues mod $n$.

For $w \in W_e$, the word $w_1w_2 \cdots w_n$ may be inserted into a tableau, and the result is an affine tableau, denoted $P(w)$ (see §5.5.1 for our tableau conventions). It is a positive affine tableau exactly when $w \in W^+_e$. By Proposition 5.4.3, the SYT $P(w_S)$ is obtained from $P(w)$ by replacing its entries with the numbers $1, \ldots, n$ so that the relative order of entries in $P(w)$ and $P(w_S)$ agree. Since $P(w_S)$ is determined by the
tableau $Q := P(w)$, independent of the chosen $w$ inserting to $Q$, we write $Q_S$ for this tableau. For example, for the given $w$ below, $w_S$, $P(w)$, and $P(w_S) = P(w)_S$ are as follows.

$$w = 21 \ 12 \ 13 \ 16 \ 4 \ 15, \quad P(w) = \begin{array}{c} 1 \ 13 \ 15 \\ 12 \ 16 \\ 21 \end{array}$$

$$w_S = 6 \ 2 \ 3 \ 5 \ 1 \ 4, \quad P(w_S) = \begin{array}{c} 1 \ 3 \ 4 \\ 2 \ 5 \\ 6 \end{array}$$

Let $Q$ be an affine tableau. The set of $w \in W_e$ inserting to $Q$ is $\{vx \in W_f : P(v) = Q_S\}$, where the word of $x$ is obtained from $Q$ by sorting its entries in decreasing order.

For any $x \in {}^8W_e$, define

$$\Gamma_Q := \{C'_{vx} : v \in W_f, \ P(v) = Q_S\} = \{C'_{w} : w \in W_e, \ P(w) = Q\}. \quad (5.25)$$

By the following result, $\Gamma_Q$ is a left cell of $\Gamma_{W_e}$, which we refer to as the left cell labeled by $Q$.

**Corollary 5.4.12.** For any $x \in {}^8W_e$, the set $\{C'_{wx} : w \in W_f\}$ is a cellular subquotient of $\text{Res}_{\mathcal{H}} \mathcal{H}$, isomorphic as a $W_f$-graph to $\Gamma_{W_f}$. In particular,

$$\Gamma_{W_e} = \bigsqcup_{Q \in AT} \Gamma_Q$$

is the decomposition of $\text{Res}_{\mathcal{H}} \mathcal{H}$ into left cells.

Note that the definition (5.9) for the preorder $\leq_E$ works just as well for any module with a distinguished basis. Write $\leq_{\mathcal{H}^+}$ for the preorder on the canonical basis of $\mathcal{H}^+$ coming from considering $\mathcal{H}^+$ as a left $\mathcal{H}^+$-module. We also refer to $\mathcal{H}^+$ as a $W^+_e$-graph and say that $\leq_{\mathcal{H}^+}$ is the preorder on the $W^+_e$-graph $\mathcal{H}^+$. The notions of
ascent and corotation edges of §5.3.4 have their obvious meanings as certain relations in $\leq_{\widehat{\mathcal{H}}}$. Similar remarks apply to the partial order on the left cells of $\widehat{\mathcal{H}}^{+}$, also denoted $\leq_{\widehat{\mathcal{H}}}$. 

**Proposition 5.4.13.** The preorder $\leq_{\widehat{\mathcal{H}}}^{+}$ is the transitive closure of the relation $\leq_{\text{Res}, \widehat{\mathcal{H}}}^{+}$ and corotation edges.

**Proof.** This is clear from the description Proposition 5.4.9 (iv) of $\widehat{\mathcal{H}}^{+}$. □

**Corollary 5.4.14.** For any $x \in \mathcal{S}^{+}_{W}$, the set $\{C'_{wx} : w \in W_{f}\}$ is a cellular subquotient of the $W_{e}^{+}$-graph $\widehat{\mathcal{H}}^{+}$. This subquotient, restricted to be a $W_{f}$-graph, is isomorphic to the $W_{f}$-graph $\Gamma_{W_{f}}$. In particular,

$$\Gamma_{W_{e}^{+}} = \bigsqcup_{Q \in \text{PAT}} \Gamma_{Q}$$

is the decomposition of $\widehat{\mathcal{H}}^{+}$ into left cells.

**Proof.** As is evident from (5.24) (iii'), the submodule $(\widehat{\mathcal{H}}^{+})_{d}^{+}$ of $\widehat{\mathcal{H}}^{+}$ is cellular. Since corotation edges increase degree by 1, Proposition 5.4.13 implies that the preorder for the cellular subquotient $(\widehat{\mathcal{H}}^{+})_{d}^{+}/(\widehat{\mathcal{H}}^{+})_{d+1}$ is the same as that of $\text{Res}_{\mathcal{H}}(\widehat{\mathcal{H}}^{+})_{d}/(\widehat{\mathcal{H}}^{+})_{d+1}$. Thus since $\{C'_{wx} : w \in W_{f}\} \subseteq (\widehat{\mathcal{H}}^{+})_{d}$ for some $d$, the result is a special case of Corollary 5.4.12. □

## 5.5 Cocyclage, catabolism, and atoms

Before going deeper into the study of the canonical basis of $\widehat{\mathcal{H}}^{+}$, we need some intricate tableau combinatorics which will be used to describe cellular subquotients
of \( \hat{H}^+ \). In this section we discuss cocyclage, define a variation of catabolizability for affine tableaux, and introduce a formalism for comparing cellular subquotients of \( \hat{H}^+ \) to certain subsets of tableaux defined in [25] and [37]. Such subsets of tableaux were referred to as super atoms in [25]; here we refer to these subsets and their variations as atoms. This section is long and heavy in definitions, so the reader may wish to skim it and refer back to it as needed; the material here is used most extensively in §5.9.

5.5.1. Let \( \Theta, \nu \) be partitions with \( \nu \subseteq \Theta \). The diagram of a (skew) shape \( \theta = \Theta / \nu \) is the subset
\[
\{(r, c) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} : c \in [\nu_r + 1, \Theta_r]\}
\]
of the array \( \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \). Diagrams are drawn in English notation so that rows (resp. columns) are labeled starting with 1 and increasing from north to south (resp. west to east). We often refer to the diagram of \( \theta \) simply by \( \theta \).

The conjugate partition \( \lambda' \) of a partition \( \lambda \) is the partition whose diagram is the transpose of that of \( \lambda \).

A tableau \( T \) of shape \( \lambda \) is a filling of \( \lambda \) with entries in \( \mathbb{Z} \) so that entries strictly increase from north to south along columns and weakly increase from west to east along rows. We write \( \text{sh}(T) \) for the shape of \( T \).

5.5.2. Let us review the definitions of cocyclage poset and related combinatorics originating in [27, 28] (see also [37]).
The cocharge labeling \( v^{cc} \) of a standard word \( v \) is the (non-standard) word obtained from \( v \) by reading the numbers of \( v \) in increasing order; labeling the 1 of \( v \) with a 0, and if the \( i \) of \( v \) is labeled by \( k \), then labeling the \( i + 1 \) of \( v \) with a \( k \) (resp. \( k + 1 \)) if the \( i + 1 \) in \( v \) appears to the right (resp. left) of \( i \). For example, the cocharge labeling of 614352 is 302120; also see Example 5.6.2.

Write \( \text{rowword}(T) \) for the row reading word of a tableau \( T \). Define the cocharge labeling \( T^{cc} \) of a tableau \( T \) to be \( P(\text{rowword}(T)^{cc}) \), where numbers are inserted as for semistandard tableaux – if two numbers are the same, then the one on the right is considered slightly bigger. The tableau \( T^{cc} \) is also \( P(w^{cc}) \) for any \( w \) inserting to \( T \). This follows from the fact that Knuth transformations do not change left descent sets.

The sum of the numbers in the cocharge labeling of a standard word \( v \) (resp. standard tableau \( T \)) is the cocharge of \( v \) (resp. \( T \)) or \( \text{cocharge}(v) \) (resp. \( \text{cocharge}(T) \)). Cocharge of semistandard words and tableaux are more subtle notions, which we do not define in the usual way here. We will come across another way of understanding this statistic in §5.9.3.

For a composition \( \eta \) of \( n \), let \( \mathcal{W}(\eta) \) and \( \mathcal{T}(\eta) \) be the sets of semistandard words and semistandard tableaux of content \( \eta \), respectively.

For a semistandard word \( w \) and number \( a \neq 1 \), \( aw \) (resp. \( wa \)) is a corotation (resp. rotation) of \( wa \) (resp. of \( aw \)). There is a cocyclage from the tableau \( T \) to the tableau \( T' \), written \( T \xrightarrow{cc} T' \), if there exist words \( u, v \) such that \( v \) is the corotation of \( u \)
and $P(u) = T$ and $P(v) = T'$. Rephrasing this condition solely in terms of tableaux, $T \xrightarrow{cc} T'$ if there exists a corner square $(r, c)$ of $T$ and uninserting the square $(r, c)$ from $T$ yields a tableau $Q$ and number $a$ such that $T'$ is the result of column-inserting $a$ into $Q$.

If $\eta$ is a partition, then the cocyclage poset $CCP(T(\eta))$ is the poset on the set $T(\eta)$ generated by the relation $\xrightarrow{cc}$. For $\eta$ not a partition, the cocyclage poset $CCP(T(\eta))$ is defined in terms of $CCP(T(\eta_+))$ using reflection operators (see [37]), where $\eta_+$ denotes the partition obtained from $\eta$ by sorting its parts in decreasing order. The cyclage poset on $T(\eta)$ is the dual of the poset $CCP(T(\eta))$, i.e. the poset obtained by reversing all relations.

**Theorem 5.5.1 ([28]).** The cyclage poset on $T(\eta)$ is graded, with rank function given by cocharge.

Similarly, define $CCP(PAT)$ to be the poset on the set of PAT generated by cocyclage-edges (see §5.3.4). It inherits a grading from that of $W_e^+$ (see (5.22)).

The covering relations of $CCP(T(\eta))$ (resp. $CCP(PAT)$) are exactly cocyclages (resp. cocyclage-edges). We consider these covering relations to be colored by the additional datum $(r, c)$, the outer corner removed in performing the cocyclage.

In preparation for the formalism of §5.5.5, we define the category *Cocyclage Posets* as follows.

**Definition 5.5.2.** An object of Cocyclage Posets, called a cocyclage poset, is allowed to be either of the following:
• A subset \( X \) of \( T(\eta) \) with a poset structure generated by the cocyclages with both tableaux in \( X \).

• A subset \( X \) of \( \text{PAT} \) with the poset structure generated by the cocyclage-edges with both ends in \( X \).

A morphism \( f \) from \( X_1 \) to \( X_2 \) is a color-preserving map from \( X_1 \cup \{0\} \) to \( X_2 \cup \{0\} \) such that \( \text{sh}(f(T)) = \text{sh}(T) \) for all \( T \in X_1 \) and \( f(0) = 0 \), where \( 0 \) is the bottom element of \( X_i \cup \{0\} \). We take the convention that for each minimal element \( T \) of \( X_i \) and outer corner \((r, c)\) of \( \text{sh}(T) \), there is a cocyclage from \( T \) to \( 0 \) with color \((r, c)\), and \( 0 \) is considered to have any shape. Thus for a minimal \( T \in X_1 \), \( f(T) = 0 \) or \( f(T) \) is minimal in \( X_2 \).

Note that with this definition, a morphism \( f : X_1 \to X_2 \) is automatically order preserving, i.e. \( T \leq T' \) implies \( f(T) \leq f(T') \).

**Definition 5.5.3.** Two cocyclage posets \( X_1, X_2 \) are **strongly isomorphic** if there exists an isomorphism \( f : X_1 \to X_2 \) in Cocyclage Posets such that the uninsertion path and insertion path corresponding to the cocyclage \( T \xrightarrow{cc} T' \) are the same as those for \( f(T) \xrightarrow{cc} f(T') \), for all \( T \xrightarrow{cc} T' \) in \( X_1 \).

See Example 5.9.15 for an example of three isomorphic cocyclage posets, two of which are strongly isomorphic to each other, but not to the other.

**5.5.3.** Here we consider an adaptation of catabolizability to affine tableaux.
For a tableau $T$ and index $r$ (resp. index $c$), let $T_{r,\text{north}}$ and $T_{r,\text{south}}$ (resp. $T_{c,\text{east}}$ and $T_{c,\text{west}}$) be the north and south (resp. east and west) subtableaux obtained by slicing $T$ horizontally (resp. vertically) between its $r$-th and $(r + 1)$-st rows (resp. $c$-th and $(c + 1)$-st columns). For a tableau $T$ and partition $\lambda \subseteq \text{sh}(T)$, let $T_\lambda$ be the subtableau of $T$ obtained by restricting $T$ to the diagram of $\lambda$. For a tableau $T$ and $a \in \mathbb{Z}$, let $T + a$ denote the tableau obtained by adding $a$ to all entries of $T$.

Let $Q$ be a PAT of shape $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\eta = (\eta_1, \ldots, \eta_k)$ a composition of $r$. Let $R_1$ be the partition $(\lambda_1, \ldots, \lambda_{\eta_1})$. If $R_1 \subseteq \text{sh}(T)$, then define the $R_1$-row catabolism of $T$, notated $\text{rcat}_{R_1}(T)$, to be

$$(n + T_{\eta_1,\text{north}}^*)T_{\eta_1,\text{south}}^*,$$

where $T^*$ is the skew subtableau of $T$ obtained by removing $T_{R_1}$.

For $Q, \eta, R$ as above, $(Q, \eta)$-row catabolizability is defined inductively as follows: $\emptyset$ is the unique $(\emptyset, (\))$-row catabolizable tableau; otherwise set $\eta = (\eta_1, \tilde{\eta})$ and define $T$ to be $(Q, \eta)$-row catabolizable if $T_{R_1} = Q_{R_1}$ and $\text{rcat}_{R_1}(T)$ is $(Q_{\eta_1,\text{south}}, \tilde{\eta})$-row catabolizable.

Column catabolizability is defined similarly: let $Q$ be a PAT of shape $\lambda$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_c)$, $\eta = (\eta_1, \ldots, \eta_k)$ a composition of $c$. Let $C_1$ be the partition $(\lambda'_1, \ldots, \lambda'_{\eta_1})'$. If $C_1 \subseteq \text{sh}(T)$, the $C_1$-column catabolism of $T$, notated $\text{ccat}_{C_1}(T)$, is the tableau

$$T_{\eta_1,\text{east}}^*(T_{\eta_1,\text{west}}^* - n),$$

where $T^*$ is the skew subtableau of $T$ obtained by removing $T_{C_1}$. 
For $Q, \eta, R$ as above, $(Q, \eta)$-column catabolizability is defined inductively as follows: $\emptyset$ is the unique $(\emptyset, (\emptyset))$-column catabolizable tableau; otherwise set $\eta = (\eta_1, \hat{\eta})$ and define $T$ to be $(Q, \eta)$-column catabolizable if $T_{C_1} = Q_{C_1}$ and $\text{ccat}_{C_1}(T)$ is $(Q_{\eta_1, \text{east}}, \hat{\eta})$-column catabolizable.

If $(n + T_{\eta_1, \text{north}}^*)$ is replaced by $T_{\eta_1, \text{north}}^*$ in the definition of row-catabolizability above and $Q$ is a superstandard tableau, then we recover the definition of catabolizability in [37].

**Example 5.5.4.** Let $A, B, C, D, E$ denote the integers 10, 11, 12, 13, 14, and maintain the convention of §5.4.2 of writing $ab$ for $na + b$, $a, b \in \mathbb{Z}$. Let

\[
Q = \begin{bmatrix}
1 & 2 & 3 & 14 & 15 & 16 \\
14 & 15 & 16 & 27 & 28 & 29 \\
3A & 3B & 3C & 4D & 4E
\end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix}
1 & 2 & 3 & 17 & 2E \\
14 & 15 & 16 & 2A \\
28 & 29 & 3E & 3D \\
5E & 5D
\end{bmatrix}.
\] (5.26)

The tableau $T$ is $(Q, \eta)$-row catabolizable for the following $\eta$: $(2, 2, 1), (2, 1, 2), (1, 2, 2)$ and all refinements of these compositions. The following computation shows $T$ to be $(Q, (1, 2, 2))$-row catabolizable: We have $T_{(3)} = Q_{(3)}$ and the tableaux $n + T_{1, \text{north}}^*$ and $T_{1, \text{south}}^*$ are the left-hand side of

\[
\begin{bmatrix}
27 & 3E & 14 & 15 & 16 & 2A \\
28 & 29 & 3E & 3D \\
3C & 3D
\end{bmatrix} \equiv \begin{bmatrix}
14 & 15 & 16 & 2A \\
27 & 28 & 29 & 3D \\
3A & 3B & 3C & 4D
\end{bmatrix}.
\] (5.27)

Letting $P$ be the tableau on the right, then $P$ is $(Q_{1, \text{south}}, (2, 2))$-row catabolizable as $P_{(3, 3)} = (Q_{1, \text{south}})_{(3, 3)}$ and the computation

\[
\begin{bmatrix}
3A & 4E & 4D \\
3E & 3C & 3D & 4D
\end{bmatrix} \equiv \begin{bmatrix}
3A & 3B & 3C & 4D & 4E
\end{bmatrix}.
\] (5.28)
Figure 5.1: The \((Q, (1, 2, 1))\)-column catabolizable tableaux with \(Q\) the tableau on the top row.

The tableau on the right is \((Q_{3, \text{south}}, (2))\)-row catabolizable.

Figure 5.1 depicts the set of \((Q, (1, 2, 1))\)-column catabolizable tableaux for \(Q\) the tableau in the top row of the figure. The \((Q, (3, 1))\)-column catabolizable tableaux are \(Q\) and the first two tableaux in the second row. The \((Q, (1, 3))\)-column catabolizable tableaux are \(Q\), the last two tableaux on the second row, and the last tableau on the third row.

5.5.4. Let \(Z^*_{\lambda}\) be the standard tableau with \(l_r + 1, l_r + 2, \ldots, l_{r+1}\) in the \(r\)-th row, where \(l_r = \sum_{i=1}^{r-1} \lambda_i\) are the partial sums of \(\lambda\) (the empty sum is understood to be 0). The tableau \(G_\lambda\) mentioned in the introduction is \(nZ^*_{\lambda} + Z^*_{\lambda}\), where the sum is taken entry-wise (see Proposition 5.6.8). The dual Garnir tableau of shape \(\lambda\) is the highest
degree occurrence of a PAT of shape \( \lambda \) in \( \mathcal{A}_{1^n} \), denoted \( G^\vee_\lambda \).

Let us briefly introduce a certain duality in \( \mathcal{A}_{1^n} \), which will be discussed more thoroughly in §5.7. For a standard word \( x = x_1 \cdots x_n \), let \( x^\dagger \) denote the word \( x_n x_{n-1} \cdots x_1 \). Then for any \( w \) with \( C'_w \in \mathcal{A}_{1^n} \), \( w_i = x_i^\text{cc} + x_i \) for all \( i \in [n] \) for some standard word \( x \) (see §5.6.1). Define the dual element \( w^\vee \) by \( w_i^\vee = (x^\dagger)_i^\text{cc} + x_i^\dagger \), \( i \in [n] \). Extend this notation to tableaux by defining \( T^\vee \) to be \( P(w^\vee) \) for any (every) \( w \) inserting to \( T \).

Note that \( G_n = G^\vee_n \) is the single row tableau \( \begin{array}{c} 1 \ 2 \ \cdots \ n \end{array} \) and \( G_1^n = G^\vee_1^n \) is the single column tableau with the entry \((r - 1) \cdot r\) in the \( r \)-th row.

**Proposition 5.5.5.**

(a) A PAT \( T \) is \((G_n, \lambda)\)-column catabolizable if and only if \( T \) is \((G_\lambda, 1^{\ell(\lambda)})\)-row catabolizable.

(b) A PAT \( T \) is \((G^\vee_\lambda, 1^{\ell(\lambda)})\)-column catabolizable if and only if \( T \) is \((G^\vee_{1^n}, \lambda)\)-row catabolizable.

**Proof.** The CCP(SYT) is isomorphic to the subposet of CCP(PAT) consisting of the left cells of \( \mathcal{A}_{1^n} \) (Proposition 5.6.8). Given this, statement (a) is the equivalence of row and column catabolizability established in [37] (see [2] for a nice proof). Statement (b) follows from the facts that \( T \) is \((G_n, \lambda)\)-column catabolizable if and only if \( T^\vee \) is \((G^\vee_{1^n}, \lambda)\)-row catabolizable and \( T \) is \((G_\lambda, 1^{\ell(\lambda)})\)-row catabolizable if and only if \( T^\vee \) is \((G^\vee_\lambda, 1^{\ell(\lambda)})\)-column catabolizable. \( \square \)

From a well-known result about catabolizability of standard tableaux, a tableau
labeling a left cell of \( R_1 \) is \((G_n, \lambda)\)-column catabolizable for a unique maximal in dominance order partition \( \lambda \). We write \( \text{ctype}(T) \) for this partition and also use this notation for the usual notion of catabolizability if \( T \) is a standard tableaux (see [37]).

We will see in §5.9.4 that for certain \( Q \) of shape \( \lambda \), the set of \((Q, \eta)\)-column catabolizable tableaux is strongly isomorphic to a dual version of the original set of tableaux \( CT(\lambda; R) \) defined in [37].

5.5.5. At the risk of being overly formal, we will define several categories which are generalizations or variations of the cocyclage posets of Lascoux and Schützenberger and the super atoms of Lascoux, Lapointe, and Morse [25]. We will primarily be concerned with the underlying sets of objects of these categories and isomorphism in these categories.

For a ring \( k \) and \( k \)-algebra \( H \), the category of \( H \)-modules with basis has objects that are pairs \((E, \Gamma)\), where \( E \) is a free \( k \)-module and an \( H \)-module (the action of \( H \) extends that of \( k \)) with \( k \)-basis \( \Gamma \). A morphism \((E, \Gamma) \to (E', \Gamma')\) is an \( H \)-module morphism \( \theta : E \to E' \) such that \( \theta(\gamma) \in \Gamma' \cup \{0\} \) for all \( \gamma \in \Gamma \).

Let \( P \) and \( P' \) be \( \text{PAT} \) and \( \Gamma_P, \Gamma_{P'} \) the corresponding cells of \( \Gamma_{W_f^+} \). By Proposition 5.3.3 together with the facts that a left cell of \( \Gamma_{W_f} \) is irreducible at \( u = 1 \) and the left cells corresponding to the same shape are isomorphic as \( W_f \)-graphs [22, Theorem
1.4], we have

(5.29.i) If \( \text{sh}(P) = \text{sh}(P') \), then there are exactly two morphisms from \( \Gamma_P \) to \( \Gamma_{P'} \):

- the 0 map and the map taking \( w \) to \( w' \) for \( w \xrightarrow{\text{RSK}} (P, Q), w' \xrightarrow{\text{RSK}} (P', Q) \)

for all SYT \( Q \) of shape \( \text{sh}(P) \).

(5.29.ii) If \( \text{sh}(P) \neq \text{sh}(P') \), then the 0 map is the only morphism from \( \Gamma_P \) to \( \Gamma_{P'} \).

The following categories will be denoted by the plural form of an object in the category, i.e., a cocyclage atom is an object in the category Cocyclage Atoms.

- **Cellular subquotients of \( \hat{H}^+ \) (CSQ(\( \hat{H}^+ \)))**: The full subcategory of \( \hat{H}^+ \)-modules with basis whose objects are cellular subquotients of \( \hat{H}^+ \) with the canonical basis.

- **Cocyclage Atoms**: Let \( \Pi^+ \) be the submonoid \( \Pi \cap W_e^+ = \langle \pi \rangle \) of \( W_e^+ \), where \( \Pi \) is as in §5.2.4. Write \( A\Pi^+ \) for the corresponding subalgebra of \( \hat{H}^+ \). A cocyclage atom is a union \( E \) of left cells of \( \hat{H}^+ \) such that \( \text{Res}_{A\Pi^+} E \) is a cellular subquotient of \( \text{Res}_{A\Pi^+} \hat{H}^+ \). A morphism \( \alpha : E \rightarrow E' \) is a morphism in the category of \( A\Pi^+ \)-modules with basis (the basis for an object being the canonical basis) such that the composition \( A\Gamma \hookrightarrow E \overset{\alpha}{	woheadrightarrow} E' \rightarrow A\Gamma' \) is of the form (5.5.5.i) or (5.5.5.ii) for \( \Gamma, \Gamma' \) left cells of \( E, E' \). Equivalently, a cocyclage atom is a convex induced subposet of \( \text{CCP}(PAT) \). A morphism is the same as a morphism in Cocyclage Posets (Definition 5.5.2).

- **Cocyclage Posets** as in Definition 5.5.2.
\[ R \times W_f\text{-Mod}. \]

**Definition 5.5.6.** For \( Q, P \in \text{PAT} \), the *cellular subquotient* \( A_{Q,P}^{\text{csq}} \) is the minimal cellular subquotient of \( \hat{H}^+ \) containing \( Q \) and \( P \).

- **Garsia-Procesi atoms** (GP atoms). The full subcategory of cellular subquotients of \( \hat{H}^+ \) with objects \( \{ A_{G_n,G_\lambda}^{\text{csq}} : \lambda \vdash n \} \). In §5.8, we will show that \( A_{G_n,G_\lambda}^{\text{csq}} \) is the \( u \)-analogue of the Garsia-Procesi module \( R/I_\lambda \) and contains exactly those left cells labeled by Proposition 5.5.5 (a).

- **dual Garsia-Procesi atoms** (dual GP atoms): A dual GP atom \( A_{G_\lambda,G_{1^n}}^{\text{GP'}} \) consists of the tableaux satisfying either catabolizability condition of Proposition 5.5.5 (b). This is conjecturally equal to (as a set of tableaux) \( A_{G_\lambda,G_{1^n}}^{\text{csq}} \). This category is the full subcategory of Cocyclage Posets with objects \( \{ A_{G_\lambda,G_{1^n}}^{\text{GP'}} : \lambda \vdash n \} \).

- **Shimozono-Weyman atoms** (SW atoms): The SW atom \( A_{Q,\eta}^{\text{SWr}} \) (resp. \( A_{Q,\eta}^{\text{SWc}} \)) is the set of \((Q, \eta)\)-row (resp. column) catabolizable tableaux. This is conjecturally a cocyclage atom and, stronger, a cellular subquotient of \( \hat{H}^+ \). This category is the full subcategory of Cocyclage Posets consisting of these atoms.

- **Lascoux-Lapointe-Morse atoms** (LLM atoms): LLM atoms will be defined in §5.9.6 as the intersection of certain SW atoms. Again, these are conjecturally cocyclage atoms and cellular subquotients of \( \hat{H}^+ \).

- **Li-Chung Chen atoms** (Chen atoms): Chen atoms are a generalization of LLM atoms, also defined as the intersection of certain SW atoms; see §5.9.5. Again,
these are conjecturally cocyclage atoms and cellular subquotients of $\mathcal{H}^+$.

5.5.6. Let us establish some basic properties of these categories. We have the following diagram of functors:

$$\begin{array}{ccccccc}
GP \text{ atoms} & \rightarrow & SW \text{ atoms} & \rightarrow & \text{Cocyclage Posets} \\
dual \text{ GP atoms} & \rightarrow & Chen \text{ atoms} & \rightarrow & CSQ(\mathcal{H}^+) & \rightarrow & \text{Cocyclage Atoms} \\
\text{LLM atoms} & \rightarrow & \text{CSQ}(\mathcal{H}^+) & \rightarrow & R \ast W_f \text{-Mod} \\
\end{array}$$

The functors on the left-hand side and the functors from SW atoms and Chen atoms to Cocyclage Posets are inclusions of full subcategories. The functors from SW atoms and Chen atoms to $CSQ(\mathcal{H}^+)$ are conjectural.

The functor $F^{cc}$ just restricts a $\mathcal{H}^+$-module with basis to an $A\Pi^+$-module with basis. The functors $F^{sp}$ just forget about $\mathcal{H}^+$ or $A\Pi^+$-module structures and restrict to the underlying poset of tableaux corresponding to the poset of left cells. By definition, the cocyclage poset $F^{sp}(E)$, $E \in CSQ(\mathcal{H}^+)$, has covering relations corresponding exactly to the cocyclage-edges of $\leq_E$ (see §5.3.4). We record the fact, immediate from Proposition 5.4.13, that

**Proposition 5.5.7.** For any $E \in CSQ(\mathcal{H}^+)$, the partial order $\leq_E$ on cells is the transitive closure of $\leq_{\Res,H} E$ and cocyclage-edges.

The functor $F^{mod}$ takes $E$ to $C \otimes_A E$ and forgets about the canonical basis, where $A \rightarrow C$ is given by $u \mapsto 1$. Thus cellular subquotients of $\mathcal{H}^+$ are a strengthening of the combinatorial Cocyclage Atoms and algebraic $R \ast W_f$-modules.
**Definition 5.5.8.** A cocyclage poset or cocyclage atom is *connected* if its poset is connected as an undirected graph.

### 5.6 A $W^+_e$-graph version of $R_{1^n}$

We exhibit a cellular subquotient $R_{1^n}$ of $\widehat{H}^+$ which is a $W^+_e$-graph version of the ring of coinvariants $R_{1^n}$. We show that under a natural identification of the left cells of $R_{1^n}$ with SYT, the subposet of $\leq_{R_{1^n}}$ consisting of the cocyclage-edges is exactly the cocyclage poset on SYT.

#### 5.6.1. There are two important theorems that give the canonical basis of $\widehat{H}$ a more explicit description. These theorems hold in arbitrary type, but we state them in type $A$ to simplify notation.

Recall that $Y_+ \subseteq Y$ is the set of dominant weights, which in type $A_{n-1}$ are weakly decreasing $n$-tuples of integers; put $Y^+_+ = Y^+ \cap Y_+$. As is customary, let $w_0$ denote the longest element of $W_f$. If $\lambda \in Y_+$, then $w_0y^\lambda$ is maximal in its double coset $W_fy^\lambda W_f$.

For $\lambda \in Y^+_{++}$, let $s_\lambda(Y) \in \widehat{H}$ denote the Schur function of shape $\lambda$ in the Bernstein generators $Y_i$.

**Theorem 5.6.1** (Lusztig [31, Proposition 8.6]). *For any* $\lambda \in Y^+_{++}$, *the canonical basis element* $C'_{w_0y^\lambda}$ *can be expressed in terms of the Bernstein generators as*

$$C'_{w_0y^\lambda} = s_\lambda(Y)C'_{w_0} = C'_{w_0}s_\lambda(Y).$$
Now define the descent monomials \( y^\beta, \beta \in D \), where \( D \subseteq Y^+ \) is defined as follows. For any \( v \in W_f \), let \( J = R(v) \). Put \( \lambda = \sum_{i \in S \backslash J} \varpi_i \), where \( \varpi_i = \sum_{k=1}^{i} \epsilon_k \) is the \( i \)-th fundamental weight. As \( \lambda \in Y^+ \) allows \( v \) to be recovered from \( v(\lambda) \), the map \( W_f \to Y^+, v \mapsto v(\lambda) \) is injective. Define \( D \) to be its image. Next, put

\[
D^S := \{ y^\beta_S : \beta \in D \},
\]

\[
D^S w_0 := \{ y^\beta_S w_0 : \beta \in D \},
\]

which are the minimal and maximal coset representatives corresponding to descent monomials. The set \( D^S w_0 \) will index a canonical basis of the coinvariants.

With \( \lambda, J \) as above, the stabilizer of \( \lambda \) under the \( W_f \)-action is the parabolic subgroup \( W_{f,J} \). And then it is not hard to see that

\[
w := v \cdot y^\lambda = y^{v(\lambda)} v
\]

is maximal in its left coset \( wW_f \), i.e., is the element of \( D^S w_0 \) corresponding to the descent monomial \( y^{v(\lambda)} \).

Descent monomials have a nice interpretation in terms of affine words. Thinking of \( v \) as the inverted window word for an element of \( W_f \) and using the notation \( v^{cc} \) of §5.5.2 for the cocharge labeling of \( v \), then

\[
v(\lambda) = v^{cc},
\]

and the word of \( w = y^{v(\lambda)} v \) is given by \( w_i = v^{cc}_i n + v_i \).

**Example 5.6.2.** For the \( v \in S_9 \) given by its word below, the corresponding \( v^{cc}, w, \)
λ, and J follow. For the computation of J, recall the convention (5.17).

\[
v = 1 6 8 4 2 9 5 7 3, \\
v(\lambda) = v^{cc} = 0 2 3 1 0 3 1 2 0, \\
w = y^{v(\lambda)} = 1 26 38 14 2 39 15 27 3, \\
\lambda = (3, 3, 2, 2, 1, 1, 0, 0, 0), \\
J = \{s_1, s_3, s_5, s_7, s_8\}.
\]

The lowest two-sided cell of \(W_e\) is the set \(\{w \in W_e : w = x \cdot w_0 \cdot z, \text{ for some } x, z \in W_e\}\), denoted \(W_\nu\). As preparation for the next theorem, we have a proposition giving the factorization of any \(w \in W_\nu \cap W_e^+\) in terms of descent monomials. This is not too hard to see from the combinatorial description, however it is more easily proved with the help of a geometric description of \(D^S\) in terms of alcoves, which we omit here.

**Proposition 5.6.3** ([3, Proposition 3.7]). For any \(w \in W_\nu \cap W_e^+\), there exists a unique expression for \(w\) of the form

\[
w = u_1 \cdot w_0 y^\lambda \cdot u_2
\]

where \(u_1, \Psi(u_2)\) are primitive and \(\lambda \in Y^+_+\) (\(\Psi\) is defined in §5.4.3).

**Proof.** This follows easily from the corresponding [3, Proposition 3.7] for \(G = SL_n\).

\[\square\]

The next powerful theorem simplifying the canonical basis of \(\hat{\mathcal{H}}^+\) is due to Xi ([41, Corollary 2.11]), also found independently by the author. In the language of
[41], the condition \( w \in D^S \) is written \( wA_{v'} \subseteq \Pi_{v'} \), where \( v' \) is a special point, \( A_{v'} \) is an alcove, and \( \Pi_{v'} \) is a box. We state here a combination of Lusztig’s theorem (Theorem 5.6.1) and Xi’s theorem.

For \( v \) such that \( v \) minimal in \( vW_f \) (resp. \( W_f v \)), define \( \widetilde{C}'_v \) (resp. \( \overrightarrow{C}'_v \)) by \( \overrightarrow{C}'_v = \overrightarrow{C}'_v C'_{w_0} \) (resp. \( C'_{w_0} = C'_{w_0} \widetilde{C}'_v \)).

**Theorem 5.6.4.** For \( w \in W(\nu) \cap W^+ \) and with \( w = u_1 \cdot w_0 y^\lambda \cdot u_2 \) as in Proposition 5.6.3, we have the factorization

\[
C'_w = s_\lambda(Y) \widetilde{C}'_{u_1} C'_{w_0} \overrightarrow{C}'_{u_2}.
\] (5.35)

**Remark 5.6.5.** This theorem is most natural for root systems associated to simply connected Lie groups \( G \) (in particular, more natural for \( G = SL_n \) than \( G = GL_n \)), because in the simply connected case, the set playing the role of \( D^S \) is naturally in bijection with \( W_f \). The result Xi proves is for the simply connected case. However, if we work with the positive part \( W^+_e \) of \( W_e \), the \( G = GL_n \) case is just as nice or nicer than the \( SL_n \) case.

**5.6.2.** Let \( e^+ = C'_{w_0} \). Then \( Ae^+ \) is the one-dimensional trivial left-module of \( \mathcal{H} \) in which the \( T_i \) act by \( u \) for \( i \in [n-1] \). The \( \widehat{\mathcal{H}}^+ = \mathcal{H}^+ \otimes \mathcal{H}^+ \) is a \( u \)-analogue of the polynomial ring \( R \); more precisely, \( \widehat{\mathcal{H}}^+ e^+ \) is a \( u \)-analogue of the left \( R \ast W_f \)-module \( R e^+ \). Without saying so explicitly, we will identify the \( \widehat{\mathcal{H}}^+ \)-module \( \widehat{\mathcal{H}}^+ e^+ \) with the cellular submodule of \( \widehat{\mathcal{H}}^+ \) spanned by \( \{ C'_w : w \text{ maximal in } wW_f \} \) as modules with basis. (It is easy to see directly that this is possible; it is also a special
case of general results about inducing $W$-graphs [1, Proposition 2.6].

Let $\mathcal{R}$ denote the subalgebra of $\widehat{\mathcal{H}}^+$ generated by the Bernstein generators $Y_i$. Thus $\mathcal{R} \cong R$ as algebras. Write $(Y^+)^W_{\geq d} \subseteq \mathcal{R}$ for the set of $W_f$-invariant polynomials of degree at least $d$. Now Theorem 5.6.4 applied to the canonical basis of $\widehat{\mathcal{H}}^+e^+$ yields the following corollary, which gives a $u$-analogue of the ring of coinvariants. Later, in §5.9.1, we will prove a more general result (Theorem 5.9.7) that uses the full power of Theorem 5.6.4.

**Corollary 5.6.6.** The $\widehat{\mathcal{H}}^+_n$-module $\widehat{\mathcal{H}}^+_n e^+$ has a cellular quotient equal to

$$\mathcal{R}_{1^n} := \widehat{\mathcal{H}}^+_n e^+/\widehat{\mathcal{H}}^+_n (Y^+_1)^{S_n} e^+$$

with canonical basis $\{C'_w : w \in D^S w_0\}$.

A careful proof of this corollary is postponed to the proof of Theorem 5.9.1.

**Example 5.6.7.** The $W^+_e$-graph $\mathcal{R}_{13}$ is drawn in Figure 5.2 with the following conventions: basis elements of the same degree are drawn on the same horizontal level; the edges with a downward component are exactly the corotation-edges (these correspond to left-multiplication by $\pi$ and increase degree by 1); for any $C'_u, C'_v$ of the same degree, an edge between them with no arrow indicates that $\mu = 1$ and $L(u) \not\subseteq L(v)$ and $L(v) \not\subseteq L(u)$; an edge from $C'_u$ to $C'_v$ indicates that $\mu = 1$ and $L(u) \not\subset L(v)$; no edge between $C'_u$ and $C'_v$ indicates that $\mu = 0$ or $L(u) = L(v)$.

**5.6.3.** Let us now describe the combinatorics of the cellular subquotient $\mathcal{R}_{1^n}$. Let $\leq_{\mathcal{R}_{1^n}}$ be the preorder of the $W^+_e$-graph $\mathcal{R}_{1^n}$, which is the restriction of the preorder
Figure 5.2: On the left is the $W^+_e$-graph of $\mathcal{R}_{1^3}$ with three labels for each canonical basis element. The bottom labels are inverted window words. On the right are the corresponding left cells and the covering relations of the partial order on left cells.

\[ \leq_{\hat{H}^+} \] on $\hat{H}^+$ to the subquotient $\hat{R}_{1^n}$. First of all, we know from Proposition 5.5.7 that the partial order $\leq_{\hat{R}_{1^n}}$ on cells is the transitive closure of $\leq_{\text{Res}_{\hat{H}} \hat{R}_{1^n}}$ and cocyclage-edges (see §5.3.4).

Let $T + T'$ denote the entry-wise sum of two tableau $T, T'$ of the same shape.

**Proposition 5.6.8.** The map

\[ \text{CCP}(\text{SYT}) \to \text{F}^{sp}(\mathcal{R}_{1^n}), T \mapsto nT^{cc} + T \]

is an isomorphism in Cocyclage Posets ($T^{cc}$ is defined in §5.5.2).

**Proof.** Since under the bijection $D^S w_0 \to W_f$, $w = v \cdot y^\lambda \mapsto v$ (see (5.32)), $w_S = v$ holds, $w = nv^{cc} + v$ (see 5.33) implies $P(w) = P(nv^{cc} + v) = nP(v)^{cc} + P(v)$. The statement is then a consequence of the stronger statement that under the bijection $v \mapsto nv^{cc} + v$ between $W_f$ and $D^S w_0$, corotation of standard words corresponds exactly to corotation of affine words. To see that a corotation of a standard word maps to a
corotation of an affine word, observe that corotating a standard word adds 1 to the cocharge label of the corotated number. To go the other way, use that $D^S w_0 \to W_f$ can be computed from $v_i = \hat{w}_i$, with $\hat{w}_i$ as in §5.4.2. Finally, observe that the last number of $v$ is 1 exactly when $\pi(nv^{cc} + v) \notin D^S w_0$. \hfill \Box

Figure 5.3 depicts the cells of the $W^+_e$-graph on $R_{1^n}$ and the partial order $\leq_{R_{1^n}}$ on cells.

5.7 A duality in $R_{1^n}$

It is well-known that there is a perfect pairing $\langle , \rangle : R_{1^n} \times R_{1^n} \to \mathbb{C}$ given by $\langle f_1, f_2 \rangle$ equal to the projection of $f_1 f_2$ onto the sign representation of $R_{1^n}$. With this, it is easy to show that an irreducible $V_\lambda \subseteq R_{1^n}$ in degree $d$ is dual to an irreducible $V_\lambda^\vee \subseteq R_{1^n}$ in degree $\left\langle \binom{n}{2} \right\rangle - d$. This duality on the character of $R_{1^n}$ is also easy to see from the cellular picture, as we will now show. However, there appears to be a stronger duality in the $W^+_e$-graph $R_{1^n}$ which is surprisingly subtle.

5.7.1. For a standard word $x = x_1 \cdots x_n$, let $x^\dagger$ denote the word $x_n x_{n-1} \cdots x_1$. For any $w \in D^S w_0$, let $x$ be the corresponding element of $W_f$ under the bijection $D^S w_0 \cong W_f$ (see (5.32)), i.e., $w_i = x_i^{cc} + x_i$ for all $i \in [n]$. Define the dual element $w^\vee$ to be the element of $D^S w_0$ that corresponds to $x^\dagger$ under bijection $D^S w_0 \cong W_f$, i.e., $w_i^\vee = (x_i^\dagger)^{cc} + x_i^\dagger$, $i \in [n]$. Extend this notation to tableaux by defining $T^\vee$ to be $P(w^\vee)$ for any (every) $w$ inserting to $T$. 
Figure 5.3: The cells of the $W^+_e$-graph on $R_{15}$. Edges are the covering relations of the partial order on cells.
From well-known properties of the insertion algorithm, the tableaux $P(x)$ and $P(x^\dagger)$ are transposes of each other for any standard word $x$. Therefore, if $T$ is a PAT labeling a cell of $R_{1^n}$, then $T$ and $T^\vee$ have shapes that are transposes of each other. Let $\rho \in Y_+^+$ be half the sum of the positive roots, i.e., $\rho = (n-1, n-2, \ldots, 0)$; note that $P(y^\rho) = G_{1^n}$. Then $\Psi(w^\vee)w = w_0\rho$ for any $w \in D^S w_0$. In particular, the sum of the degrees of $T$ and $T^\vee$ is $(n \choose 2)$. Thus we have shown that $^\vee$ corresponds to a duality on the character of $R_{1^n}$.

We have the following conjectural duality for the $W_e^+$-graph $R_{1^n}$. The first part of this conjecture is proved below.

**Conjecture 5.7.1.** For any $x, w \in D^S w_0$,

(a) if $x = \pi w$, then $x^\vee = \pi^{-1} w^\vee$.

(b) $\mu(x, w) = \mu(w^\vee, x^\vee)$.

One route to proving this conjecture is to exhibit a perfect pairing on $R_{1^n}$ that respects canonical bases. This does not seem to work in a straightforward way, however the following approach seems promising.

For $h \in \widehat{H}^+$ write $[C'_w]h$ for the coefficient of $C'_w$ of $h$ written as an $A$-linear combination of $\{C'_w : w \in W_e^+\}$. Define $\langle , \rangle : \widehat{H}^+ \times \widehat{H}^+ \to A$ by

$$\langle h_1, h_2 \rangle = [C'_{w_0 y^\rho}]h_1 h_2.$$ 

Let $j : A \to A$ be the ring automorphism determined by $j(u) = -u^{-1}$, and also denote by $j$ the involution of $\widehat{H}$ given by $j(\sum_x a_x T_x) = \sum_x j(a_x) T_x$. The unprimed
canonical basis element $C_w$, $w \in W_e$, is related to the primed $C'_w$ by $j(C'_w) = C_w$.

**Conjecture 5.7.2.** For $x \in W_e^{+S}w_0$ and $w \in \Psi(W_e^{+S}w_0)$,

$$
\langle C_x, C'_w \rangle = \begin{cases} 
1 & \text{if } w \in D^S w_0, \ x \in \Psi(D^S w_0), \text{ and } \Psi(x) = w^\vee, \\
0 & \text{otherwise}.
\end{cases}
$$

As introduced in §5.4.3, there is an automorphism $\Delta$ of $W_e$ given on generators by $s_i \mapsto s_{n-i}$, $\pi \mapsto \pi^{-1}$.

**Proposition 5.7.3.** Conjecture 5.7.2 implies Conjecture 5.7.1 holds in the case $L(x) \cap S \nsubseteq L(w) \cap S$.

Recall that $L(x) \cap S \nsubseteq L(w) \cap S$ if and only if the edge weight $\mu(x, w)$ matters for the structure of $R_{1n}$ as an $\hat{\mathcal{H}}^+$-module, and therefore the main case we are interested in.

**Proof of Proposition 5.7.3.** It is easy to see that corotating and then applying $\dagger$ to a standard word is the same as applying $\dagger$ and then rotating. Part (a) then follows from the bijection $W_f \cong D^S w_0$ and the fact that this takes corotations to corotation-edges (see Proposition 5.6.8 and its proof).

For any $x \in D^S w_0$, the following are straightforward from the definitions of $\Psi$ and $\vee$:

$$
R(\Psi(x)) = \Delta(L(x)),
$$

$$
L(x^\vee) \cap S = \{\Delta(s) : s \in S \backslash L(x)\},
$$

$$
R(\Psi(x^\vee)) \cap S = \{s : s \in S \backslash L(x)\}.
$$

(5.36)
Suppose \( x, w \in D^{S_{w_0}}, L(x) \cap S \not\subseteq L(w) \cap S \), and let \( s \) be any element of \( S \cap L(x) \setminus L(w) \).

We have

\[
\mu(w^\vee, x^\vee) = \mu(\Psi(w^\vee), \Psi(x^\vee)) = [C_{\Psi(w^\vee)}]C_{\Psi(x^\vee)}T_s
\]

\[
= \langle C_{\Psi(x^\vee)}T_s, C_s' \rangle = \langle C_{\Psi(x^\vee)}T_sC_s', C_s' \rangle = [C_s']T_sC_s' = \mu(x, w). \tag{5.37}
\]

The first equality holds because \( \Psi \) is an anti-automorphism of the extended Coxeter group \( W_e \). The third and fifth equalities use Conjecture 5.7.2. The last equality follows from \( s \in S \cap L(x) \setminus L(w) \) and the definition of a \( W \)-graph (5.7). Noting that (5.36) implies \( s \in S \cap R(\Psi(w^\vee)) \setminus R(\Psi(x^\vee)) \), the second equality follows for the same reason.

\[\square\]

## 5.8 Extending the Garsia-Procesi approach using finite-dimensional \( \widehat{H} \)-modules studied by Bernstein and Zelevinsky

The Garsia-Procesi approach to understanding the \( R_\lambda = R/I_\lambda \) realizes \( I_\lambda \) as the ideal of leading forms of functions vanishing on an orbit \( S_n a \), for certain \( a \in \mathbb{C}^n = \text{Spec } R \).

We adapt this approach to the Hecke algebra setting using certain representations of \( \widehat{H} \) studied by Bernstein and Zelevinsky. Our main result, Theorem 5.8.8, shows that the \( u \)-analogues \( \mathcal{R}_\lambda \) of the \( R_\lambda \) are actually cellular.

Let \( \mathcal{C}_n^\mathbb{Z} \) (resp. \( \mathcal{C}_n^{+\mathbb{Z}} \)) be the category of finite-dimensional \( \widehat{H}_n \)-modules (resp. \( \widehat{H}^+ \)-
modules) in which the $Y_i$'s have their eigenvalues in $u^{2\mathbb{Z}}$. In the next subsection, we review the needed results about the category $C^\mathbb{Z}_n$, referring the reader to [29, 40] for a more thorough treatment.

5.8.1. For $\eta = (\eta_1, \eta_2, \ldots, \eta_r)$ an $r$-composition of $n$, write $l_j = \sum_{i=1}^{j-1} \eta_i, j \in [r+1]$ for the partial sums of $\eta$ (where the empty sum is defined to be 0). Let $B_j$ be the interval $[l_j + 1, l_j + 1]$, $j \in [r]$, and define

$$J_\eta = \{ s_i : \{i, i+1\} \subseteq B_j \text{ for some } j \}$$

(5.38)

so that $S_{n,J_\eta} \cong S_{\eta_1} \times \cdots \times S_{\eta_r}$.

Let $\widehat{H}_\eta$ be the subalgebra of $\widehat{H}$ generated by $H_{I_\eta}$ and $Y_i^{\pm 1}, i \in [n]$. The algebra $\widehat{H}_\eta$ is isomorphic to $\widehat{H}_{\eta_1} \times \cdots \times \widehat{H}_{\eta_r}$. Similarly, let $\widehat{H}_\eta^+$ be the subalgebra of $\widehat{H}^+$ generated by $H_{J_\eta}$ and $Y_i, i \in [n]$. For $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$, let $C_{\eta,a}$ be the 1-dimensional representation of $\widehat{H}_\eta^+$ on which $H_{I_\eta} \subseteq \widehat{H}_\eta^+$ acts trivially ($T_i$ acts by $u$ for $s_i \in J_\eta$) and $Y_{l_i+1}$ acts by $u^{2a_i}, i \in [r]$. The relations in $\widehat{H}_\eta^+$ demand that $Y_{l_i+k}$ acts by $u^{2(a_i-k+1)}$ for $l_i + k \in B_i$. Note that our conventions differ from those in [29] since we use the right affine Hecke algebra while they use the left.

Next define $M_{\eta,a}$ to be the induced module

$$M_{\eta,a} = \widehat{H}_\eta^+ \otimes_{\widehat{H}_\eta^+} C_{\eta,a}.$$  

(5.39)

For $M$ in $C^\mathbb{Z}_n$ or $C^{+\mathbb{Z}}_n$, the points of $M$ are the joint generalized eigenspaces for the action of the $Y_i$. The coordinates of a point $v$ of $M$ is the tuple $(c_1, \ldots, c_n)$ of
generalized eigenvalues, i.e. \((Y_i - c_i)^k_i v = 0\) for some \(k_i\) and all \(i \in [n]\). The tuple \((c_1, \ldots, c_n)\) is also identified with the word \(c_1 c_2 \cdots c_n\).

We are interested in the case where the points of \(M_{\eta, a}\) are 1-dimensional.

**Proposition 5.8.1.** If intervals \([a_i - \eta_i, a_i]\) are disjoint, then the points of \(M_{\eta, a}\) are 1-dimensional with coordinates

\[
S_n J_\eta (u^{2a_1}, u^{2(a_1-1)}, \ldots, u^{2(a_1-\eta_1)}, u^{2a_2}, \ldots, u^{2(a_2-\eta_2)}, \ldots, u^{2a_r}, u^{2(a_r-1)}, \ldots, u^{2(a_r-\eta_r)}),
\]

where \(s_i\) acts on an \(n\)-tuple by swapping its \(i\)-th and \((i + 1)\)-st entries. Equivalently, the coordinates of the points of \(M_{\eta, a}\) are shuffles of the words

\[
u^{2a_1} u^{2(a_1-1)} \cdots u^{2(a_1-\eta_1)}, u^{2a_2} \cdots u^{2(a_2-\eta_2)}, \ldots, u^{2a_r} u^{2(a_r-1)} \cdots u^{2(a_r-\eta_r)}.
\]

**Proof.** This is a special case of well-known results about inducing modules in \(C_n^\mathbb{Z}\) (see [40, §5]).

We complete this section with a couple more algebraic generalities, further preparing us for our main result Theorem 5.8.8. Given any left \(\widehat{\mathcal{H}}^+\)-module \(M\), the annihilator \(\text{Ann} M = \{h \in \widehat{\mathcal{H}}^+ : h M = 0\}\) is a 2-sided ideal of \(\widehat{\mathcal{H}}^+\).

For any two-sided ideal \(N\) of \(\widehat{\mathcal{H}}^+\), \(N\) has a filtration

\[
0 \subseteq N_{\leq 0} \subseteq \cdots \subseteq N_{\leq d} \subseteq \ldots, \tag{5.40}
\]

where \(N_{\leq d} = (\widehat{\mathcal{H}}^+)_{\leq d} \cap N\). We can form the associated graded

\[
\text{gr}(N) := \oplus_{d \geq 0} N_d/N_{d-1} \subseteq \oplus_{d \geq 0} (\widehat{\mathcal{H}}^+)_{\leq d}/(\widehat{\mathcal{H}}^+)_{<d} = \widehat{\mathcal{H}}^+.
\]
It is an ideal of $\hat{H}^+$, isomorphic to $N$ as an $H$-module. We also have that $\hat{H}^+/N$ is isomorphic to $\hat{H}^+/\text{gr}(N)$ as an $H$-module. For $h \in N$, define $\text{in}(h)$ to be the leading homogeneous component of $h$, i.e., the image of $h$ in $N_d/N_{d-1} = \text{gr}(N)_d$, where $d$ is the smallest integer so that $h \in (\hat{H}^+)_d$.

**Proposition 5.8.2.** Let $M_{\eta,a}$ be as above. If $M_{\eta,a}$ is irreducible, then it contains an element $v^+$ such that, setting $N = \text{Ann } v^+$, $\hat{H}^+e^+/Ne^+ \cong M_{\eta,a}$ as $\hat{H}^+$-modules.

Thus by the discussion above, $\hat{H}^+e^+/\text{gr}(N)e^+ \cong M_{\eta,a}$ as $H$-modules.

**Proof.** As an $H$-module, $M_{\eta,a}$ is the induced module $H \otimes_{\mathcal{H}_{\lambda}} e^+J_{\lambda}$, and this contains a copy of the trivial $H$-module as a submodule. Let $v^+$ span this submodule. Thus there is a map $\hat{H}^+e^+ \to M_{\eta,a}$, $e^+ \mapsto v^+$. It is surjective by the irreducibility assumption and has kernel $Ne^+$, hence the proposition. \qed

**Remark 5.8.3.** The assumption that $M_{\eta,a}$ is irreducible cannot be dropped.

5.8.2. The ideals $I_{\lambda}$ are generated by certain elementary symmetric functions in subsets variables, also known as Tanisaki generators (see [8, 13]). We show that certain $C_w \in \mathcal{R}_{1^n}$ are essentially these generators. This will relate the ideals $\text{gr}(\text{Ann } M_{\eta,a})e^+$ to the canonical basis of $\hat{H}^+e^+$. Indeed, this was our original motivation for applying the Garsia-Procesi approach to understand cellular submodules of $\mathcal{R}_{1^n}$.

Let us make the inclusion $\hat{H}_{n-1} \hookrightarrow \hat{H}_{(n-1,1)} \hookrightarrow \hat{H}_n$ of §5.8.1 completely explicit. Recall that $S = \{s_1, s_2, \ldots, s_{n-1}\}$ and let $S'$ be the subset $\{s_1, s_2, \ldots, s_{n-2}\}$ of simple
reflections of \( W_f \). On the level of groups,

\[ \iota_n : \hat{S}_{n-1} \hookrightarrow \hat{S}_n \]  

(5.41)

is given on generators by

\[ \iota_n(y_i) = y_i, \quad i \in [n-1], \]  

\[ \iota_n(s_i) = s_i, \quad s_i \in S', \]  

(5.42)

from which it follows

\[ \iota_n(s_0) = s_{n-1}s_0s_{n-1}, \quad \iota_n(\pi) = \pi s_{n-1}. \]  

(5.43)

Since \( \iota_n(s_0) \notin K \), this is not a morphism of Coxeter groups. This inclusion of groups restricts to an inclusion of monoids \( \iota_n : \hat{S}^+_{n-1} \hookrightarrow \hat{S}^+_n. \)

It is immediate from (5.15) and (5.18) that \( \iota_n \) is given in terms of words by

\[ \lambda_1.x_1 \lambda_2.x_2 \cdots \lambda_{n-1}.x_{n-1} \mapsto \lambda_1.(x_1 + 1) \lambda_2.(x_2 + 1) \cdots \lambda_{n-1}.(x_{n-1} + 1) \quad 1, \]  

(5.44)

where \( x_i \in [n-1] \) and \( \lambda_i \in \mathbb{Z} \) (where, with the convention of §5.4.2, \( a.b = an + b \) in the top line and \( a.b = \pi(n-1) + b \) in the bottom line).

The corresponding morphism of algebras \( \iota_n : \hat{H}_{n-1} \rightarrow \hat{H}_n \) is given by

\[ \iota_n(Y_i) = Y_i, \quad i \in [n-1], \]  

\[ \iota_n(T_i) = T_i, \quad s_i \in S'. \]  

(5.45)

from which it follows

\[ \iota_n(\pi) = \iota_n(Y_1T_1^{-1}T_2^{-1}\cdots T_{n-2}^{-1}) = Y_1T_1^{-1}T_2^{-1}\cdots T_{n-2}^{-1} = \pi T_{n-1}, \]  

\[ \iota_n(T_0) = \iota_n(\pi^{-1}T_1\pi) = T_{n-1}^{-1}\pi^{-1}T_1\pi T_{n-1} = T_{n-1}^{-1}T_0 T_{n-1}. \]  

(5.46)

This map restricts to a map \( \iota_n : \hat{H}^+_{n-1} \rightarrow \hat{H}^+_n. \)
Lemma 5.8.4. For \( k, d \in [n] \) such that \( d \leq k \), let \( \lambda = \epsilon_{k-d+1} + \ldots + \epsilon_k \). Then \( y^\lambda S = v\pi^d \) for some \( v \in S_n \).

Proof. The word of \( (y_1y_2 \ldots y_d)_S \) is

\[
\pi^d = 1.d \ 1.(d-1) \ldots 1.1 \ n \ n-1 \ldots d+1,
\]

and the word of \( y^\lambda S \) is

\[
n \ n-1 \ldots n-(k-d)+1 \ 1.d \ 1.(d-1) \ldots 1.1 \ n-(k-d) \ n-(k-d)-1 \ldots d+1.
\]

This word is obtained from the word of \( \pi^d \) by a sequence of left-multiplications by \( s_i \in S \) that increase length by 1. This sequence yields the desired \( v \in S_n \).

Recall that for any \( w \) maximal in its coset \( wW_f \), we can write \( C'_w = \overline{C}'_z C'_{w_0} \), where \( w = z \cdot w_0 \). We have \( \overline{C}'_z = \sum_x \overline{P}_{x,z} T_x \), where the sum is over \( x \leq z \) such that \( x \cdot w_0 \) is reduced and \( \overline{P}_{x,z} := P'_{xw_0,zw_0} \) (see [1] for the more general construction of which this is a special case).

Theorem 5.8.5. For \( k, d \in [n] \) such that \( d \leq k \), put \( \lambda = \epsilon_{k-d+1} + \ldots + \epsilon_k \) and \( w = y^\lambda S w_0 \in D^S w_0 \). Then

\[
C'_w = u^{d(k-n)} s_{1d}(Y_1, \ldots, Y_k) C'_{w_0}.
\]

Proof. We proceed by induction on \( n \). If \( k = n \), then this is a special case of Theorem 5.6.1. Otherwise, by induction, the following holds in \( \widehat{\mathcal{H}}^n_{n-1} \):

\[
C'_w = u^{d(k-n+1)} s_{1d}(Y_1, \ldots, Y_k) C'_{w_0 S'},
\]
where \( w' = y^\lambda_{S'} w_{0S'} \in D^S w_0 \). Putting \( z = y^\lambda_{S'} \), we have \( C'' = \sum_{x \leq z} \overrightarrow{P}_{x,z} T_x C'_{w_0 S'} \).

Applying \( \iota_n \) to both sides, using Lemma 5.8.4 (\( z = u \pi^d, u \in S_{n-1} \) implies any \( x \leq z \) has a similar form) and then (5.46), we obtain

\[
\iota_n(C''') = \sum_{x \leq z} \overrightarrow{P}_{x,z} \iota_n(T_x) C'_{w_0 S'} = \sum_{x \leq z} \overrightarrow{P}_{x,z} T_x T_{n-d} \ldots T_{n-1} C'_{w_0 S'} ,
\]

(5.51)

Multiplying on the right by \( \overrightarrow{C'}_{s_{n-1} \ldots s_1} \), we obtain

\[
u^d \sum_{x \leq z} \overrightarrow{P}_{x,z} T_x C''_0 .
\]

(5.52)

Next, we show that

\[
\sum_{x \leq z} \overrightarrow{P}_{x,z} T_x C''_0 = C''
\]

(5.53)

using the characterization of the canonical basis from Theorem 5.3.1. It is not hard to see that \( w' s_{n-1} s_{n-2} \ldots s_1 = w \) using (5.44) and the affine word computation in the proof of Lemma 5.8.4. Then the left-hand side of (5.53) is certainly in \( T_w + u^{-1} \mathcal{L} \) (\( \mathcal{L} \) as in Theorem 5.3.1). To see that it is \( \pi \)-invariant, use that

\[
\sum_{x \leq z} \overrightarrow{P}_{x,z} T_x \pi^d C''_{w_0 S'} = \sum_{x \leq z} \overrightarrow{P}_{x,z} T_{x \pi^{-d}} \pi^d C''_{w_0 S'}
\]

(5.54)
as an equation in \( \hat{\mathcal{H}}^{n-1} \). Since \( x \pi^{-d} \in S_{n-1}, \iota_n(x \pi^{-d}) = \iota_n(x \pi^{-d}) \). Hence applying \( \iota_n \) to this equation and then multiplying on the right by \( u^{-d} \overrightarrow{C'}_{s_{n-1} \ldots s_1} \) yields \( \pi \)-invariance for the left-hand side of (5.53).

Finally, the theorem follows by applying \( \iota_n \) to both sides of (5.50) and multiplying on the right by \( u^{-d} \overrightarrow{C'}_{s_{n-1} \ldots s_1} \) to obtain (5.49). □
5.8.3. In the next proposition, we relate the descriptions of \( \text{gr}(\text{Ann } M_{\lambda, \alpha}) \) in terms of elementary symmetric polynomials in subsets of the variables to catabolizability. Let \( T^{d,k} \) = \( P(y_k - d y_{k-d} \ldots y_1) \). Under the isomorphism between \( \text{CCP}(\text{SYT}) \) and \( F^{sp}(\mathcal{R}_1^\infty) \), \( T^{d,k} = nT^{\text{rec}} + T' \) for some SYT \( T' \). The tableau \( T^{\text{rec}} \) has at most two rows and is filled with 0’s and 1’s; it has \( n - d \) 0’s in the first row and \( \min(d, n - k) \) 1’s in the second row. Set \( \mu = \text{ctype}(T^{d,k}) \) (see §5.5.3 for the definition of \( \text{ctype} \)).

**Proposition 5.8.6.** With \( \mu \) as defined above, \( d, k \in [n], d \leq k, \) and \( \lambda \vdash n \), the following are equivalent:

(a) \( d > k - n + \lambda'_1 + \cdots + \lambda'_{n-k} \),

(b) \( d > k - \sum_i (\lambda_i - (n-k))_{\geq 0} \),

(c) \( \mu \not\vdash \lambda \),

(d) \( T^{d,k} \) is not \( (G_{\lambda}, 1^{\ell(\lambda)}) \)-row catabolizable,

where for \( c \in \mathbb{Z}, (c)_{\geq 0} \) denotes \( c \) if \( c \geq 0 \) and 0 otherwise.

**Proof.** The equivalence of (a) and (b) comes from counting the number of boxes in the first \( n - k \) columns of the diagram of \( \lambda \) in two different ways. The equivalence of (c) and (d) is well-known (see [37]). It is easy to see that the catabolizability of \( T^{d,k} \) is \( \mu = (n-d, \mu_2, \mu_2, \ldots, \mu_2, r) \), where \( \mu_2 = \min(d, n-k) \) and \( r \) is the unique integer such that \( r \leq \mu_2 \) and \( \mu \vdash n \).

Next, let \( l \) be the number of parts of \( \lambda \) that are greater than \( n - k \). Rewriting
condition (b), and using the computation of $\mu$, we have

$$\sum_{i=1}^{l} \lambda_i > k - d + l(n - k) = n - d + n - k(l - 1) \geq \mu_1 + \sum_{i=2}^{l} \mu_i. \quad (5.55)$$

This shows the equivalence of (b) and (c).

A result of Garsia-Procesi ([8, Proposition 3.1]) carries over to this setting virtually unchanged. For a composition $\eta$, let $\eta_+$ denote the partition obtained from $\eta$ by sorting its parts in decreasing order.

**Proposition 5.8.7.** Suppose $\eta$ is an $r$-composition of $n$ with $\lambda := \eta_+$, and $k, d \in [n]$, $d \leq k$, such that any (all) of the conditions in Proposition 5.8.6 are satisfied. If $M_{\eta,a}$ satisfies the hypotheses of Proposition 5.8.1, then

$$s_{1^{d}}(Y_1, \ldots, Y_k) \in \text{gr}(\text{Ann } M_{\eta,a}).$$

**Proof.** Let $p_k^Y(t) = \prod_{i=1}^{k}(t + Y_i)$, thought of as a univariate polynomial in the indeterminate $t$. By Proposition 5.8.1, the points of $M_{\eta,a}$ are shuffles of words of length $\eta_1, \eta_2, \ldots, \eta_r$. Thus the word of length $\eta_i$ must intersect the first $k$ letters of the shuffle in size at least $(\eta_i - (n - k))_{\geq 0}$. Therefore the value of $p_k^Y(t)$ on any point of $M_{\eta,a}$ is divisible by

$$g(t) := [t + u^{2\alpha}]_{(\eta_1 - (n - k))_{\geq 0}}[t + u^{2\alpha_2}]_{(\eta_2 - (n - k))_{\geq 0}} \cdots [t + u^{2\alpha_r}]_{(\eta_r - (n - k))_{\geq 0}}, \quad (5.56)$$

where

$$[t + u^a]_c := (t + u^a)(t + u^{a-2}) \cdots (t + u^{a-2(c-1)}). \quad (5.57)$$
Put $m = \sum_i (\lambda_i - (n - k)) \geq 0$ and define $p^*_m(t) = \prod_{i=1}^m (t + z_i)$, thought of as a polynomial in $t$ with $z := (z_1, \ldots, z_m) \in A^m$. Divide $p^*_k(t)$ by $p^*_m(t)$ to obtain

$$p^*_k(t) = q(t)p^*_m(t) + r(t),$$

(5.58)

where $r(t) = \sum_{i=0}^{m-1} c_i t^i$ is a polynomial in $t$ of degree less than $m$ with coefficients $c_i \in A[Y_1, \ldots, Y_k]$. We will make use of the fact that equation (5.58) is homogeneous of degree $k$ if $t$, the $Y_i$’s, and the $z_i$’s have degree 1.

The coefficient $c_{k-d}$ exists as Proposition 5.8.6 (b) is equivalent to $k - d < m$ and, for a certain $z$, it is the element of $\text{Ann } M_{\eta,a}$ we are looking for: on the one hand, if $z$ is chosen so that $g(t) = p^*_m(t)$, then $c_{k-d}$ evaluates to 0 on every point of $M_{\eta,a}$ as $p^*_k(t)$ evaluated at any point of $M_{\eta,a}$ is divisible by $p^*_m(t)$. On the other hand, the leading component in$(c_{k-d})$ of $c_{k-d}$ is obtained from $c_{k-d}$ by setting $z = 0$. Then since setting $z = 0$ results in $p^*_m(t) = t^m$, (5.58) shows that in$(c_{k-d}) = s_{1d}(Y_1, \ldots, Y_k)$.

5.8.4. For $h \in \widehat{\mathcal{H}}^+$, write $[C'_w]h$ for the coefficient of $C'_w$ of $h$ written as an $A$-linear combination of $\{C'_w : w \in W_e^+\}$. Define $\langle , \rangle_\lambda : \widehat{\mathcal{H}}^+ \times \widehat{\mathcal{H}}^+e^+ \rightarrow A$ by

$$\langle h_1, h_2 \rangle_\lambda = \langle C'_g \rangle h_1h_2,$$

(5.59)

where $g_\lambda = \text{rowword}(G_\lambda)$.

We now come to our main result.

**Theorem 5.8.8.** Suppose $M_{\eta,a}$ satisfies the hypotheses of Propositions 5.8.1 and 5.8.2 and maintain the notation of Proposition 5.8.2. Then the following submodules of $\widehat{\mathcal{H}}^+e^+$ are equal.
(i) $\mathcal{I}_\lambda^o := \text{gr}(\text{Ann } v^+)e^+$,

(ii) $\mathcal{I}_\lambda^T := \hat{H}^+ \{ s_{1^d}(Y_1, \ldots, Y_k) : d, k \text{ satisfy (a)-(d) of Proposition 5.8.6} \} e^+$,

(iii) $\mathcal{I}_\lambda^{\text{pair}} := \{ v \in \hat{H}^+ e^+ : \langle \hat{H}^+, v \rangle_\lambda = 0 \}$,

(iv) $\mathcal{I}_\lambda^{\text{cell}} := \text{The maximal cellular submodule of } \hat{H}^+ e^+ \text{ not containing } \Gamma_{G_\lambda}$,

(v) $\mathcal{I}_\lambda^{\text{cat}} := \text{A}\{ C'_{w} : w \text{ is not } (G_\lambda, 1^{\ell(\lambda)})\text{-row catabolizable} \}$.

Note that $\mathcal{I}_\lambda^{\text{cat}}$ is not obviously a submodule but will be shown to be one. The abbreviations o, T, pair, are shorthand for orbit, Tanisaki, and pairing. Also note that modules $M_{\eta, a}$ satisfying the hypotheses of Propositions 5.8.1 and 5.8.2 exist by the general theory. For instance, if $|a_i - a_j| >> 0$ for all $i \neq j$, then these hypotheses are satisfied.

Given the theorem, define $\mathcal{R}_\lambda$ to be $\hat{H}^+ e^+ / \mathcal{I}_\lambda^{\text{cell}}$ for $\mathcal{I}_\lambda$ equal to any (all) of the submodules above.

Write $A_{\geq 0}$ for the semiring $\mathbb{Z}_{\geq 0}[u, u^{-1}] \subseteq A$ and $A_{>0}$ for the subset $\mathbb{Z}_{>0}[u, u^{-1}] \subseteq A_{\geq 0}$. Through the work of Kazhdan-Lusztig and Beilinson-Bernstein-Deligne-Gabber we have (see, for instance, [32])

**Theorem 5.8.9.** If $(W, S)$ is crystallographic, then the structure coefficients $\beta_{x,y,z} = [C'_z]C'_x C'_y$ belong to $A_{\geq 0}$.

The next two corollaries could be phrased as general facts about any algebra with basis in which the structure coefficients are positive, however, we state them for the
special cases that we need. Recall the notation of §5.3.3 and the definition (5.9) of \( \delta \leftarrow_{\Gamma} \gamma \). Note that
\[
\delta \leftarrow_{\Gamma} \gamma \iff \beta_{x,\gamma,\delta} \neq 0 \text{ for some } x \in W
\] (5.60)
as \([\delta]h\gamma \neq 0\) for some \(h = \sum_{x \in W} a_x C'_x \in \mathcal{H}, a_x \in A\) implies \([\delta]C'_x \gamma = \beta_{x,\gamma,\delta} \neq 0\) for some \(x\).

**Corollary 5.8.10.** For any \(W^+_e\)-graph \(\Gamma \subseteq \Gamma_{W^+_e}\) (i.e., the \(W^+_e\)-graph of some cellular subquotient of \(\widehat{\mathcal{H}}^+\)), \(\delta \leq_{\Gamma} \gamma\) if and only if \(\delta \leftarrow_{\Gamma} \gamma\).

**Proof.** The “if” direction is part of the definition of \(\leq_{\Gamma}\). For the “only if” direction, suppose \(\gamma_3 \leftarrow_{\Gamma} \gamma_2 \leftarrow_{\Gamma} \gamma_1\). Then by (5.60) there exist \(x_1, x_2 \in W^+_e\) such that \(\beta_{x_1,\gamma_1,\gamma_2} \neq 0\) and \(\beta_{x_2,\gamma_2,\gamma_3} \neq 0\). Applying Theorem 5.8.9 yields
\[
x_1 \gamma_1 \in A_{>0} \gamma_2 + A_{\geq 0} \Gamma \text{ and } (5.61)
x_2 \gamma_2 \in A_{>0} \gamma_3 + A_{\geq 0} \Gamma, \quad (5.62)
\]
which imply
\[
x_2 x_1 \gamma_1 \in A_{>0} \gamma_3 + A_{\geq 0} \Gamma. \quad (5.63)
\]
Thus \(\gamma_3 \leftarrow_{\Gamma} \gamma_1\).

The general case then follows by induction as \(\delta \leq_{\Gamma} \gamma\) means there exists \(\delta = \gamma_n, \gamma_{n-1}, \ldots, \gamma_1 = \gamma\) such that \(\gamma_{i+1} \leftarrow_{\Gamma} \gamma_i\).

**Corollary 5.8.11.** If \(\gamma \in T^\text{pair}_\lambda, \gamma \in \Gamma_{W^+_e}\), then \(\delta \leq_{\widehat{\mathcal{H}}^+} \gamma\) (\(\delta \in \Gamma_{W^+_e}\)) implies \(\delta \in T^\text{pair}_\lambda\), i.e., the cellular submodule generated by \(\gamma\) is contained in \(T^\text{pair}_\lambda\).
Proof. Suppose for a contradiction that $\delta \notin I_\lambda^{\text{pair}}$. Then by definition of $I_\lambda^{\text{pair}}$, $g_\lambda \leftarrow \delta$.

Applying Corollary 5.8.10 to this and the assumption $\delta \leq \hat{\mathcal{H}} + \gamma$ implies $g_\lambda \leftarrow \hat{\mathcal{H}} + \delta$, contradicting $\gamma \in I_\lambda^{\text{pair}}$.

Proof of Theorem 5.8.8. First we have $I_\lambda^T \subseteq I_\lambda^o$ by Proposition 5.8.7 and the inclusion $\text{Ann } M_{\eta,a} \subseteq \text{Ann } v^+$. We know by Proposition 5.8.2 that $\text{Res}_{\mathcal{H}} \hat{\mathcal{H}}^+ e^+ / I_\lambda^o$ affords the representation $\mathcal{H}_n \otimes \mathcal{H}_{\lambda} e^+ j_\lambda$. Next, an argument of the same flavor as Proposition 5.8.7 yields $I_\lambda^o \subseteq I_\lambda^{\text{pair}}$: for $\mu \in Y_+^+$, define

$$f_\mu(Y_1, \ldots, Y_n) = \prod_{i=1}^n \prod_{j=1}^{\mu_i} (Y_i - u^{2a_j}).$$

Assume that $\eta = \eta_+^+$; if not, the following argument works with the indices of the $a_j$ in the above expression permuted. If $f_\mu \notin \text{Ann } M_{\eta,a}$, then $w^1 \subseteq [\mu'_1 + 1, n]$, $w^2 \subseteq [\mu'_2 + 1, n], \ldots, w^r \subseteq [\mu'_r + 1, n]$, where $w^1 \sqcup \ldots \sqcup w^r = [n]$, $|w^r| = \eta_i$ determine the coordinates of a point in $M_{\eta,a}$ by specifying the positions of the shuffled words of Proposition 5.8.1. Thus for $l \in [r]$, $w^1 \sqcup w^2 \sqcup \cdots \sqcup w^l \subseteq [\mu'_l + 1, n]$ implying $\lambda_1 + \cdots + \lambda_l \leq n - \mu'_l$, or equivalently, $\mu'_l \leq \lambda_{l+1} + \cdots + \lambda_r$. In particular, adding up these inequalities yields $|\mu| \leq n(\lambda)$. Thus $\{Y^\mu : \mu \in Y_+^+, |\mu| > n(\lambda)\} \subseteq \text{gr}(\text{Ann } M_{\eta,a}) \subseteq \text{gr}(\text{Ann } v^+)$. 

By specializing to $u = 1$, it is easy to see that $\{\mathcal{H}_n Y^\mu : \mu \in Y_+^+, |\mu| = d\} = (\hat{\mathcal{H}}^+)_d$. Therefore, $(\hat{\mathcal{H}}^+)^{> n(\lambda)} \subseteq I_\lambda^o$. Since $\hat{\mathcal{H}}^+ e^+ / I_\lambda^o$ contains a single copy of the representation of shape $\lambda$, we must have $\Gamma_{G_\lambda} \not\subseteq I_\lambda^o$. Note that $I_\lambda^{\text{pair}}$ is also the maximal submodule of $\hat{\mathcal{H}}^+ e^+$ not containing $\Gamma_{G_\lambda}$. Hence we have $I_\lambda^o \subseteq I_\lambda^{\text{pair}}$.

Now that the inclusion $I_\lambda^T \subseteq I_\lambda^{\text{pair}}$ is established, Theorem 5.8.5 and Corollary
5.8.11 show that $I^T_\lambda$ is cellular. So $I^T_\lambda$ is a cellular submodule not containing $\Gamma_{G,\lambda}$, and hence $I^T_\lambda \subseteq I^{cell}_\lambda$. It follows from the algorithm for catabolizability in [2], or alternatively as a special case of Proposition 5.9.6, that there is a sequence of ascent-edges and corotation-edges from $w$ to $g_\lambda$ for any $w$ with $P(w) = (G_\lambda, 1^{\ell(\lambda)})$-row catabolizable. Then by Corollary 5.8.10, $I^{cell}_\lambda \subseteq I^{cat}_\lambda$.

We have shown that $I^T_\lambda \subseteq I^{cell}_\lambda \subseteq I^{cat}_\lambda$ and $I^T_\lambda \subseteq I^o_\lambda \subseteq I^{pair}_\lambda$. The $u = 1$ results of Garsia-Procesi and Bergeron-Garsia (see [13]) establish that $\text{rank}_A(\mathcal{H}^+ e^+ / I^T_\lambda) = \text{rank}_A(\mathcal{H}^+ e^+ / I^{pair}_\lambda) = \binom{n}{\lambda_1, \ldots, \lambda_r}$. The standardization map of Lascoux (see [37] and §5.9.1) shows that $\text{rank}_A(\mathcal{H}^+ e^+ / I^{cat}_\lambda) = \binom{n}{\lambda_1, \ldots, \lambda_r}$. Thus we must have equality $I^T_\lambda = I^{cell}_\lambda = I^o_\lambda = I^{pair}_\lambda$.

5.9 Atoms

We are primarily interested in subquotients of the coinvariants $R_1^n$, however it appears that there are many other copies of these subquotients in $\mathcal{H}^+$, outside of $R_1^n$. The realization of an $R \star W_f$-module $E$ as a cellular subquotient has genuinely different combinatorics depending on which element of $(F^{\text{mod}})^{-1}(E)$ is chosen. It is reasonable to guess that two cellular subquotients of $\mathcal{H}^+$ are isomorphic if their images in Co-cyclage Posets are connected and isomorphic, and this has been our empirical way of identifying isomorphic copies of atoms.

One fundamental problem which we hope to make some steps towards in this section is to find a combinatorial object that describes an atom, but without reference to
a specific set of tableaux, just as a Knuth equivalence graph encodes a representation of $\mathcal{E}$ without realizing it as the set of words with a fixed insertion tableau.

5.9.1. There are several examples in the literature of identifying cocyclage posets of different sets of tableaux. One example is Lascoux’s standardization map from tableaux of content $\lambda \vdash n$ to standard tableaux, which has image $\{T : \text{ctype}(T) \supseteq \lambda\}$ [27] (see also [37, §4]). This map can be defined by taking any subposet of $\text{CCP}(T(\lambda))$ that is connected as an undirected graph. The standardization map is then defined by fixing it on one tableau, for instance, the single row shape is mapped to the single row shape, or $Z_\lambda$, the superstandard tableau of shape and content $\lambda$, is mapped to the standard tableau $Z^*_\lambda$ (defined in §5.5.4). Standardization is then defined on the other tableaux by forcing the map to be a color preserving isomorphism of cocyclage posets. Miraculously, the map then preserves all cocyclages, not only those used to define it.

Another example of this are the copies of super atoms of Lascoux, Lapointe, and Morse [25]. A super atom $A^{(k)}_\lambda$ and some connected subposet of its cocyclage poset is given. A selected tableau of the super atom (usually the largest cocharge) is mapped to some other tableau, and this map is extended to the entire super atom as above. Then again miraculously, it appears that the map is an isomorphism in Cocyclage Posets.

There appear to be many more instances of this, and this has been our empirical way of constructing what might be isomorphic copies of cellular subquotients,
Cocyclage Atoms, etc. that occur in the coinvariants.

Let $Q$ be a standard tableau of shape $\lambda \vdash n$ and $\Gamma_Q$ the corresponding left cell of $\mathcal{H} = \mathcal{H}(S_n)$. Next define $\mathbb{A}_Q$ by

$$A_Q := \hat{\mathcal{H}}^+ \otimes \mathcal{H} A \Gamma_Q = A\{C_w : P(s w) = Q\}. \tag{5.64}$$

This equality, which shows that $A_Q$ is a cellular subquotient, is a special case of [1, Proposition 2.6].

For a standard tableau $Q$ of shape $\lambda$, one can try to map the cocyclage poset $A_{G_\lambda}^{GP'}$ to some subset of tableaux of $A_Q$, requiring that $G_\lambda^\vee$ map to $Q$. This appears to always work, and moreover something more general seems to hold.

For a PAT $Q$, let $\text{sgn}(Q)$ denote the filling of $1^n$ obtained by stacking the columns of $Q$ on top of each other (in order, from left to right) and adding $n(c - 1)$ to the entries in the $c$-th column of $Q$. For example,

$$\begin{array}{cccc}
1 & 4 & 6 & 15 \\
2 & 3 & & \\
13 & & & \\
\end{array}
\quad \text{if } Q = 12461523, \quad \text{then } \text{sgn}(Q) = \begin{array}{cccc}
1 & 2 & 14 & 23 \\
13 & 20 & 29 & 32 \\
\end{array}
$$

If this filling of the shape $1^n$ is not a tableau, then say that $\text{sgn}(Q)$ is undefined. As a special case, if $Q = G_\lambda^\vee$, then $\text{sgn}(Q) = G_1^\vee$.

5.9.2. Here we give some conjectural descriptions of the cells of the subquotients $A_{Q, \text{sgn}(Q)}^{csq}$ and some partial results towards these conjectures.
Algorithm 5.9.1. This algorithm depends on a PAT $Q$ of shape $\lambda$ such that $\text{sgn}(Q)$ exists and integers $d_1 \leq d_2 \leq \cdots \leq d_{\lambda_1}$. It takes as input an affine word $w$ and outputs true or false.

We define a function $f$ below, which takes a pair consisting of a part of an affine word and a PAT with one column to another such pair. Let $x = za$, $z$ a word and $a$ a number. Let $C' := a \rightarrow C$ be the result of column-inserting $a$ into $C$. If $a \rightarrow C$ has more than one column, let $a'$ be the entry in the second column of $C'$.

\[ f(x, C) = \begin{cases} (z, C') & \text{if } C' \text{ has one column,} \\ (1.a' z, C'_{|C|}) & \text{otherwise.} \end{cases} \] (5.65)

The algorithm has a counter $c$ which begins at 1. The algorithm repeatedly applies $f$ to the pair $(w, \emptyset)$; each application of $f$ is a step of the algorithm. After $d_c$ steps it is checked whether $C^*_1 = \text{sgn}(Q)_1$, where $(x^*, C^*) = f^{d_c}(w, \emptyset)$ and $l_j$ are the partial sums of $\lambda'$. The algorithm returns false if this is not so and otherwise increases $c$ by 1 and continues applying $f$. If the algorithm reaches the $d_{\lambda_1}$-th step and $f^{d_{\lambda_1}}(w, \emptyset) = (\emptyset, \text{sgn}(Q))$, then it returns true.

We are interested in applying this algorithm with $d_1 = |w|$ and for $c > 1$, $d_c := d_{c-1} + |x|$, where $x$ is the word after the $d_{c-1}$-th step.

Conjecture 5.9.2. Suppose $Q$ is a PAT of shape $\lambda$ such that $\text{sgn}(Q)$ is defined. Then the following are equivalent for a PAT $T$:

(i) $T$ is $(\text{sgn}(Q), \lambda)$-row catabolisable.
(ii) Algorithm 5.9.1 for $Q$ and $d_i$ as above with input $\text{rowword}(T)$ returns true.

(iii) Algorithm 5.9.1 for $Q$ and $d_i$ as above with input $\text{colword}(T)$ returns true.

(iv) $T$ is $(Q, 1^{\lambda_1})$-column catabolizable.

(v) There is a sequence of Knuth transformations and corotation-edges from any word inserting to $T$ to $\text{rowword}(\text{sgn}(Q))$ and there is a sequence of Knuth transformations, corotation-edges, and ascent-edges from $\text{rowword}(Q)$ to any word inserting to $T$.

We can show that any of (i)-(iv) implies (v).

**Definition 5.9.3.** The dual Garsia-Procesi atom copy $A^{GP'}_{Q,\text{sgn}(Q)}$ is the set of PAT satisfying condition (v) in Conjecture 5.9.2.

**Example 5.9.4.** Figure 5.4 depicts the atom copy $A^{GP'}_{Q,\text{sgn}(Q)}$ for $Q = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 \end{bmatrix}$.

**Conjecture 5.9.5.** If $Q, Q'$ are PAT such that $\text{sgn}(Q)$ and $\text{sgn}(Q')$ are defined, then

(i) the posets $A^{GP'}_{Q,\text{sgn}(Q)}$ and $A^{GP'}_{Q',\text{sgn}(Q')}$ are isomorphic in Cocyclage Posets.

(ii) the subquotients $A^{csq}_{Q,\text{sgn}(Q)}$ and $A^{csq}_{Q',\text{sgn}(Q')}$ are isomorphic in $CSQ(\hat{H}^+)$.

(iii) $F^{sp}(A^{csq}_{Q,\text{sgn}(Q)}) = A^{GP'}_{Q,\text{sgn}(Q)}$.

Note that (ii) and (iii) imply (i). Our computer experimentation provides substantial evidence for (i), and our main reason for believing (ii) is primarily this evidence as well.
Figure 5.4: The atom copy $A^{GP}_{Q,\text{sgn}(Q)}$ for $Q$ the tableau on the top.
The “easy half” of part (iii) of this conjecture is immediate from the definition of $A_{Q,sgn(Q)^{\prime}}^G$:

**Proposition 5.9.6.** There holds $F^{sp}(A_{Q,sgn(Q)}^{csq}) \supseteq A_{Q,sgn(Q)}^{GP^\prime}$.

We are now in a position to state the generalization of Corollary 5.6.6 that uses the full power of Theorem 5.6.4. For $\lambda \in Y^{+_+}$ and $u_2 \in W_e^+$ such that $\Psi(u_2) \in D^S$, put

$$N_{\lambda,u_2} := A\{s_{\lambda}(Y)C_{u_1w_0u_2} : u_1 \in D^S\}. \quad (5.66)$$

**Theorem 5.9.7.** Suppose $w \in W_e$ is maximal in its coset $W_f w$ and let $w = w_0 y^\nu u'$ with $\nu \in Y^{+_+}$, $\Psi(u') \in D^S$ be the factorization of Proposition 5.6.3. Let $Q$ be the single-row tableau $P(w)$. Then

(a) $N_{\nu,u'}$ and $A\{C_{uw} : u \in D^S\}$ are equal and are cellular subquotients of $\hat{H}^{+}$.

(b) $N_{\nu,u'}$ is isomorphic to $A_{1^n}$ in $CSQ(\hat{H}^{+})$ (and therefore isomorphic to any $N_{\lambda,u_2}$).

(c) $F^{sp}(N_{\nu,u'}) = A_{Q,sgn(Q)}^{GP^\prime}$.

(d) $N_{\nu,u'} = A_{Q,sgn(Q)}^{csq}$.

**Proof.** The equality of $N_{\nu,u'}$ and $A\{C_{uw} : u \in D^S\}$ is immediate from Theorem 5.6.4. For $\lambda \in Y^{+_+}$ and $u_2 \in W_e^+$ such that $\Psi(u_2) \in D^S$, define

$$N_{\geq \lambda,u_2} := \bigoplus_{\mu \geq \lambda} N_{\mu,u_2}, \quad N_{\lambda} := \bigoplus_{\Psi(u_2) \in D^S} N_{\lambda,u_2}.$$
By Theorem 5.6.4 and the Littlewood-Richardson rule, $N_{\geq \lambda}$ and $N_{\geq \lambda}$ are submodules of $\hat{\mathcal{H}}^+$. Thus $N_{\lambda} = N_{\geq \lambda}/N_{\geq \lambda}$ is a cellular subquotient of $\hat{\mathcal{H}}^+$. Moreover, as $\hat{\mathcal{H}}^+ e^+ \cong N_{\geq 0, id}$ is a submodule of the left $\hat{\mathcal{H}}^+$-module $\hat{\mathcal{H}}^+$, $N_{\geq 0, u_2} = N_{\geq 0, id} \tilde{C}_u^+$ is as well, where the equality is by Theorem 5.6.4. Since the intersection of two cellular subquotients is a cellular subquotient, $N_{\lambda, u_2} = N_{\geq 0, u_2} \cap N_{\lambda}$ is a cellular subquotient of $\hat{\mathcal{H}}^+$. This proves (a).

For (b), define the map $f : N_{\geq 0, id} \rightarrow N_{\geq \lambda, u_2}$ by requiring $C_{w_0} u_2 \mapsto s_\lambda(Y)C_{w_0} u_2$. This implies $f(C_{w_0} u_2) = s_\lambda(Y)C_{w_0} u_2$ for all $u \in D^S$ and $f(N_{\geq 0, id}) \subseteq N_{\geq \lambda, u_2}$ by Theorem 5.6.4. Thus $f$ gives rise to the isomorphism

$$N_{0, id} = N_{\geq 0, id}/N_{\geq 0, id} \cong N_{\geq \lambda, u_2}/N_{\geq \lambda, u_2} = N_{\lambda, u_2}. \quad (5.67)$$

This proves (b) as $N_{0, id} = \mathcal{R}_1$.

Statement (c) is clear for $w = w_0$. Then by (b), $F^{sp}(N_{\nu, u'}) \cong F^{sp}(\mathcal{R}_1)$ so (c) follows from the defining condition for $A_{Q, sgn(Q)}^{GP'}$.

Since $N_{\nu, u'}$ contains the left cells $Q$ and $sgn(Q)$, $N_{\nu, u'} \supseteq A_{Q, sgn(Q)}^{csq}$. Then by (c) and Proposition 5.9.6, $F^{sp}(N_{\nu, u'}) = A_{Q, sgn(Q)}^{GP'} \supseteq F^{sp}(A_{Q, sgn(Q)}^{csq})$. This implies $N_{\nu, u'} \subseteq A_{Q, sgn(Q)}^{csq}$, hence (d) is proved.

**Problem 5.9.8.** Give a combinatorial description of the tableaux labeling the left cells of any cellular subquotient of $\hat{\mathcal{H}}^+$ isomorphic to $A_{Q, sgn(Q)}^{csq}$, along the lines of Conjecture 5.9.2.
5.9.3. Here we show that the dual of CCP($T(\eta)$) is strongly isomorphic to a subposet of CCP($PAT$) (see Definition 5.5.3).

Given an $r$-composition $\eta$ of $n$, let $\hat{S}_n^+ = \hat{S}_{\eta_1}^+ \times \hat{S}_{\eta_2}^+ \times \cdots \times \hat{S}_{\eta_r}^+$. Let $l_j = \sum_{i=1}^{j-1} \eta_i$, $j \in [r + 1]$ be the partial sums of $\eta$ and $B_j$ be the interval $[l_j + 1, l_{j+1}]$ for $j \in [r]$.

Recall that the notation $\hat{a}$ denotes the element of $[n]$ congruent to the integer $a$ mod $n$.

Define $\alpha : \hat{S}_n^+ \rightarrow \hat{S}_\eta^+$ by $\alpha(w) = x^1 \times x^2 \times \cdots \times x^r$, where (identifying $\hat{S}_n^+$ and $\hat{S}_\eta^+$ with affine words and $r$-tuples of affine words) $x^j$ is determined as follows: let $\tilde{w}$ be the word $w_1 \ w_2 \ \cdots \ \ w_n$ sorted in increasing order, i.e., $\tilde{w} = w_{0\ldots\ n}$. Let $w^j$, $j \in [r]$ be the subword of the word of $w$ consisting of the numbers in $\{ \tilde{w}_i : i \in B_j \}$; $w$ is a shuffle of its subwords $w^1, \ldots, w^r$. Then $x^j$ is determined by $w^j$ from the conditions

$$\hat{x}^j_1 \hat{x}^j_2 \ \cdots \ \hat{x}^j_{\eta_j} \text{ has the same relative order as } \hat{w}^j_1 \hat{w}^j_2 \ \cdots \ \hat{w}^j_{\eta_j},$$

and if $x^j_i = a.b$, then $w^j_i = a.c$.

**Example 5.9.9.** Suppose $n = 9$ and $\eta = (2, 2, 1, 4)$. Then for the given $w$, $\alpha(w)$ is computed below.

$$w = \begin{array}{cccccccccccc}
1 & 3 & 4 & 4 & 9 & 31 & 12 & 25 & 46 & 7 & 8 \\
\end{array}$$

$$w^1 \times w^2 \times \cdots \times w^r = \begin{array}{cccccccccccc}
7 & 8 & 9 & 12 & 13 & 44 & 31 & 25 & 46 \\
\end{array}$$

$$\alpha(w) = \begin{array}{cccccccccccc}
x^1 \times x^2 \times \cdots \times x^r = 1 & 2 & 2 & 11 & 11 & 42 & 31 & 23 & 44 \\
\end{array}$$

For $D = (D_1, D_2, \ldots, D_r)$ an ordered partition of the set $[n]$ with $|D_i| = \eta_i$, define another map $\overline{\alpha}_D : \hat{S}_n^+ \rightarrow \hat{S}_\eta^+$ by $\overline{\alpha}_D(w) = x^1 \times x^2 \times \cdots \times x^r$, where $x^j$ is defined in
terms of $w^j$ as above and $w^j$ is the subword of $w$ consisting of those $w_i$ such that $\hat{w}_i \in D_j \ (i \in [n])$.

**Example 5.9.10.** Suppose $n = 9$ and $\eta = (2,2,1,4)$ and $D = \{1,5\} \cup \{2,9\} \cup \{6\} \cup \{3,4,7,8\}$. Then $\overline{\pi}_D$ is computed below.

$$w = 13 44 9 31 12 25 46 7 8$$

$$w^1 \times w^2 \times \cdots \times w^r = 31 25 \times 9 12 \times 46 \times 13 44 7 8$$

$$x^1 \times x^2 \times \cdots \times x^r = 31 22 \times 2 11 \times 41 \times 11 42 3 4$$

Recall that $W(\eta)$ denotes the set of words of content $\eta$. Let $W(\eta)^\vee$ be the set of semistandard words of content $\eta$, but with the convention that if two numbers in such a word are the same, then the one on the left is slightly bigger; let $T(\eta)^\vee$ be the set of transposed tableaux of content $\eta$, or equivalently, the set of insertion tableaux of $W(\eta)^\vee$; let $\text{CCP}(T(\eta)^\vee)$ be the cocyclage poset obtained from $\text{CCP}(T(\eta))$ by transposing tableaux and reversing edges. Let $W' \subseteq W^e_+$ consist of those $w$ such that $\alpha(w) = x^1 \times \cdots \times x^r$ with $x^j$ decreasing. For $w \in W'$, denote by $\beta(w)$ the unique element of $W(\eta)^\vee$ such that $w_S$ and $\beta(w)$ have the same relative order. This defines a map $\beta : W' \to W(\eta)^\vee$.

For $D = (D_1, D_2, \ldots, D_r)$ an ordered partition of the set $[n]$ with $|D_i| = \eta_i$, define a map $\overline{\beta}_D : W^e_+ \to W(\eta)$, similar to $\overline{\pi}_D$, as follows: $\overline{\beta}_D(w)$ has word $x = x_1 x_2 \ldots x_n$, where $x_i = j$ if and only if $\hat{w}_i \in D_j$.

**Example 5.9.11.** If $n = 9$, $\eta = (2,2,1,4)$, $D_j = [l_j + 1, l_{j+1}]$ with $l_j = \sum_{i=1}^{j-1} \eta_i$, and...
$w$ is as shown, then $\beta(w)$ and $\overline{\beta}_D(w)$ follow

$$w = 2 \ 23 \ 48 \ 35 \ 47 \ 1 \ 46 \ 39 \ 14,$$

$$w_S = 2 \ 4 \ 9 \ 5 \ 8 \ 1 \ 7 \ 6 \ 3,$$

$$\beta(w) = \overline{\beta}_D(w) = 1 \ 2 \ 4 \ 3 \ 4 \ 1 \ 4 \ 4 \ 2.$$

Let $W'_D := \{w \in W : \beta(w) = \overline{\beta}_D(w)\}$. A weak corotation of a semistandard word is the same as a corotation except we allow the number 1 to be co-rotated.

**Proposition 5.9.12.** The map $\beta$ (or $\overline{\beta}_D$) restricted to $W'_D$ commutes with left-multiplication by $s \in S$, preserves left descent sets, and commutes with weak corotations.

**Proof.** Since $\beta(w)$ and $w_S$ have the same relative order, $\beta$ certainly commutes with left-multiplications by $s \in S$ and preserves left descent sets. On the other hand, $\overline{\beta}_D$ certainly commutes with weak corotations.

Also write $\beta$ for the map of tableau $P(w) \mapsto P(\beta(w))$, which is well-defined by the proposition.

Put $\lambda = \eta_+$, the partition obtained by sorting the parts of $\eta$. Note that the word of $\pi^k$, $k \in \mathbb{Z}$ is decreasing and that $A\{\pi^k : k \in [i, j]\}$, $0 \leq i \leq j$, is a cellular subquotient of $\widehat{\mathcal{H}}^+$. Suppose $a \in \mathbb{Z}_{\geq 0}$ and $[a_1/\eta_1] < \cdots < [a_r/\eta_r]$. There is a unique tableau $Q$ of shape $\lambda'$ with $\alpha(w) = \overline{\alpha}_D(w) = \pi^{a_1} \times \pi^{a_2} \times \cdots \times \pi^{a_r}$ for any $w$ inserting to $Q$. For example, with $\eta = (3, 2, 2, 1)$, $D_j = [l_j + 1, l_{j+1}]$, and $a = (0, 2, 4, 3)$, the resulting $Q$ is the tableau in the first row and second column of Figure 5.5.
For a cocyclage poset $A$, let $W(A)$ denote the set $\{w : P(w) \in A\}$.

**Proposition 5.9.13.** With $Q$ defined in terms of $a$ as above, if $\eta = \eta_+$ and $a_1/\eta_1 \ll a_2/\eta_2 \ll \cdots \ll a_r/\eta_r$, then $\text{sgn}(Q)$ is defined and $W(A^G_{Q, \text{sgn}(Q)}) \subseteq W'_D$.

**Proof.** With the present hypotheses, the column reading word of $Q$ is $w^1 w^2 \ldots w^r$, where $w^i$ is of the form
\[
a.c_1 a.c_2 \cdots a.c_k (a - 1).c_{k+1} (a - 1).c_{k+2} \cdots (a - 1).c_{\eta_i},
\]
where $a = \lceil a_i/\eta_i \rceil$ and $D_i = \{c_1, \ldots, c_{\eta_i}\}$. Then $a_1/\eta_1 \ll \cdots \ll a_r/\eta_r$ implies $\text{sgn}(Q)$ is defined.

First observe that $\text{rowword}(\text{sgn}(Q))$ is a shuffle of words $w^1, \ldots, w^r$, where $w^i$ is of the form (5.68) for $a$ not too far from $\lceil a_i/\eta_i \rceil$. Any word obtained from $\text{rowword}(\text{sgn}(Q))$ by sequence of Knuth transformations and a small number of rotation-edges is also a shuffle of a similar form. It is easy to see that such a shuffle belongs to $W'_D$. For any $w \in W(A^G_{Q, \text{sgn}(Q)})$, there is a path from $\text{rowword}(\text{sgn}(Q))$ to $w$ consisting of Knuth transformations and at most $\binom{n}{2}$ rotation-edges, hence the desired result. \hfill $\Box$

For the next theorem, we will assume Conjecture 5.9.2.

**Theorem 5.9.14.** With $Q$ as above and if $\text{sgn}(Q)$ exists and $W(A^G_{Q, \text{sgn}(Q)}) \subseteq W'_D$, then there exists a section $\beta' : W(\eta)^\vee \rightarrow W_e^+$ of $\beta$ with image $W(A^G_{Q, \text{sgn}(Q)})$. In particular, if $a$ is as in Proposition 5.9.13, then there exists such a section. Moreover,
if such a section exists and \( \eta \) is a partition, then \( \beta \) is a strong isomorphism between 
\( A_{Q,\text{sgn}(Q)}^{GP'} \) and \( \text{CCP}(T(\eta)^\vee) \).

**Proof.** For any \( T_1, T_2 \in A_{Q,\text{sgn}(Q)}^{GP'} \), we will show by induction on \( \deg(\text{sgn}(Q)) - \deg(T_i) \) that \( \beta(T_1) = \beta(T_2) \) implies \( T_1 = T_2 \). The base case is \( \text{sh}(\beta(T_1)) = \text{sh}(\beta(T_2)) = 1^n \). In this case, it is easy to see that \( T_1, T_2 \) being \( (\text{sgn}(Q), \lambda) \)-column catabolizable implies \( T_1 = \text{sgn}(Q) = T_2 \). For \( \beta(T_1) = \beta(T_2) \) not of shape \( 1^n \), induction, the fact that every \( w \in \mathcal{W}(A_{Q,\text{sgn}(Q)}^{GP'}) \) has a path of Knuth transformations and corotation-edges to \( \text{rowword}(\text{sgn}(Q)) \), and Proposition 5.9.12 implies \( T_1 = T_2 \). Thus \( \beta \) restricted to \( A_{Q,\text{sgn}(Q)}^{GP'} \) is injective. A similar inductive argument shows that \( \beta \) restricted to \( A_{Q,\text{sgn}(Q)}^{GP'} \) is surjective. Thus we can define \( \beta' \) to be the inverse of \( \beta \) restricted to \( A_{Q,\text{sgn}(Q)}^{GP'} \).

Since \( \beta \) and \( \beta(w) \) have the same relative order, it is immediate that \( \beta \) is a strong isomorphism of cocyclage posets provided we check that \( \beta(w) \) ends in a 1 if and only if \( \pi w \notin \mathcal{W}(A_{Q,\text{sgn}(Q)}^{GP'}) \). This follows from either catabolizability condition of Conjecture 5.9.2 ((i) and (iv)) and the fact \( P(\pi w)_{1\eta_1} \neq Q_{1\eta_1} \).

**Example 5.9.15.** Suppose \( n = 8, \eta = (3, 2, 2, 1) \) and \( D_j = [l_j + 1, l_{j+1}] \). Let \( Q, \tilde{Q}, \beta(Q) \) be the three tableaux in the first row of Figure 5.5 and let \( T^{r,c} \) be the tableau in the \( r \)-th row and \( c \)-th column. In each column is a selection of tableaux from the cocyclage posets \( A_{Q,\text{sgn}(Q)}^{GP'}, A_{\tilde{Q},\text{sgn}(\tilde{Q})}^{GP'}, \text{CCP}(T(\eta)^\vee) \) such that the \( T^{r,1} \leftrightarrow T^{r,2} \leftrightarrow T^{r,3} \) under the isomorphisms between these posets. In the first column, we see that \( \beta_D = \beta \) on the second row but not on the third and there is a cocyclage-edge from \( T^{2,1} \) to \( T^{3,1} \). The \( a_i \) are large enough so that \( \beta(T^{r,2}) = \beta_D(T^{r,2}) = T^{r,3} \).
for all $r$; this is in contrast to $\beta(T^{5,1}) \neq \beta_D(T^{5,1}) = T^{5,3}$. Also the cocyclage-edges $T^{4,1} \xrightarrow{cc} T^{5,1}$ and $T^{4,3} \xrightarrow{cc} T^{5,3}$ demonstrate that $A^\text{GP'}_{Q,\text{sgn}(Q)}$ and $\text{CCP}(T(\eta)\vee)$ are not strongly isomorphic.

5.9.4. Here we discuss Shimozono-Weyman atoms in more detail. Recall from §5.5.5 that the Shimozono-Weyman atom $A^\text{SWr}_{Q,\eta}$ (resp. $A^\text{SWc}_{Q,\eta}$) is the set of $(Q, \eta)$-row (resp. column) catabolizable tableaux. These atoms first appear in [37], where they are conjectured to give a combinatorial description of certain generalized Hall-Littlewood polynomials. These symmetric polynomials with coefficients in $\mathbb{C}[t]$ are defined as the
formal characters of the Euler characteristics of certain \( \mathbb{C}[\mathfrak{gl}_n] \)-modules supported in nilpotent conjugacy class closures. The generalized Hall-Littlewood polynomials are known to be \( t \)-analogues of the character of certain induced modules, however as far as we know, it is yet to be proved that SW atoms also satisfy this property.

**Conjecture 5.9.16.** Suppose that \( f : \mathbb{A}^{GP}_{Q, \text{sgn}(Q)} \cong \mathbb{A}^{GP}_{Q', \text{sgn}(Q)'} \) is an isomorphism in Cocyclage Posets and \( \text{sh}(Q) = \text{sh}(Q') = \lambda \). Then for all \( T \in \mathbb{A}^{GP}_{Q, \text{sgn}(Q)} \) and \( c \in [\lambda_1] \), \( T_{C_1} = Q_{C_1} \) if and only if \( f(T)_{C_1} = Q'_{C_1} \), where \( C_1 := (\lambda'_1, \ldots, \lambda'_c)' \).

**Proposition 5.9.17.** Assuming Conjecture 5.9.16 and with its notation, \( f \) restricts to an isomorphism \( \mathbb{A}^{SWc}_{Q, \eta} \cong \mathbb{A}^{SWc}_{Q', \eta} \).

**5.9.5.** Here we give the definition of Chen’s atoms as the intersection of certain Shimozono-Weyman atoms.

For a skew shape \( \theta = \Theta/\nu \), define \( \text{row}(\theta) \) (resp. \( \text{column}(\theta) \)) to be the composition given by the row (resp. column) lengths of \( \theta \), i.e., \( \text{row}(\theta)_i = \Theta_i - \nu_i \) and \( \text{column}(\theta)_i = \Theta'_i - \nu'_i \).

**Definition 5.9.18.** If \( \theta = \Theta/\nu \) is a skew shape with \( |\theta| = n \) and \( \lambda, \mu \vdash n \), then \( \lambda \) is **skew-linked to** \( \mu \) by \( \theta \), written \( \lambda \stackrel{\theta}{\rightarrow} \mu \), if \( \text{row}(\theta) = \lambda \) and \( \text{column}(\theta) = \mu' \). See Figure 5.6.

Let \( V_\lambda \) be the Specht module of shape \( \lambda \), \( J_\mu \subseteq S \) be as in §5.8.1, \( S_\mu \) the parabolic subgroup \( S_{n,J_\mu} \), and \( e^+_\mu := e^+_J \mu \) the trivial module for \( \mathbb{C}S_\mu \).
Proposition 5.9.19 ([5]). The following are equivalent.

(a) There is a skew shape \( \theta \) such that \( \lambda \rightarrow^\theta \mu \),

(b) There exists a non-negative integer \( d(\mu, \lambda) \) such that in the \( S_n \)-module \( R \otimes \mathbb{C} (\text{Ind}_{\mu}^{S_n} e^+_\mu) \), \( V_\lambda \) occurs with multiplicity 1 in degree \( d(\mu, \lambda) \) and this is the unique occurrence of any \( V_\nu \) with \( \nu \sqsubseteq \lambda \) in degree less than or equal to \( d(\mu, \lambda) \).

Definition 5.9.20. For \( \lambda, \mu \) satisfying any (all) of the conditions in Proposition 5.9.19, let \( A_{\mu,\lambda}^{\text{mod}} \) be the minimal quotient of \( R \otimes \mathbb{C} (\text{Ind}_{\mu}^{S_n} e^+_\mu) \) containing \( V_\lambda \). By the proposition, this determines a unique \( R \ast W_f \)-module.

Conjecture 5.9.21. Suppose \( \lambda \rightarrow^\theta \mu \). The following are equivalent for a PAT \( Q \) of shape \( \mu \) and a PAT \( P \) of shape \( \lambda \):

(a) The PAT \( Q \) is the unique tableau in degree \( \text{deg}(P) - d(\mu, \lambda) \) of shape \( \mu \) in the minimal cellular quotient of \( \hat{\mathcal{H}}^+ \) containing \( \Gamma_P \).

(b) The PAT \( P \) is the unique tableau in degree \( \text{deg}(Q) + d(\mu, \lambda) \) of shape \( \lambda \) in the minimal cellular submodule of \( \hat{\mathcal{H}}^+ \) containing \( \Gamma_Q \).
**Definition 5.9.22.** If \( P \) and \( Q \) satisfy (a) and (b) of Conjecture 5.9.21, then \( P \) is skew-linked to \( Q \) by \( \theta \), or \( P \) and \( Q \) are skewed-linked.

For a skew shape \( \theta = \Theta/\nu \) with \( \lambda \rightarrow \mu \), define the intervals \([b_r, d_r]\) for \( r \in [\ell(\lambda)] \) as follows: let \( c = \nu_r \). If \( \nu_r \neq 0 \), then \( b_i = \mu'_{c+1}, d_i = \mu'_{c} \); if \( \nu_r = 0 \), then \( b_r = d_r = \ell(\lambda) + 1 - r \) (which is \( \leq \mu'_1 \)).

**Example 5.9.23.** Let \( \theta \) be as shown.

The intervals \([b_1, d_1]\ldots[b_{10}, d_{10}]\) are

\[ [4, 5] \quad [4, 5] \quad [5, 5] \quad [5, 6] \quad [6, 6] \quad [5, 5] \quad [4, 4] \quad [3, 3] \quad [2, 2] \quad [1, 1]. \]

**Definition 5.9.24.** Let \( \lambda \rightarrow \mu \) and \( b_r, d_r \) be as above. Suppose \( G_{\lambda} \) is skew-linked to \( Q \) by \( \theta \). The Li-Chung Chen atom \( A_{Q,G_{\lambda}}^{Chen} \) is the intersection of \( A_{G_{\lambda},\eta}^{SWr} \) over those \( \ell(\lambda) \)-compositions \( \eta \) such that \( \eta_i \leq d_{l+1} \) for \( i \in [\ell(\lambda)] \), where \( l_j = \sum_{i=1}^{j-1} \eta_i \).

Assuming Conjecture 5.9.16, then \( A_{Q,G_{\lambda}}^{Chen} \) is isomorphic to Chen’s original definition of these atoms in terms of semistandard tableau \([5]\).

Also, if we take \( Q = \beta'(Z'_\mu) \) with \( Z'_\mu \) the transpose of the superstandard tableau of shape and content \( \mu \) and \( \beta' \) as in Theorem 5.9.14, then there is a unique \( P \) of shape \( \lambda \) in \( A_{Q,\text{sgn}(Q)}^{GP'} \) in degree \( \deg(Q) + d(\mu, \lambda) \). Let \( A_{Q,P}^{Chen} \) be defined analogously to \( A_{Q,G_{\lambda}}^{Chen} \) with \( A_{Q,\eta}^{SWc} \) in place of \( A_{G_{\lambda},\eta}^{SWr} \). Then \( A_{Q,P}^{Chen} \) is the original definition of Chen atoms under the correspondence \( \beta \).
**Example 5.9.25.** The leftmost cocyclage poset in the top row of Figure 5.7 is the Chen atom $A_{Q,G(3,2,1)}^{Chen}$, $Q = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{array}$, corresponding to the skew linking shape $\begin{array}{cccc} 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}$. In this case we checked that this poset is $F^{sp}(A_{Q,G(3,2,1)}^{csq})$. Moreover, the top row of Figure 5.7 contains all cellular subquotients of $\mathcal{R}_1$ isomorphic to $A_{Q,G(3,2,1)}^{csq}$. The bottom row contains two other cellular subquotients of $\mathcal{H}^+$ isomorphic to $A_{Q,G(3,2,1)}^{csq}$. See Example 5.9.33 for a proof that the two subquotients on the bottom row are isomorphic.

**5.9.6.** The Lascoux-Lapointe-Morse super atoms of [25] are conjecturally a special case of Chen atoms. Let us see how this comes about.

A partition $\lambda$ is $k$-bounded if its parts have length $\leq k$. For a $k$-bounded partition $\lambda$ with $r$ parts, the skew shape $\theta^k(\lambda) = \Theta/\nu$ is defined uniquely by the condition $\text{row}(\theta) = \lambda$ and the inductive conditions: $\Theta_r = \lambda_r$, $\nu_r = 0$; $\theta^k(\lambda)_{<1} = \theta^k(\lambda_{<1})$ and $\nu_1$ is the smallest non-negative integer such that the hook lengths of $\theta^k(\lambda)$ are $\leq k$. See (5.69).

$$ \begin{array}{c} \lambda \\ \theta^4(\lambda) \end{array} \quad (5.69) $$

**Proposition 5.9.26 ([25, Property 33]).** For a $k$-bounded partition $\lambda$, $\lambda \xrightarrow{\theta^k} \mu$ for some partition $\mu$ whose conjugate is also $k$-bounded.

In the language of [25], the $k$-conjugate of $\lambda$ is $\mu'$ in the proposition.
Figure 5.7: Copies of Chen atoms.
Definition 5.9.27. For $\mu, \lambda$ as in Proposition 5.9.26 and tableaux $G_\lambda, Q$ that are skew-linked by $\theta^k$, the LLM atom of $G_\lambda, Q$ is $A^\text{Chen}_{Q,G_\lambda}$.

5.9.7. One easy way to show that two cellular subquotients of $\widehat{\mathfrak{g}}^+$ are isomorphic is using right star operations.

For the following definitions, let $(W, S)$ be an (extended) Coxeter group. Let $s$ and $t$ be in $S$ such that $st$ has order 3. Define

$$D_L(s, t) = \{w \in W : |L(w) \cap \{s, t\}| = 1\},$$
$$D_R(s, t) = \{w \in W : |R(w) \cap \{s, t\}| = 1\}.$$

The left star operation with respect to $\{s, t\}$ is the involution $D_L(s, t) \rightarrow D_L(s, t)$, $w \mapsto *w$, where $*w$ is the single element of $D_L(s, t) \cap \{sw, tw\}$. Similarly, the right star operation with respect to $\{s, t\}$ is the involution $D_R(s, t) \rightarrow D_R(s, t)$, $w \mapsto w^*$, where $w^*$ is the single element of $D_R(s, t) \cap \{ws, wt\}$. We use the convention of writing $* = \{s, t\}$ to signify that the star operation is with respect to $\{s, t\}$. We will need the following results from the original Kazhdan-Lusztig paper, the second of which is quite crucial to the theory that originated there.

Proposition 5.9.28 ([22, Proposition 2.4]). Suppose $x, w \in \Gamma_W$.

(i) If $x$ and $w$ belong to the same left cell, then $R(x) = R(w)$.

(ii) If $x$ and $w$ belong to the same right cell, then $L(x) = L(w)$.

Theorem 5.9.29 ([22, Theorem 4.2]). With the convention of Remark 5.3.2, $* = \{s, t\} \subseteq S$, and $st$ of order 3,
(i) if \( x, w \in D_L(s, t) \), then \( \mu(x, w) = \mu(*x, *w) \),

(ii) if \( x, w \in D_R(s, t) \), then \( \mu(x, w) = \mu(x^*, w^*) \).

For \((W, S) = (W_e, K)\) and elements identified with affine words, left and right star operations are Knuth transformations and dual Knuth transformations \((\sim_KT \text{ and } \sim_{DKT})\). See [38, A1] for an introduction to this combinatorics in the case \(W = S_n\).

Knuth transformations look like those for standard words. A Knuth transformation of an affine word \(w \in W_e\) is a transformation of one of the following forms:

\[
\cdots 1.b 1.a 1.c \cdots b a c \cdots \xrightarrow{K_T} \cdots 1.b 1.c 1.a \cdots b c a \cdots , \tag{5.70}
\]

\[
\cdots 1.a 1.c 1.b \cdots a c b \cdots \xrightarrow{K_T} \cdots 1.c 1.a \cdots b c a \cdots , \tag{5.71}
\]

for \(a, b, c \in \mathbb{Z}, a < b < c\). These pictures are to be interpreted to mean that for every \(k \in \mathbb{Z}\), the adjacent numbers \(k.a\) and \(k.c\) are transposed. These Knuth transformations correspond to the left star operation with respect to \(\{s_i, s_{i+1}\}\) (subscripts taken mod \(n\)), where, for the first line, \(i\) is the position of \(b\), and, for the second line, \(i\) is the position of the \(a\) on the left-hand side.

To see a dual Knuth transformation of an affine word \(w\), it is not enough to only examine the window \(w_1 \ldots w_n\). A dual Knuth transformation of an affine word \(w \in W_e\) is a transformation of one of the following forms:

\[
\cdots i \cdots i + 2 \cdots i + 1 \cdots \xrightarrow{DKT} \cdots i + 1 \cdots i + 2 \cdots i \cdots , \tag{5.72}
\]

\[
\cdots i + 1 \cdots i \cdots i + 2 \cdots \xrightarrow{DKT} \cdots i + 2 \cdots i \cdots i + 1 \cdots . \tag{5.73}
\]
These pictures are to be interpreted to mean that a similar transformation is performed on the numbers $k.i$, $k.i + 1$, $k.i + 2$ for every $k \in \mathbb{Z}$. These dual Knuth transformations correspond to the right star operation with respect to $\{s_{n-i}, s_{n-(i+1)}\}$ (recall the unusual convention of (5.17)).

**Example 5.9.30.** For $n = 5$, the following are examples of a Knuth transformation corresponding to the left star operation with respect to $\{s_1, s_2\}$ and a dual Knuth transformation corresponding to the right star operation with respect to $\{s_1, s_2\}$ (recall the unusual convention of (5.17)).

$$
\begin{array}{cccccc}
13 & 1 & 42 & 14 & 5 \\
\end{array} \sim_{\text{KT}} \begin{array}{cccccc}
13 & 42 & 1 & 14 & 5 \\
\end{array}
$$

$$
\begin{array}{cccccc}
13 & 1 & 42 & 14 & 5 \\
\end{array} \sim_{\text{DKT}} \begin{array}{cccccc}
13 & 1 & 42 & 15 & 4 \\
\end{array}
$$

The next proposition relates connectivity of atoms to the left cells of $W_e$. Xi’s paper [42] is a nice guide to the current knowledge of cells of $W_e$. We remark that the left cells of $W_e$ are typically infinite in contrast to those of $W_e^+$ and $W_f$. For a subset $\Gamma$ of $W_e^+$, let $\hat{\Gamma}$ be the minimal element of $\text{CSQ}(\hat{\mathcal{H}}^+)$ containing $\Gamma$.

**Proposition 5.9.31.** If $\Gamma$ is a union of left cells of $W_e^+$ such that the undirected graph on $\Gamma$ consisting of cocyclage-edges is connected, then $\Gamma$ and, stronger, $\hat{\Gamma}$ are contained in a left cell of $W_e$.

*Proof.* The connectivity assumption and our knowledge of the left cells of $\hat{\mathcal{H}}^+$ (Corollary 5.4.14 and [22, Theorem 1.4]) show that the undirected graph on $\Gamma$ consisting of Knuth transformations and corotation edges is connected. Thus $\Gamma$ is contained in a
left cell $\Lambda$ of $W_e$. Here we are using that $\pi w$ and $w$ are contained in the same left cell of $W_e$ for any $w \in W_e$. Since left cells are cellular subquotients, the minimal cellular subquotient of $\hat{H}$ containing $\Gamma$ is $\Lambda$. Moreover, the minimal cellular subquotient of $\hat{H}$ containing a subset of $W_e$ certainly contains the minimal cellular subquotient of $\hat{H}^+$ containing that subset, hence $\hat{\Gamma}$ is contained in $\Lambda$. \hfill \blacksquare

**Proposition 5.9.32.** Suppose $A\Gamma \in CSQ(\hat{H}^+)$ and $F^{cc}(A\Gamma)$ is connected. Let $* = \{s, t\} \subseteq S$ with $st$ of order 3. Then

(i) if $\gamma \in D_R(s, t)$ for some $\gamma \in \Gamma$, then $\Gamma \subseteq D_R(s, t)$,

(ii) If $\Gamma \subseteq D_R(s, t)$, then $A\Gamma^* \in CSQ(\hat{H}^+)$ and $\Gamma \cong \Gamma^* := \{\gamma^* : \gamma \in \Gamma\}$,

(iii) $A\Gamma \pi \in CSQ(\hat{H}^+)$ and $\Gamma \cong \Gamma \pi := \{\gamma \pi : \gamma \in \Gamma\}$.

**Proof.** Propositions 5.9.31 and 5.9.28(i) imply $R(\gamma) = R(\gamma')$ for all $\gamma, \gamma' \in \Gamma$, hence (i).

For (ii), note that by Theorem 5.9.29 (ii) and Proposition 5.9.28 (ii) the edges $\mu$ with both ends in $\Gamma$ are the same as those of $\Gamma^*$, so it remains to show $A\Gamma^* \in CSQ(\hat{H}^+)$. This argument also shows that the edges $\mu$ in $\hat{\Gamma}^*$ are the same as those of $\hat{\Gamma}^*$ since $\hat{\Gamma}^* \subseteq D_R(s, t)$ by Propositions 5.9.31 and 5.9.28(i). Then if $Y_3 \leq_{\hat{H}^+} Y_2 \leq_{\hat{H}^+} Y_1$ for $Y_2 \subseteq \hat{\Gamma}^* \setminus \Gamma^*$, $Y_1, Y_3 \in \Gamma^*$ for some left cells $Y_i$ of $\hat{H}^+$, we would have $Y_3^* \leq_{\hat{H}^+} Y_2^* \leq_{\hat{H}^+} Y_1^*$ for $Y_2^* \subseteq \hat{\Gamma}^* \setminus \Gamma$, $Y_1^*, Y_3^* \in \Gamma$, contradiction. Thus $\hat{\Gamma}^* = \Gamma^*$.

Statement (iii) is immediate from the identity $\mu(x, w) = \mu(x \pi, w \pi)$ for all $x, \pi \in W_e$. \hfill \blacksquare
Example 5.9.33. Letting $\Gamma$ be the cellular subquotient on the left-hand side of the bottom row of Figure 5.7, the cellular subquotient on the right of the bottom row is equal to $(\Gamma \pi)^{\ast_1 \ast_2}$, where $\ast_1 = \{s_{n-1}, s_{n-2}\}$, $\ast_2 = \{s_{n-2}, s_{n-3}\}$ ($n = 6$).

5.9.8. Here we make some further conjectures about atoms.

Problem 5.9.34. Let $G_\lambda, Q$ be skew-linked and put $\mu = \text{sh}(Q)$. Assuming that $F^\text{sp}(\Lambda^\text{csq}_{Q,G_\lambda}) = \Lambda^\text{Chen}_{Q,G_\lambda}$ and $F^\text{mod}(\Lambda^\text{csq}_{Q,G_\lambda}) = \Lambda^\text{mod}_{\mu,\lambda}$, give a reasonable combinatorial condition characterizing which $Q', P' \in \text{PAT}$ satisfy $\Lambda^\text{csq}_{Q',P'} \cong \Lambda^\text{csq}_{Q,G_\lambda}$ in $\text{CSQ}(\hat{\mathcal{H}}^+)$. Is the set of $Q', P'$ such that this isomorphism holds the same as those pairs $P', Q'$ that are skew-linked?

Remark 5.9.35. For $\lambda = 1^n, Q = G_\mu^\vee$, this is essentially Problem 5.9.8.

Consider the following property of an atom $A$:

$$Q \text{ is the unique tableau of lowest degree in } A \text{ and } P \text{ is the unique tableau of highest degree in } A. \quad (5.74)$$

Conjecture 5.9.36.

(a) If $G_\lambda, Q$ are skew-linking, then $F^\text{sp}(\Lambda^\text{csq}_{Q,G_\lambda}) = \Lambda^\text{Chen}_{Q,G_\lambda}$. This has been checked for $n = 6$.

(b) If $P, Q$ are skew-linked, then $F^\text{mod}(\Lambda^\text{csq}_{Q,P}) = \Lambda^\text{mod}_{\mu,\lambda}$.

(c) If $\Lambda^\text{csq}_{Q,P}$ and $\Lambda^\text{csq}_{Q',P'}$ satisfy (5.74), then $\Lambda^\text{csq}_{Q,P} \cong \Lambda^\text{csq}_{Q',P'}$ in $\text{CSQ}(\hat{\mathcal{H}}^+)$. 
Bibliography


[34] T. Nakashima, Crystal base and a generalization of the Littlewood-Richardson


