Traveling waves and weak solutions for an equation with degenerate dispersion

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Abstract
We consider the following family of equations:

$$u_t = 2uu_{xxx} - u_xu_{xx} + 2ku_x.$$  

Here $k \neq 0$ is a constant and $x \in [-L_0, L_0]$. We demonstrate that for these equations: (a) there are compactly supported traveling wave solutions (which are in $H^2$) and (b) the Cauchy problem (with $H^2$ initial data) possesses a weak solution which exists locally in time. These are the first degenerate dispersive evolution PDE where (a) and (b) are known to hold simultaneously. Moreover, if $k < 0$ or $L_0$ is not too large, the solution exists globally in time.

1 Introduction

Nonlinear dispersive PDE, such as the nonlinear Schrödinger equation or Korteweg-deVries equations, have been the subject of intense mathematical scrutiny. The importance of linear dispersive effects in understanding the dynamics of these equations cannot be understated, particularly in establishing the existence of special solutions such as solitons and also in more general studies of the associated Cauchy problem. However, there are number of dispersive equations in which the mechanism which generates the dispersive effects is itself nonlinear. The most famous example is the $K(m,n)$ family of equations developed in [12]. (Some other such equations are degenerate
variations of NLS equations [17], Klein-Gordon equations [13], model equations for granular media [9, 5] and magma dynamics [16], and equations which arise in the numerical analysis of the KdV equation [4].) The most striking effect of degenerate dispersion is the existence of coherent structures with compact support. While there are numerous careful numerical studies ([6, 3, 14, 15]) of compact traveling waves (i.e. compactons) and other types of compact structures, there are very few rigorous results concerning the existence and behavior of general solutions to the Cauchy problem ([1, 3]). In this document we consider the following family of equations:

\[
    u_t = 2uu_{xxx} - u_x u_{xx} + 2kuu_x
\]

Here \( k \neq 0 \) is a constant and \( t > 0 \). We take \( x \in X := [-L_0, L_0] \) and assume \( u \) satisfies periodic boundary conditions. We will be considering the Cauchy problem:

\[
    u(x, 0) = \phi(x) \in H^2.
\]

By rescaling the time and space variables we can set \( k = \pm 1 \). Note this has the additional effect of changing width of \( X \), but since we view \( L_0 \) as a parameter in the problem, this change is not substantial. Note that (1) is a modification of the \( K(2, 2) \) equation:

\[
    v_t = 2vv_{xxx} + 6v_x v_{xx} + 2vv_x.
\]

We demonstrate (a) that (1) has compactly supported traveling wave solutions and (b) the Cauchy problem (with \( H^2 \) initial data) for (1) possesses a weak solution which exists locally in time. Equations (1) are the first degenerate dispersive evolution PDE where (a) and (b) are known to hold simultaneously. Since many studies of degenerate dispersive equations focus on interactions between compactly supported traveling waves, it is of fundamental importance that the Cauchy problem for initial data in the same class as the traveling waves be understood. In our case, our traveling wave solutions are in fact \( H^2 \).

In Section 2 we first prove some \textit{a priori} estimates for smooth solutions of (1). Namely, we show that the \( H^2 \) norm is bounded uniformly in terms of the initial data, at least for finite times. In certain situations, the \( H^2 \) norm is bounded for all \( t > 0 \). Subsequently we use those estimates to prove:

**Theorem 1.** For \( k = \pm 1 \), for all \( L_0 > 0 \) and all \( \phi \in H^2(X) \), there exists \( T^* = T^*(k, L_0, \phi) \in (0, \infty) \) so that for all \( 0 < T < T^* \) there is a function

\[
    u \in L^2(0, T; H^{7/4})
\]
which is a weak solution of (1). Specifically \( u \) satisfies (4) below. Additionally, if \( k = -1 \) or \( L_0 < \pi/\sqrt{2} \) then \( T^* = +\infty \).

Surprisingly, we find that for each wave speed \( c > 0 \), there are multiple solitary wave solutions to (1) with a continuum of different amplitudes. As is the case for the \( K(2,2) \) compactons, our traveling waves are not smooth, having a jump in the second derivative. There are several substantial differences between the \( k = 1 \) and \( k = -1 \) cases. If \( k = 1 \), there are solitary waves of arbitrarily large amplitude, though their width is bounded by a universal constant, independent of the domain size. On the other hand, if \( k = -1 \) the solitary waves have a maximal amplitude, but can be as wide as the whole of \( X \). In the case where \( X = \mathbb{R} \), we find front solutions whose support is a half-line. In Section 3 we derive explicit formulae for the traveling wave solutions.

2 Existence of weak solutions

2.1 Energy estimates

In this section we prove several \( a \) priori estimates for (1). Let

\[
E[f] := \int_X \left( \frac{1}{2} f^2(x) - \frac{5k}{4} f_x^2(x) + \frac{1}{2} f_{xx}^2(x) \right) dx.
\]

**Proposition 1.** If \( u \) is a sufficiently smooth solution of (1) with initial condition (2), we have for all \( t \)

\[
E[u(t)] = E[\phi].
\]

**Proof.** Let

\[
E_0(t) := \frac{1}{2} \int_X u^2(x,t)dx, \quad E_1(t) := -\frac{5k}{4} \int_X u_x^2(x,t)dx
\]

\[
E_2(t) := \frac{1}{2} \int_X u_{xx}^2(x,t)dx.
\]

Differentiating \( E_0 \) with respect to time, using (10) and then integrating by parts gives:

\[
\dot{E}_0 = \int_X uu_t dx
\]

\[
= \int_X u(2uu_{xx} - \frac{3}{2} u_x^2 + ku^2) dx
\]

\[
= - \int_X u_x (2uu_{xx} - \frac{3}{2} u_x^2 + ku^2) dx.
\]
Since \(2u_x u_{xx} = u(u_x^2)_x\) and \(u^2 u_x = 1/3(u^3)_x\), we can rewrite the last line above as:
\[-\int_X (u(u_x^2)_x - \frac{3}{2} u_x^3 + \frac{k}{3}(u^3)_x)dx.\]

Now, \(\int_X (u^3)_x\) vanishes due to the periodic boundary conditions. A final integration by parts then gives:
\[\dot{E}_0 = \frac{5}{2} \int_X u_x^3 dx.\]

Similarly for \(E_1\), we have after integrating by parts in \(x\) one time:
\[\dot{E}_1 = -\frac{5k}{2} \int_X u_x u_{xx} dx\]
\[= -\frac{5k}{2} \int_X u_x (2u u_{xxx} - \frac{3}{2} u_x^2 + ku^2)_{xxx} dx\]
\[= \frac{5k}{2} \int_X u_{xxx} (2u u_{xxx} - \frac{3}{2} u_x^2 + ku^2) dx.\]

Applying the derivative and multiplying out the integrand converts the last line above to:
\[\frac{5k}{2} \int_X u_{xxx} (-u_x u_{xx} + 2uu_{xxx} + 2ku u_x) dx\]
\[= \frac{5k}{2} \int_X (-u_x u_x^2 + 2uu_{xxx} u_{xxx} + 2ku u_x u_{xx}) dx.\]

We then use the fact that \(2uu_{xxx} u_{xxx} = u(u_x^2)_x\) to get:
\[\dot{E}_1 = -5k \int_X u_x u_x^2 dx - \frac{5k^2}{2} \int_X u_x^3 dx.\]

And for \(E_2\), we apply the time derivative, use (10) and integrate in parts one time:
\[\dot{E}_2 = \int_X u_{xxx} u_{xx} dx\]
\[= \int_X u_{xxx} (2u u_{xx} - \frac{3}{2} u_x^2 + ku^2)_{xxx} dx\]
\[= -\int_X u_{xxx} (2u u_{xx} - \frac{3}{2} u_x^2 + ku^2)_{xxx} dx.\]
If we apply the two \( x \) derivatives to the quantity in parentheses above we arrive at:

\[
- \int_X u_{xxx}(-u_x u_{xx} + 2u u_{xxx} + 2k u u_x) \, dx
\]

\[
= - \int_X u_{xxx}(-u^2_{xx} + u_x u_{xxx} + 2u u_{xxxx} + 2k u u_x + 2k u_x^2) \, dx
\]

\[
= - \int_X (-u_{xxx} u^2_{xx} + u_x u^2_{xxx} + 2u u_{xxxx} u_{xxx} + 2k u u_{xx} u_{xxx} + 2k u_x^2 u_{xxx}) \, dx
\]

Since \( u_{xxx} u^2_{xx} = 1/3(u^3_{xx})_x \) and \( u_x u^2_{xxx} + 2u u_{xxxx} u_{xxx} = (u u^2_{xxx})_x \) we have:

\[
- \int_X (-\frac{1}{3} u^3_{xx})_x + (u u^2_{xxx})_x + 2k u u_{xx} u_{xxx} + 2k u_x^2 u_{xxx}) \, dx.
\]

The first two terms are perfect derivatives and thus vanish upon integration due to the periodic boundary conditions. Observing that \( 2u u_{xx} u_{xxx} = u(u^2_{xx})_x \) and integrating by parts the term \( 2k u_x^2 u_{xxx} \) gives:

\[
- \int_X (2k u u_{xx} u_{xxx} + 2k u_x^2 u_{xxx}) \, dx
\]

\[
= - \int_X (k u^2_{xx})_x - 4k u u_x^2 \, dx
\]

A final integration by parts gives:

\[
\dot{E}_2 = 5k \int_X u_x u^2_{xx} \, dx.
\]

Since \( k = \pm 1 \), we have \( k^2 = 1 \) and therefore

\[
\dot{E} = \dot{E}_0 + \dot{E}_1 + \dot{E}_2 = 0.
\]

\( \square \)

When \( k = -1 \), it is clear that \( E^{1/2} \) is equivalent to the usual norm on \( H^2 \). On the other hand, when \( k = 1 \) then \( E_1 \) is negative and \( E^{1/2} \) is not necessarily equivalent to the \( H^2 \) norm. We have

**Lemma 2.** If \( k = 1 \), then \( E^{1/2} \) is equivalent to the usual norm on \( H^2 \) if and only if \( 0 < L_0 < \frac{\sqrt{2} \pi}{2} \).
Proof. Since $X = [-L_0, L_0]$, by Plancherel’s theorem we have

$$E(t) = 2L_0 \sum_{n \in \mathbb{Z}} \left( \frac{1}{2} - \frac{5}{4} \left( \frac{n\pi}{L_0} \right)^2 + \frac{1}{2} \left( \frac{n\pi}{L_0} \right)^4 \right) |\hat{u}_n|^2$$

where $\hat{u}_n = (2L_0)^{-1} \int_{-L_0}^{L_0} u(x, t) e^{-in\pi x/L_0} dx$ are the usual Fourier coefficients.

The polynomial

$$\mu(n) := \frac{1}{2} - \frac{5}{4} \left( \frac{n\pi}{L_0} \right)^2 + \frac{1}{2} \left( \frac{n\pi}{L_0} \right)^4$$

has $\mu(n) > 0$ if and only if $|n\pi/L_0| > \sqrt{2}$ or $|n\pi/L_0| < \sqrt{2}/2$. Therefore, if there are no integers $n$ with $|n|$ in the set $[\sqrt{2}L_0/2\pi, \sqrt{2}L_0/\pi]$ we have $\mu(n) > 0$ for all $n \in \mathbb{Z}$. This in turn implies that there exists $C > 1$ such that for all $n \in \mathbb{Z}$

$$C^{-1} \left( 1 + \left( \frac{n\pi}{L_0} \right)^4 \right) < \mu(n) < C \left( 1 + \left( \frac{n\pi}{L_0} \right)^4 \right).$$

This then gives the equivalence of $E^{1/2}$ to the normal $H^2$ norm.

Thus we only need to determine when there are no integers $n$ with $|n|$ in the set $[\sqrt{2}L_0\pi/2, \sqrt{2}L_0\pi]$. This happens precisely when $L_0 < \sqrt{2}\pi/2$. On the other hand, if $L_0 \geq \sqrt{2}\pi/2$, $\mu(n)$ is not strictly bounded above zero on the integers, and we cannot have the equivalence of the norms.

The following corollary gives a useful $a\ priori$ estimate in situations where $E^{1/2}$ is not a norm on $H^2$. Let

$$N[f] := \frac{1}{2} \|f\|_{H^2}^2 = \frac{1}{2} \left( \|f\|_{L^2}^2 + \|f_{xx}\|_{L^2}^2 \right).$$

**Corollary 3.** For $k = \pm 1$ and any $L_0 > 0$, we have for any sufficiently smooth solution of (1):

$$\frac{d}{dt} N[u(t)] \leq CN^{3/2}[u(t)].$$
Proof. Since $N = N[u(t)] = E_0(t) + E_2(t)$, we have:

$$\dot{N} = \frac{5}{2} \int_X u_x^3 dx + 5k \int_X u_x u_{xx}^2 dx \leq C \|u_x\|_{L^\infty} \int_X (u_x^2 + u_{xx}^2) dx \leq C \|u\|_3^3 \|H^2\|_H^2 = CN^{3/2}.$$ (3)

2.2 Construction of weak solutions

Let $\psi : X \times [0, \infty) \to \mathbb{R}$ be a compactly supported test function. Since $\psi$ is compactly supported, there exists $T > 0$ such that $\psi(x, t) = 0$ for all $t \geq T$.

If we multiply (1) by $\psi$, integrate over $X \times [0, \infty)$ and subsequently integrate by parts in each of $x$ and $t$ as appropriate, we arrive at:

$$\int_0^T \int_X \left[ -(\partial_t \psi)u + (\partial^3_{x} \psi + k \partial_x \psi)(u^2) - \frac{7}{2} (\partial_x \psi)(\partial_x u)^2 \right] \, dx dt = \int_X \psi(x, 0) \phi(x) \, dx. \quad (4)$$

We say that $u \in L^2(0, T; H^{7/4})$ is a weak solution of (1) provided this relation holds for all test functions $\psi$. We now prove Theorem 1.

Proof. Step 1, Existence of regularized approximate solutions: We begin by introducing a regularized version of equation (1). Let $\varepsilon > 0$ be given, and let $\mathcal{J}_\varepsilon$ be a family of Friedrichs mollifiers (for instance, convolution with an appropriate smooth family of functions [8]). Then consider:

$$u_t = 2 \mathcal{J}_\varepsilon \left( (\mathcal{J}_\varepsilon u)(\partial^3_x \mathcal{J}_\varepsilon u) \right) - \mathcal{J}_\varepsilon \left( (\partial_x \mathcal{J}_\varepsilon u)(\partial^2_x \mathcal{J}_\varepsilon u) \right) + 2k \mathcal{J}_\varepsilon \left( (\mathcal{J}_\varepsilon u)(\partial_x \mathcal{J}_\varepsilon u) \right). \quad (5)$$

We use the same initial data as above, (2). It is sometimes useful to rewrite this as:

$$\partial_t u = \partial^3_x \mathcal{J}_\varepsilon (\mathcal{J}_\varepsilon u)^2 - \partial_x \mathcal{J}_\varepsilon \left( \frac{7}{2} (\partial_x \mathcal{J}_\varepsilon u)^2 \right) + k \partial_x \mathcal{J}_\varepsilon (\mathcal{J}_\varepsilon u)^2. \quad (6)$$

With the abundance of mollifiers above, it is clear that the right hand side of (5) is a bounded and continuous map from $H^2$ into itself. Thus the Picard theorem for ODEs on a Banach space applies: there is a solution of (5), denoted $u_\varepsilon(x, t)$, in $C^1(0, T_\varepsilon; H^2)$ for some $T_\varepsilon > 0$. 

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Step 2. Uniform time of existence: That the time of existence for $u_\varepsilon$
depends upon $\varepsilon$ is a problem, as we would like to take $\varepsilon \to 0$. Here we prove
the following:

Lemma 4. For $k = \pm 1$ and $L_0 > 0$ there exists $T^* = T^*(k, L_0, \phi) \in (0, \infty)$
such that for all $T \in (0, T^*)$ and $\varepsilon > 0$, the solutions $u_\varepsilon$ of (5) satisfy:

$$u_\varepsilon \in C^1(0, T; H^2) \subset L^2(0, T; H^2)$$

and

$$\partial_t u_\varepsilon \in C(0, T; H^{-1}) \subset L^2(0, T; H^{-1}).$$

These functions are bounded uniformly in $\varepsilon$ in these spaces. Finally, if $k = -1$ or $L_0 < \pi/\sqrt{2}$ then $T^* = \infty$ for all $\phi$.

Proof. This lemma follows from the energy estimates derived in the previous
section. The placement of mollifiers in (5) is done in such a way so that these
energy estimates carry over to the regularized equation. Specifically we have
for all $t \in [0, T_\varepsilon)$

$$E[u_\varepsilon(t)] = E[\phi]$$

(7)

and

$$\frac{d}{dt} N[u_\varepsilon(t)] \leq CN^{3/2}[u_\varepsilon(t)].$$

(8)

Of course, the verification of (7) and (8) follows along lines similar to
the proofs of Proposition 1 and Corollary 3 and we will suppress most of
the detail. The key point is to show that the mollifiers do not affect the
structure of the energy argument.

If we set $F_0(t) := \frac{1}{2} \int_X u_\varepsilon^2 dx$ and $v_\varepsilon = J_\varepsilon u_\varepsilon$, then:

$$\dot{F}_0 = \int_X u_\varepsilon \partial_t u_\varepsilon dx$$

$$= \int_X u_\varepsilon J_\varepsilon (2v_\varepsilon v_\varepsilon,xx - \frac{3}{2} v_\varepsilon^2 + kv_\varepsilon^2)dx$$

$$= \int_X (J_\varepsilon u_\varepsilon)(2v_\varepsilon v_\varepsilon,xx - \frac{3}{2} v_\varepsilon^2 + kv_\varepsilon^2)dx$$

$$= \int_X v_\varepsilon (2v_\varepsilon v_\varepsilon,xx - \frac{3}{2} v_\varepsilon^2 + kv_\varepsilon^2)dx$$

In going from the second to the third line we have made use of the fact that
$J_\varepsilon$ is self-adjoint. Notice that the final line here is the same as that which

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appears in the expression for $\dot{E}_0$ in the proof of Proposition 1 except with $v_\varepsilon$ appearing instead of $u$. Thus the same steps used there lead us to:

$$\dot{F}_0 = \frac{5}{2} \int_X v^3_{\varepsilon,x} dx.$$ 

In exactly the same fashion, if we set $F_1(t) := -\frac{5k}{4} \int_X u^2_{\varepsilon,x} dx$ and $F_2(t) := \frac{1}{2} \int_X u^2_{\varepsilon,xx} dx$ then

$$\dot{F}_1 = -5k \int_X v_{\varepsilon,x} v^2_{\varepsilon,xx} dx - \frac{5k^2}{2} \int_X v^3_{\varepsilon,x} dx, \quad \dot{F}_2 = 5k \int_X v_{\varepsilon,x} v^2_{\varepsilon,xx} dx.$$ 

From this, (7) and (8) follow immediately.

If $k = -1$ or $L_0 < \pi/\sqrt{2}$, Lemma 2 tells us that $E_{1/2}[u_\varepsilon(t)]$ an equivalent norm for $H^2$. Thus (7) tells us that the $H^2$ norm of the solution cannot blow up at any time. Therefore, by the continuation theorem for autonomous ODEs on a Banach space, for any $0 < T < \infty$ the solutions $u_\varepsilon$ of (5) exist on the interval $[0, T)$, and are in the space $C^1(0, T; H^2(X))$, bounded uniformly in $\varepsilon$.

On the other hand if $k = 1$ and $L_0 \geq \pi/\sqrt{2}$, then $E_{1/2}$ is not a norm on $H^2$. Instead, we integrate the differential inequality (8) to find that there exists a time $T^* > 0$, which is independent of $\varepsilon$, before which $N[u_\varepsilon(t)] = \frac{1}{2} \|u_\varepsilon\|^2_{H^2}$ must remain finite. That is, in this case, for any $0 < T < T^*$ the solutions $u_\varepsilon$ of (5) exist on the interval $[0, T)$, and are in the space $C^1(0, T; H^2(X))$, bounded uniformly in $\varepsilon$.

That $\partial_t u_\varepsilon$ are bounded uniformly in $C(0, T; H^{-1})$ follows by examining the right hand side of (6). For instance, $u_\varepsilon \in H^2$ implies $u^2_\varepsilon \in H^2$, which in turn implies $(u^2_\varepsilon)_{xxx} \in H^{-1}$. Thus the first term on the right hand side is in $C(0, T; H^{-1})$. All the other terms are similar. \hfill \Box

**Step 3, Convergence of $u_\varepsilon$ to a weak solution:** Fix $T < T^*$. The Aubin-Lions lemma states (see [2])

**Lemma 5.** Suppose that $B_0$, $B_1$, and $B_{-1}$ are three separable reflexive Banach spaces with $B_1 \subset B_0$ and $B_0$ is continuously embedded in $B_{-1}$. Suppose that $f_m$ is a bounded sequence in $L^p(0, T; B_1)$ and $\partial_t f_m$ is a bounded sequence in $L^q(0, T; B_{-1})$. Here $1 < p, q < \infty$. Then there a subsequence $f_{m'}$ which converges in $L^p(0, T; B_0)$. 

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$X$ is a compact and thus $H^2 \subset H^{7/4} \subset H^{-1}$. Therefore this Lemma together with our Lemma 4 imply that there is a function $u$ such that

$$u_\varepsilon \rightarrow u$$

in $L^2(0,T;H^{7/4})$ as $\varepsilon \rightarrow 0$, along a subsequence. This function $u$ is the weak solution we are looking for; here is the calculation.

By multiplying (6) by a compactly supported test function, integrating in both space and time and integrating by parts, we find that the solutions $u_\varepsilon$ satisfy:

$$\int_0^T \int_X \left[ -(\partial_t \psi)u_\varepsilon + (\partial^3_x \psi + k\partial_x \psi)(J_\varepsilon(J_\varepsilon u_\varepsilon)^2) - (\partial_x \psi)\left(\frac{7}{2} J_\varepsilon(\partial_x J_\varepsilon u_\varepsilon)^2\right) \right] dxdt = \int_X \psi(x,0)\phi(x) \, dx. \quad (9)$$

Since we have $u_\varepsilon$ converging strongly to $u \in L^2(0,T;H^{7/4})$, we have enough regularity to ensure that $J_\varepsilon(J_\varepsilon u_\varepsilon)^2$ converges strongly to $u^2$ and $J_\varepsilon(\partial_x J_\varepsilon u_\varepsilon)^2$ converges strongly to $(\partial_x u)^2$. We can therefore take the limit as $\varepsilon \rightarrow 0$ in (9), finding

$$\int_0^T \int_X \left[ -(\partial_t \psi)u + (\partial^3_x \psi + k\partial_x \psi)(u^2) - \frac{7}{2}(\partial_x \psi)(\partial_x u)^2 \right] dxdt = \int_X \psi(x,0)\phi(x) \, dx. \quad (11)$$

Thus $u$ is a weak solution and we are done.

\[\square\]

3 Existence of compactly supported traveling waves

Note that (1) can be rewritten as

$$u_t = \partial_x \left( 2uu_{xx} - \frac{3}{2}u_x^2 + ku^2 \right). \quad (10)$$

Making the traveling wave Ansatz $u(x,t) = Q(s)$ where $s = x - ct$, we find:

$$-cQ = 2Q\ddot{Q} - \frac{3}{2}\dot{Q}^2 + kQ^2. \quad (11)$$

We have integrated in $s$ one time.
We rewrite (11) as a system:
\[
\begin{align*}
\dot{Q} &= P \\
\dot{P} &= \frac{1}{2Q} \left( -cQ - kQ^2 + \frac{3}{2} P^2 \right).
\end{align*}
\] (12)

Our goal is to find a solution which is homoclinic to the origin and for which duration of the trajectory is finite. We are able to find explicit formulae for such solutions. Notice that (12) has two equilibria: one at the origin (which is degenerate) and another at \( Q = -c/k, P = 0 \).

Notice that when \( P < 0 \) we have \( \dot{Q} < 0 \). Therefore we can assert that a solution which lies in the lower half of the \( QP \)-plane has a trajectory which is a graph over the \( Q \) axis. That is,
\[
P(t) = f(Q(t))
\]
for an as yet unspecified function \( f \). We differentiate this relationship and use (12):
\[
\dot{P} = f'(Q)\dot{Q} = f'(Q)P = f'(Q)f(Q) = \frac{1}{2} \frac{d}{dQ} \left( f^2(Q) \right).
\]
The second equation in (12) then becomes:
\[
\frac{1}{2} \frac{d}{dQ} \left( f^2(Q) \right) = \frac{1}{2Q} \left( -cQ - kQ^2 + \frac{3}{2} f^2(Q) \right).
\]
Letting \( F = f^2 \) and rearranging terms leads to the following ODE:
\[
F' - \frac{3}{2Q} F = -c - kQ
\]
This can be solved explicitly for \( F \):
\[
F(Q) = 2cQ - 2kQ^2 - \beta_0 Q^{3/2}.
\]
\( \beta_0 \in \mathbb{R} \) is an arbitrary constant.

Thus the solution of (11) lies on the graph \( P = -\sqrt{F(Q)} \) and so:
\[
\dot{Q} = -\sqrt{2cQ - 2kQ^2 - \beta_0 Q^{3/2}}
\] (13)
This equation can be solved explicitly, though the details vary depending on the sign of \( k \).
3.1 Traveling pulses and fronts for \( k = -1 \)

Separating variables and integrating (13) give:

\[
\int_{A}^{Q(s)} \frac{dQ}{\sqrt{2cQ + 2Q^2 - \beta_0 Q^{3/2}}} = -s
\]

\[
\Rightarrow \sqrt{2} \ln \left( \frac{-\sqrt{2}\beta_0/4 + \sqrt{2}Q(s) + \sqrt{2c + 2Q(s) - \beta_0\sqrt{Q(s)}}}{-\sqrt{2}\beta_0/4 + \sqrt{2A + \sqrt{2c + 2A - \beta_0\sqrt{A}}}} \right) = -s
\]

If we take \( \beta_0 \) so that \( Q(0) = A > 0 \), we find:

\[
Q(s) = \frac{1}{4A} \left[ c + A - (c - A) \cosh \left( \frac{\sqrt{2}}{2} s \right) \right]^2.
\] (14)

This function diverges exponentially quickly as \( s \to \infty \) and so does not immediately give the profile of a compactly supported traveling wave for (1). Notice that this function has two zeros when \( 0 < A < c \) at

\[
s = L := \pm \sqrt{2} \ln \left( \frac{c + A + 2\sqrt{cA}}{c - A} \right).
\]

Since \( Q(s) \) is nonnegative and in \( C^\infty \), it follows that \( \dot{Q}(\pm L) = 0 \) and \( \ddot{Q}(\pm L) \) is finite and positive. Thus we can cut the function \( Q \) off for for \( |s| \geq L \) and the resulting truncated function is still a solution of (11), since it is degenerate at \( Q = 0 \).

That is to say, we have a pulse solution to (11) of the form:

\[
Q(s) = Q^{c,A}_-(s) := \frac{1}{4A} L(s) \left[ A + c + (A - c) \cosh \left( \frac{\sqrt{2}}{2} s \right) \right]^2
\]

Here \( 1_L(s) \) is the characteristic function of \([-L, L]\). Clearly we require \( L \leq L_0 \).

We make the following observations about \( Q^{c,A}_- \).

- The maximum of \( Q^{c,A}_- \) is \( A \), thus for any speed \( c > 0 \) there are compactly supported traveling waves with any amplitude \( A \in (0,c) \).
- As \( A \to c^- \), notice that \( L \to +\infty \).
- \( \lim_{s \to L^-} \frac{d^2}{ds^2} Q^{c,A}_-(s) = \lim_{s \to -L^+} \frac{d^2}{ds^2} Q^{c,A}_-(s) = c. \)
Figure 1: Profiles for traveling waves, including the front and back, of (1) when $k = -1$. Here, all waves move with same speed $c = 1$ but vary in amplitude. All traveling waves are asymptotic to the same parabola as they approach 0. Also observe that pulses approach the front solution was their amplitude approaches 1.

There are also front solutions of (1) with speed $c$, if we allow $X = \mathbb{R}$. These correspond to the two heteroclinic solutions of (12) connecting the origin to the equilibrium at $(c,0)$. These are given by:

$$Q_{c,\text{front}}(s) = \begin{cases} 
  c \left[ 1 - e^{\frac{c}{2}s} \right]^2, & \text{if } s < 0 \\
  0, & \text{otherwise.}
\end{cases}$$

and

$$Q_{c,\text{back}}(s) = Q_{c,\text{front}}(-s).$$

Observe that $\lim_{s \to 0^-} \frac{d^2}{ds^2} Q_{c,\text{front}}(s) = \lim_{s \to 0^+} \frac{d^2}{ds^2} Q_{c,\text{back}}(s) = c$. We plot the various pulse and front solutions in Figures 1 and 2.
3.2 Traveling pulses for $k = 1$

We can likewise compute explicit formulae for pulses when $k = 1$; the details are only slightly different than above and so we only provide the end result:

$$Q_{c,A}^+(s) := \frac{1_M(s)}{4A} \left[ A - c + (A + c) \cos \left( \frac{\sqrt{2}}{2}s \right) \right]^2.$$  

Here $1_M(s)$ is the characteristic function of $[-M, M]$ and

$$M := \sqrt{2} \arccos \left( \frac{c - A}{c + A} \right).$$

Clearly we require $M \leq L_0$.

We make the following observations about $Q_{c,A}^+$.

- The maximum of $Q_{c,A}^+$ is $A$, and unlike when $k = -1$, there is no upper limit on $A$; for any speed $c > 0$ there are compactly supported traveling waves with any amplitude $A > 0$.

- As $A \to \infty$, notice that $M \to \sqrt{2} \arccos(-1) = \sqrt{2} \pi$ monotonically. Thus pulses have a maximum width in this setting.
Figure 3: Profiles for traveling waves when $k = 1$. Here, all waves move with same speed $c = 1$ but vary in amplitude. Observe that the large amplitude solution has width approaching $4.4 \sim \sqrt{2\pi}$.

- Similarly, for any amplitude $A > 0$, there is a standing pulse ($c = 0$) which is of maximum width.

- As before: $\lim_{s \to -M} \frac{d^2}{ds^2} Q_{c,A}^+(s) = \lim_{s \to -M^+} \frac{d^2}{ds^2} Q_{c,A}^+(s) = c$.

We plot the various pulse solutions in Figures 3 and 4.

**Remark 1.** Note that for any $A \in \mathbb{R}$ and $c \in \mathbb{R}$

$$\hat{Q}(s) := \frac{1}{4A} \left[ A - c + (A + c) \cos \left( \frac{\sqrt{2}}{2} s \right) \right]^2$$

gives the profile of a periodic traveling wave for (1) when $k = 1$ and the period of $\hat{Q}$ is compatible with the width of the domain $X$. When $c > 0$ and $A < 0$, these periodic waves are negative valued and are strictly bounded away from zero; in particular we have $\hat{Q} \in \left[ \frac{\epsilon^2}{4A}, A \right]$.

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Figure 4: Profiles for a variety of traveling waves for the $k = 1$ equation when $A = 1$ and $c$ varies. Note the $c = 0$ wave has width exactly $4.4 \sim \sqrt{2\pi}$.

References


