Chapter 11.1: Limits

**Intuitive Approach**

**Motivational Example 1**

Let \( f(x) = x + 3 \)

Notice: \( f(1) = 1 + 3 = 4 \)

As \( x \) gets close to 1, let's see what happens to the function values:

<table>
<thead>
<tr>
<th>From the left</th>
<th>From the Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( f(x) )</td>
</tr>
<tr>
<td>( 0.9 )</td>
<td>3.9</td>
</tr>
<tr>
<td>( 0.99 )</td>
<td>3.99</td>
</tr>
<tr>
<td>( 0.999 )</td>
<td>3.999</td>
</tr>
<tr>
<td>( 1.001 )</td>
<td>4.001</td>
</tr>
</tbody>
</table>

Note: When we get close to 1 from the left, we choose \( x \)-values slightly less than 1.

No matter how we approach 1 along the \( x \)-axis, the function \( f(x) \) values approach 4.

We say:

"The limit of \( f(x) \) as \( x \) approaches 1 is 4."

And we write:

\[
\lim_{{x \to 1}} f(x) = 4
\]

Graphically:

[Graph of \( f(x) = x + 3 \)]

**Motivational Example 2**

Let \( g(x) = \frac{x^2 + 2x - 8}{x - 1} \)

Notice here that \( g(x) \) does not exist at \( x = 1 \)

\( g(1) \) DNE
Let's see what happens as $x$ gets close to 1.

**From the Left**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.9$</td>
<td>$3.9$</td>
</tr>
<tr>
<td>$0.99$</td>
<td>$3.99$</td>
</tr>
<tr>
<td>$0.999$</td>
<td>$3.999$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

**From the Right**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.01$</td>
<td>$4.01$</td>
</tr>
<tr>
<td>$1.001$</td>
<td>$4.001$</td>
</tr>
</tbody>
</table>

\[ \downarrow \]

As $x$ gets close to 1 from both sides, we see that $g(x)$ gets close to 4.

Also, the closer $x$ gets to 1, the closer $g(x)$ gets to 4.

Thus, \[ \lim_{{x \to 1}} g(x) = 4. \]

**Graphically:**

![Graph showing a hole at (1, 4)](image_url)

It doesn't matter that $g(x)$ does. When we compute limits, we only care about what happens close to $x=1$.

**Warning:** When we compute \[ \lim_{{x \to a}} f(x), \] we do not care about the value of $f(a)$ or if it even exists! We only care about $x$ values "close" to $a$.

**Motivational Example 3**

Let \[ h(x) = \begin{cases} x^2 & x \geq 0 \\ x-1 & x < 0 \end{cases} \]

This is called a piecewise function.
Graphically:

Notice: \( h(0) = 0^2 = 0 \). [Be careful which piece you use!!]

Let's look at what happens as \( x \) gets close to 0 from the left!

* If we trace the graph of \( h(x) \) from left to right, we see that as \( x \) approaches 0, the function gets closer to -1.

* We say: "the limit as \( x \) approaches 0 from the left of \( h(x) \) is -1".

* We write: \[ \lim_{{x \to 0^-}} h(x) = -1 \]

Let's look at what happens as \( x \) gets close to 0 from the right!

* If we trace the graph of \( h(x) \) from right to left, we see that as \( x \) approaches 0, the function gets closer to 0.

* We say: "the limit as \( x \) approaches 0 from the right of \( h(x) \) is 0".

* We write: \[ \lim_{{x \to 0^+}} h(x) = 0 \].

This is bad! Our function is approaching \( \frac{3}{2} \) different values.
\[ \lim_{x \to 0^-} h(x) = -1 \quad \text{but} \quad \lim_{x \to 0^+} h(x) = 0 \]

the function is not getting close to any particular number as \(x\) gets close to 0.

* We say \[ \lim_{x \to 0} h(x) \text{ DNE} \]

* FACT: If \( \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \), then \( \lim_{x \to a} f(x) \text{ DNE} \)

**Motivational example #4**

Let \( f(x) = \frac{1}{x^2} \)

Graphically:

\[ \text{Notice} \cdot f(0) \text{ DNE.} \]

- \( \lim_{x \to 0^-} f(x) = +\infty \)
- \( \lim_{x \to 0^+} f(x) = +\infty \)

\[ \text{So} \quad \lim_{x \to 0} f(x) = +\infty \]

[Recall from Math 101 that if \( f(x) \to \pm \infty \) at \( x = a \),

then \( x = a \) is a vertical asymptote

So \( x = 0 \) is a vertical asymptote

- \( \lim_{x \to \infty} f(x) = 0 \)
- \( \lim_{x \to -\infty} f(x) = 0 \)

So \( f(x) \) has a horizontal asymptote at \( y = 0 \) since the function "levels off" at the line \( y = 0 \).
II. Computing Limits (at a point)

Theorems:

a) If \( k \) is a constant then \( \lim_{x \to a} k = k \)

b) \( \lim_{x \to a} x = a \)

c) If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) then

\[ \lim_{x \to a} [f(x) \pm g(x)] = L \pm M \]

\[ \lim_{x \to a} [f(x) \cdot g(x)] = LM \]

\[ \lim_{x \to a} \left[ \frac{f(x)}{g(x)} \right] = \frac{L}{M} \] (if \( M \neq 0 \))

Note: Informally this theorem tells us that we can compute \( \lim_{x \to a} f(x) \) by simply "plugging in" and computing \( f(a) \).

\[ \lim_{x \to 3} x^2 = 3^2 = 9 \]

\[ \lim_{x \to -1} (x^2 + 2x) = (-1)^2 + 2(-1) = -1 \]

\[ \lim_{x \to 3} \frac{x^2 - 2x}{x+1} \rightarrow \frac{3^2 - 2(3)}{3+1} = \frac{9-6}{4} = \frac{3}{4} \]

\[ \lim_{x \to 1} \frac{x-1}{2x+1} = \frac{1-1}{2(1)+1} = \frac{0}{3} = 0 \]
This method only works if our answer makes sense.

\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 0 \quad \text{← This doesn't make sense!!}
\]

\[
\text{[\(\frac{0}{0}\) is called indeterminate]}
\]

\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1} (x+1) = 2
\]

[Check Graphically]

(math technically: by cancelling the \(x-1\)'s we are really dividing the numerator and denominator by \(x-1\). This is OK because \(x\to 1\) means \(x\neq 1\).

\[
\lim_{x \to 8} \frac{x^2 - 3x - 3}{x^2 - 9} \to \frac{3^2 + 2(3) - 3}{3^2 - 9} = \frac{0}{0}
\]

[\text{indeterminate}]

\[
\text{\underline{\text{Factor + Cancel!!}}}
\]

\[
= \lim_{x \to 3} \frac{(x-3)(x+3)}{(x+3)(x-3)} = \lim_{x \to 3} \frac{x+1}{x+3} = \frac{4}{6} = \frac{2}{3}
\]

\[
\lim_{x \to 0} \frac{\frac{1}{x} - \frac{1}{3}}{x} \to \frac{\frac{1}{0} - \frac{1}{3}}{0} = \frac{1}{3} - \frac{1}{3} = 0
\]

[\text{indef.}]

\[
\text{factor/cancel won't work. Instead, we'll clean up the complex fraction.}
\]

\[
\lim_{x \to 0} \frac{\frac{1}{x} - \frac{1}{3}}{x} = \lim_{x \to 0} \frac{\frac{1}{x} - \frac{1}{3}}{x} \cdot \frac{(3+x)(3)}{(x+3)(3)}
\]

\[
= \lim_{x \to 0} \frac{-x}{x(3+x)(3)}
\]

\[
= \lim_{x \to 0} \frac{-1}{x+3}
\]
\[ \lim_{x \to 0} \frac{-1}{(2+x)(2)} = \frac{-1}{2} \]

**Example**

Let \( f(x) = \begin{cases} x^2 & x > 0 \\ x-1 & x < 0 \end{cases} \)

**Compute** \( \lim_{x \to 0} f(x) \)

**Note** \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x-1 = 0-1 = -1 \)

**Then** \( x \)-values smaller than 0 \( (x < 0) \)

\( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0^2 = 0 \)

**Then** \( x \)-values larger than 0 \( (x > 0) \)

Since \( \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \) we know \( \lim_{x \to 0} f(x) \) **DNE**

**Warning!** Just because a function is piecewise doesn't mean that the limit will fail to exist.

**Example**

\( f(x) = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases} \)

Compute \( \lim_{x \to 0} f(x) \)

Since 0 is the "breaking point" we compute the left and right limits.

\( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = -0 = 0 \)
\[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x) = 0 \]
\[ (x > 0) \]

Since \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0 \)

we know \( \lim_{x \to 0} f(x) = 0 \).

III. Computing limits (at infinity).

A. Recall \( f(x) = \frac{1}{x} \)

\[ \lim_{x \to \infty} \frac{1}{x} = 0 \] and \( \lim_{x \to -\infty} \frac{1}{x} = 0 \).

FACT if \( n > 0 \), then \( \lim_{x \to \infty} \frac{1}{x^n} = 0 \) and \( \lim_{x \to -\infty} \frac{1}{x^n} = 0 \)

[i.e. as we divide by larger values of \( x \), the overall]

Fun goes to 0.

B. Polynomials:

1. Take \( f(x) = x^2 \)

\[ \lim_{x \to \infty} x^2 = \infty \]
\[ \lim_{x \to -\infty} x^2 = \infty \]

FACT if \( n > 0 \) and \( n \) is EVEN, then \( \lim_{x \to \infty} x^n = \infty \) and \( \lim_{x \to -\infty} x^n = \infty \)

2. Let \( f(x) = x^3 \)

\[ \lim_{x \to \infty} x^3 = \infty \]
\[ \lim_{x \to -\infty} x^3 = -\infty \]
Fact: If \( n > 0 \) and \( n \) is **ODD**, then \( \lim_{x \to \infty} x^n = \infty \) and \( \lim_{x \to -\infty} x^n = -\infty \).

3. \[ \lim_{x \to \infty} (3x^3 - 7x + 2) = \lim_{x \to \infty} (3x^3) = \infty \]

Note: We are really computing the **end behavior**! The \( 3x^3 \) is the dominant term which controls the end behavior.

\[ \lim_{x \to \infty} (-2x^4 - 9x^3 + x - 2) = \lim_{x \to \infty} (-2x^4) = -\infty \]

\[ \lim_{x \to \infty} \frac{2x^3 - 9x}{3x^3 - 1} \to \infty \]

Think **end behavior**!

[From 10.1, know that if degree \( \text{top} = \text{degree bottom} \) take leading term for \( \lim_{x \to \infty} \) and guess \( y = \frac{2}{3} \).]

So, guess that this limit is \( \frac{2}{3} \) since the two levels off at this line.

Calculus way of taking this limit:

\[ \lim_{x \to \infty} \frac{2x^3 - 9x}{3x^3 - 1} \]

\[ = \lim_{x \to \infty} \frac{2 - \frac{9}{x^3}}{3 - \frac{1}{x^3}} \]

\[ = \frac{2 - 0}{3 - 0} = \frac{2}{3} \]
General Strategy: Divide all terms by the highest power of x which occurs in the denominator

Alternate Approach

same problem:

\[ \lim_{x \to \infty} \frac{2x^3 - 9x}{3x^3 - 1} \]

end behavior

\[ = \lim_{x \to \infty} \frac{2x^3}{3x^3} \]

\[ = \lim_{x \to \infty} \frac{2}{3} = \frac{2}{3} \text{ same!} \]

So, we can look at the ratio of the "dominant terms" instead.

For this course, I will use the 2nd method.

For all "end behavior" problems,

\[ \lim_{x \to \infty} \frac{x + 7}{3x^2 - 9} \]

\[ = \lim_{x \to \infty} \frac{x}{3x^2} \]

\[ = \lim_{x \to \infty} \frac{1}{3x} = \frac{1}{3x} \]

\( \text{acts like } \frac{1}{x} \)

\[ \lim_{x \to \infty} \frac{4x^3 + 7x - 1}{3x^2 - 9} \]

\[ = \lim_{x \to \infty} \frac{4x^3}{3x^2} \]

\[ = \lim_{x \to \infty} \frac{4x}{3} \to \infty \]

because as \( x \) gets larger and larger \( \frac{4x}{3} \) also gets larger.