Vector-Valued Functions: Practice Problems

EXPECTED SKILLS:

- Be able to describe, sketch, and recognize graphs of vector-valued functions (parameterized curves).
- Know how to differentiate vector-valued functions. And, consequently, be able to find the tangent line to a curve (as a vector equation or as a set of parametric equations).
- Be able to determine angles between tangent lines.
- Know how to use differentiation formulas involving cross-products and dot products.
- Give a curve defined parametrically in terms of \( t \), be able to compute the unit tangent vectors \( T(t) \), the principal unit normal vectors \( N(t) \), and the binormal vectors \( B(t) \).
- Be able to evaluate indefinite and definite integrals of vector-valued functions as well as solve vector initial-value problems.
- Be able to calculate the arc length of a smooth curve between two moments in time. Also, be able to find a parameterization of the curve in terms of arc length (i.e., in terms of the distance travelled along the curve).

PRACTICE PROBLEMS:

1. Consider the curve \( C : \mathbf{r}(t) = (-5 + t, -4 + 2t) \), shown below.

(a) Sketch the following position vectors: \( \mathbf{r}(-1), \mathbf{r}(0), \mathbf{r}(1), \mathbf{r}(2), \) and \( \mathbf{r}(3) \).
(b) Indicate the orientation of the curve (i.e., the direction or increasing $t$).

2. Sketch the following vector valued functions. Also, describe the curve in words.

   (a) $\vec{r}(t) = \langle 4 \cos t, 4 \sin t, 5 \rangle$, $0 \leq t \leq 4\pi$
   
   The curve is a circle in the $z = 5$ plane which has a radius of 4 and a center at $(0, 0, 5)$, traversed twice counterclockwise.

   (b) $\vec{r}(t) = \langle 4 \cos t, 4 \sin t, t \rangle$, $0 \leq t \leq 4\pi$.
   
   The curve is a helix on the cylinder $x^2 + y^2 = 16$ which climbs from the point $(4, 0, 0)$ to the point $(4, 0, 4\pi)$

3. Consider $\mathbf{r}(t) = \langle t, t^2 \rangle$

   (a) Sketch $\mathbf{r}(t)$ and indicate the direction of increasing $t$.
   (b) On your sketch, draw $\mathbf{r}(1)$ and $\mathbf{r}'(1)$. 
4. Consider \( \mathbf{r}(t) = \langle 3 \cos t, 2 \sin t \rangle \)

(a) Sketch \( \mathbf{r}(t) \) and indicate the direction of increasing \( t \).

(b) On your sketch, draw \( \mathbf{r}(\pi) \) and \( \mathbf{r}'(\pi) \).

5. For each of the following, find an equation of the line which is tangent to the given curve at the indicated point.

(a) \( \mathbf{r}(t) = \langle \ln t, 2\sqrt{t}, t^2 \rangle \) at \( (x, y, z) = (0, 2, 1) \)

\[
\mathbf{\ell}(t) = \langle 0, 2, 1 \rangle + t\langle 1, 1, 2 \rangle
\]

(b) \( \mathbf{r}(t) = \langle \sin t, \cos t, \tan t \rangle \) when \( t = \pi \)

\[
\mathbf{\ell}(t) = \langle 0, -1, 0 \rangle + t\langle -1, 0, 1 \rangle
\]

6. Find all points on the curve \( \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \) where its tangent line is parallel to the vector \( 2\mathbf{i} + 8\mathbf{j} + 24\mathbf{k} \).

The tangent line will be parallel to the given vector when \( t = 2 \) which corresponds to the point \( (x, y, z) = (2, 4, 8) \)
7. The following vector valued functions describe the paths of two bugs flying in space.

\[ \mathbf{r}_1(t) = \langle t^2, 2t + 3, t^2 \rangle \]
\[ \mathbf{r}_2(t) = \langle 5t - 6, t^2, 9 \rangle \]

At some moment in time, the two bugs collide.

(a) Determine the moment in time when the bugs collide as well as the location in space where the bugs collide.

The bugs intersect when \( t = 3 \). This corresponds to the point \((x, y, z) = (9, 9, 9)\).

(b) What is the angle between their paths at the point of collision? (Hint: The angle between the paths is the angle between the tangent vectors.)

\[ \cos^{-1} \left( \frac{42}{\sqrt{76} \sqrt{61}} \right) \]

8. Consider the helix \( \mathbf{r}(t) = \langle 2 \cos t, t, 2 \sin t \rangle \).

(a) Compute \( \mathbf{T}, \mathbf{N}, \) and \( \mathbf{B} \) when \( t = \pi \).

\[ \mathbf{T}(\pi) = \left\langle 0, \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle; \quad \mathbf{N}(\pi) = \langle 1, 0, 0 \rangle; \quad \mathbf{B}(\pi) = \left\langle 0, -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle \]

(b) **Definition:** The plane determined by the unit normal and binormal vectors \( \mathbf{N} \) and \( \mathbf{B} \) at a point \( P \) on a curve \( C \) is called the **normal plane** of \( C \) at \( P \). It consists of all lines that are orthogonal to the tangent vector \( \mathbf{T} \).

Compute an equation of the normal plane of the helix described above at the point which corresponds to \( t = \pi \).

\[ y - 2z = \pi \]

(c) **Definition:** The plane determined by the unit tangent and normal vectors \( \mathbf{T} \) and \( \mathbf{N} \) at a point \( P \) on a curve \( C \) is called the **osculating plane** of \( C \) at \( P \). From the latin “Osculum,” meaning to kiss, this is the plane that comes closest to containing the part of the curve near \( P \).

Compute an equation of the osculating plane of the helix described above at the point which corresponds to \( t = \pi \).

\[ 2y + z = 2\pi \]
9. Solve the following vector initial value problems:

\[
\begin{align*}
\frac{dr}{dt} &= e^{-t}i + 3t^2j \\
r(0) &= 2i - 8j
\end{align*}
\]

\[r(t) = (-e^{-t} + 3, t^3 - 8)\]

10. A particle moves through 3-space in such a way that its velocity is \(\mathbf{v}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}\). If the particle’s initial position at time \(t = 0\) is \((1, 2, 3)\), what is the particle’s position when \(t = 1\)? (Hint: set up an initial value problem.)

The position of the particle at time \(t = 1\) is \((x, y, z) = \left(\frac{3}{2}, \frac{7}{3}, \frac{13}{4}\right)\).

11. Find the arc length of the curve described parametrically by \(r(t) = \langle \cos t, \sin t, \ln(\cos t) \rangle\) from \(t = 0\) to \(t = \frac{\pi}{4}\).

\[\ln\left(1 + \sqrt{2}\right)\]

12. Find the arc length parameterization of the curve described parametrically by \(r(t) = \langle e^t \cos t, e^t \sin t \rangle\) measured from \(t_0 = 0\).

\[r(s) = \langle \left(\frac{s}{\sqrt{2}} + 1\right) \cos \left(\ln \left(\frac{s}{\sqrt{2}} + 1\right)\right), \left(\frac{s}{\sqrt{2}} + 1\right) \sin \left(\ln \left(\frac{s}{\sqrt{2}} + 1\right)\right) \rangle\]