The generating function of simultaneous $s/t$-cores

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Abstract

Previous work on partitions simultaneously $s$-core and $t$-core has enumerated such partitions broadly: the number of them overall is $\frac{1}{s+t} \binom{s+t}{s}$, and the largest is $\frac{(s^2-1)(t^2-1)}{24}$. However, missing from the literature is a description of the generating function that would count them. We can produce one, following part of Jaclyn Anderson’s original proof of the overall number of $s/t$-cores. For general $s$ and $t$, the function appears combinatorially unwieldy; the situation improves in some special cases of interest.
We will use the fourth quadrant for partitions and identify the unit square by its lower right corner at \((i, -j)\).

The hooklengths \(\{h_{ij}\}\) of a partition \((\lambda_1, \ldots, \lambda_r)\) are the values \(\lambda_i - j + \lambda'_j - i + 1\).

A partition is \(s\)-core if its Ferrers diagram contains no hook of size \(s\).
The generating function that counts $s$-cores by weight is

$$\prod_{j \geq 1} \frac{(1 - q^{js})^s}{1 - q^j} = \frac{(q^s; q^s)_\infty}{(q; q)_\infty}.$$ 

Core partitions are of interest in areas including enumerative combinatorics and representation theory.
A useful description of partitions for discussing s-cores is their abacus. This represents their profile, the set of bottom and right borders of squares in the partition. We adopt the convention that the first horizontal step is numbered 0. If convenient, we can preface the profile with an indefinitely long string of vertical steps, and terminate it with an indefinitely long string of horizontal steps. The hook lengths of the partition are the set of differences between vertical steps and lower horizontal steps.
Representing horizontal steps as white spacers and vertical steps as black beads, we list the steps of the profile on a runner which we can wrap to a desired length. Here, we have wrapped to length 5, so hooks of length 5 would appear anywhere a spacer has a bead below it. We see several, so this partition is not a 5-core. (In fact, it is not s-core for any s at or below 11, its largest hook.)
Thus the abaci of $s$-cores, wrapped on runners of length $s$, consist of uninterrupted columns of beads, with no beads in the 0 column.

Any such abacus gives an $s$-core, and all $s$-cores are given by these abaci.
To consider partitions that are both $s$-core and $t$-core for $s$ and $t$ coprime, $s < t$, we use the $t$-abacus. Let’s observe the possible positions of beads for such a partition, using $5/7$-cores as an example.

There are 7 columns to begin with. As with any core, no beads can be entered into the first column. Let’s denote such forbidden spaces with ×.
To consider partitions that are both $s$-core and $t$-core for $s$ and $t$ coprime, $s < t$, we use the $t$-abacus. Let’s observe the possible positions of beads for such a partition, using 5/7-cores as an example.

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\times & - & - & - & - & - & - \\
\times & - & - & - & - & - & - \\
\times & - & - & - & - & - & - \\
\end{array}
\]

We also can’t follow any spacer with a bead 5 places later; for instance, the spacer at 0 can’t be followed with a bead in place 5.
To consider partitions that are both $s$-core and $t$-core for $s$ and $t$ coprime, $s < t$, we use the $t$-abacus. Let’s observe the possible positions of beads for such a partition, using $5/7$-cores as an example.

Then the places above this spacer can’t contain beads.
$s/t$-cores

To consider partitions that are both $s$-core and $t$-core for $s$ and $t$ coprime, $s < t$, we use the $t$-abacus. Let’s observe the possible positions of beads for such a partition, using 5/7-cores as an example.

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× − − × − × −
× − − − − × −
× − − − − × −
× − − − − × −
× − − − − × −

Five places later than place 5 is place 10, or 7+3; the third entry in the second row.
To consider partitions that are both \(s\)-core and \(t\)-core for \(s\) and \(t\) coprime, \(s < t\), we use the \(t\)-abacus. Let’s observe the possible positions of beads for such a partition, using 5/7-cores as an example.

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\times & - - \times - \times - \\
\times & - - \times - \times -
\end{align*}
\]

Then the places above this...
To consider partitions that are both $s$-core and $t$-core for $s$ and $t$ coprime, $s < t$, we use the $t$-abacus. Let’s observe the possible positions of beads for such a partition, using 5/7-cores as an example.

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\begin{align*}
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\end{align*}
\]

The column 5 places after these...
To consider partitions that are both $s$-core and $t$-core for $s$ and $t$ coprime, $s < t$, we use the $t$-abacus. Let’s observe the possible positions of beads for such a partition, using $5/7$-cores as an example.

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\times & \times & - & \times & - & \times & \times \\
\end{array}
\]

And then this column – starting in the same row!
To consider partitions that are both $s$-core and $t$-core for $s$ and $t$ coprime, $s < t$, we use the $t$-abacus. Let’s observe the possible positions of beads for such a partition, using $5/7$-cores as an example.

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\begin{array}{cccccccccc}
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\end{array}
\]

Five more...
To consider partitions that are both $s$-core and $t$-core for $s$ and $t$ coprime, $s < t$, we use the $t$-abacus. Let’s observe the possible positions of beads for such a partition, using 5/7-cores as an example.

\[
\begin{array}{cccccccc}
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\]

And finally this. 5 has generated the additive group mod 7. The line in the lattice with a slope of 5/7 forms the bottom of the profile. (This also gives the basic form of Anderson’s proof that the number of such cores is finite.)
It’s possible to fill every allowed space with beads. In fact, it’s not too difficult to show that this partition gives the $s/t$-core of largest size calculated by Olsson and Stanton, $\frac{(s^2-1)(t^2-1)}{24}$:

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On the other hand, not every filling with a 7-core gives a 5-core:

\[
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\times & \times & \bullet & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times \\
\end{array}
\]

The condition of being a 7-core requires that columns are uninterrupted. The condition of being a 5-core requires uninterrupted beads along the line of slope 5/7 in this abacus. In other words, if we arrange the columns in decreasing order of algebraic order (and thus size), and vertically justify the removed beads, the removed beads must form a partition.
The previous partition gives this pattern.
On the other hand, this is a 5/7-core.

Partitions strictly above the line from \((0, -s)\) to \((t, 0)\) are exactly the lattice paths that Anderson counts in [1] to enumerate \(s/t\)-cores. Here, having interpreted them as patterns of removal of beads, we can determine the contribution to the size of the partition made by removing each bead, and thereby produce a generating function.
Changing a bead to a spacer removes the part the bead denoted, and adds 1 to all larger parts. The part removed is of size equal to the number of spacers below this position in the abacus, and the number of parts to which 1 is added is the number of beads above.
Of course, once we remove multiple beads, we have to worry about not increasing the size of removed parts. But we can handle this separately. So what change in the size of the partition is caused by a given removal?
Start with the largest partition, which we know has size \( \frac{(s^2-1)(t^2-1)}{24} \). Begin removing parts with the largest part. There are no higher beads, so we only need to know what size of part is removed.
J. J. Sylvester told us this: the last number which is not a linear combination of a positive number of coprime $s$ and $t$ is $st - s - t + 1$. Furthermore, the semigroup generated by $s$ and $t$ is symmetric with its complement in the interval $[0, st - s - t + 1]$, so exactly half the places below are beads. Thus the largest part is $\frac{(s-1)(t-1)}{2}$. 
Now suppose we remove a lower bead. If we move backwards along the abacus $k$ spaces, then the size of the part removed is decreased by the number of spacers passed, while the number of parts with size increased will grow by the number of beads passed, include the bead from which we started.
Thus if we remove a bead one row above in a column in the $t$-abacus, the net change in amount lost is $-t$, while removing a bead one column earlier along the same $\frac{s}{t}$-sloped line (that is, horizontal in the lattice we constructed) is $-s$. 
Finally, suppose we remove more than one bead. Then the number of parts with increased size above a given bead is reduced by the number of beads in higher places already removed. Unfortunately, this is not a static trait for a given bead, but if we remove $n$ beads, then the sum of such losses will be $\binom{n}{2}$. 
Thus if we remove the bead at position \((i, -j)\) in the lattice below the \(s/t\) line, neglecting the effect of previously-removed parts, the total weight change is

\[
- \frac{(s - 1)(t - 1)}{2} + s(i - 1) + t(j - 1) .
\]
The quantities $i - 1$ and $j - 1$ are the co-leg and co-arm of the position respectively; call their sum the co-hook, and the weighted value $s(i - 1) + t(j - 1)$ the $s, t$-weighted co-hook. For a partition $\lambda$, denote the sum of all $s, t$-weighted co-hooks in the partition by

$$CH_{s,t}(\lambda) = s \sum_{\lambda_i} \binom{\lambda_i}{2} + t \sum_{\lambda'_i} \binom{\lambda'_i}{2}$$
The $s/t$-core generating function

\[ \phi_{s,t} = \]

So what happens when we remove a partition $\lambda \vdash n$ in beads?
The $s/t$-core generating function

$$\phi_{s,t} = q^{\frac{(s^2-1)(t^2-1)}{24}}$$

The full abacus starts by representing $\frac{(s^2-1)(t^2-1)}{24}$. 
The $s/t$-core generating function

$$\phi_{s,t} = q^{\frac{(s^2-1)(t^2-1)}{24}} \sum_{n=0}^{\frac{(s-1)(t-1)}{2}}$$

$n$ can be somewhere between 0 and $\frac{(s-1)(t-1)}{2}$. 
The $s/t$-core generating function

$$\phi_{s,t} = q^{(s^2-1)(t^2-1)/24} \sum_{n=0}^{2} \sum_{\lambda \triangleleft \pi \atop \lambda \vdash n} \frac{(s-1)(t-1)}{24}$$

Each partition of these $n$ that fits under the $s/t$ line yields an $s/t$-core.
The basic amount removed by removing a bead is $\frac{(s-1)(t-1)}{2}$.
The $s/t$-core generating function

\[ \phi_{s,t} = q \frac{(s^2-1)(t^2-1)}{24} \sum_{n=0}^{2} q^{-n\left(\frac{(s-1)(t-1)}{2}\right) - \binom{n}{2}} \sum_{\lambda \vdash \pi, \lambda \vdash n} \]

The amount removed is increased by a total of $\binom{n}{2}$ as repeated parts are removed.
The $s/t$-core generating function

$$
\phi_{s,t} = q^{\frac{(s^2-1)(t^2-1)}{24}} \sum_{n=0}^{2} q^{-n\left(\frac{(s-1)(t-1)}{2}\right)} - \binom{n}{2} \sum_{\lambda \vdash n} q^{CH_{s,t}(\lambda)}.
$$

And beads at various positions decrease the amount removed by values which total the $s$, $t$-weighted co-hooks at those positions.
Before delving into some specializations that might be of interest, I’d like to focus on some tangential notions related to this factor:

$$\sum_{\lambda \in \pi} q^{CH_{s,t}(\lambda)}$$
One way to describe this sum is as generating function for Ferrers diagrams of partitions where boxes are weighted by $s$ and $t$:

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The weight associated to a Ferrers diagram is the total of the entries of the boxes. For instance, the diagram in the marked squares below would have a weight of \(5s + 6t\).

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The generating function for standard partitions would weight Ferrers diagrams with a 1 in each box:

The entries in this lattice are constant.
Partitions with two colors would have a periodic labeling as one reads up columns:

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This partition is still (3, 2, 2), but the two 2s have different colors (the latter 2 has the same color as the 3).
As a description of partitions, the framework seems quite adaptable. Partitions into odd parts:

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Could we find a weight-preserving map between the two lattices? (It would not preserve the number of boxes; diagrams with three boxes are weighted 3, 5, and 9 in one, and 3, 5, and 6 in the other.)
The lattice we started with is the simplest planar lattice in which the two directions have independent slopes. What if they were the same? It seems like finding a generating function for this lattice should be an easier problem to tackle.

An algorithmic approach to any "nice" class of such lattices could give a unified way of approaching many partition identities.
Specializations

\[
\phi_{s,t} = q^{\frac{(s^2-1)(t^2-1)}{24}} \sum_{n=0}^{2} q^{-n\left(\frac{(s-1)(t-1)}{2}\right)} \binom{n}{2} \sum_{\lambda \vdash n} q^{CH_{s,t}(\lambda)}.
\]

As generating functions go, this one is pretty unwieldy. Having one (fairly complicated) term per object being counted means it doesn’t make an efficient counting function. However, we might hope that it has structural properties that could be of use in proving facts about \(s/t\)-cores, and in some cases of interest it does specialize.
On one end of the spectrum, we can restrict $s$ to be very small. The generating function for $2/t$-cores is trivial (it is $\sum_{1 \leq i \leq (t+1)/2} q^{\binom{i}{2}}$ since the 2-cores are the staircase partitions), so $3/t$-cores are the first nontrivial case.
Specializations

The $3/t$-abacus will consist of one long and one short row, the short of length $\lfloor \frac{t}{3} \rfloor$, and the longer of length $2\lfloor \frac{t}{3} \rfloor + \epsilon$, where $\epsilon = 0$ if $t \equiv 1 \mod 3$ and 1 if $t \equiv 2 \mod 3$.

Thus the partition of removed beads will consist of $i$ parts of size 1, and $\frac{n-i}{2}$ parts of size 2.
This gives us

\[ \phi_{3,t} = q^{t^2 - 1 \over 3} \sum_{n=0}^{t-1} q^{-{n \choose 2} + (t-1)n} \sum_{0 \leq i \leq \min(n,2\lfloor {t \over 3} \rfloor + \epsilon) \atop i \equiv n(2)} q^{t \left( {n-i \over 2} \right) + 3 \left( \left( {n-i \over 2} \right) + \left( {n+i \over 2} \right) \right)} \]

which, with a little algebra, can be reduced to

\[ \phi_{3,t} = q^{t^2 - 1 \over 3} \sum_{(n, i) \atop 0 \leq n \leq t - 1 \atop M(n) \leq i \leq N(n) \atop i \equiv n(2)} q^{-t \frac{n+i}{2} + \frac{n^2 + 3i^2}{4}} \]

where \( M(n) = \max(0, n - 2\lfloor {t \over 3} \rfloor) \), \( N(n) = \min(n, 4\lfloor {t \over 3} \rfloor + 2\epsilon - n) \).
This can be pretty rapidly calculated. Here are the coefficients (from $q^1$ to $q^{833}$) of the generating function for 3/50-cores:

Looks like a significant amount of structure.
And here are the coefficients (from $q^1$ to $q^{83333}$) for 3/500-cores:

Looks like similar structure! Can we write down an algorithm based on $t$ and $n$ to determine this coefficient directly?
Another set of simultaneous $S$-cores, that has arisen in the work of Lam, Shimozono, LaPointe, and Morse are the $t/t + 1$-cores. These have the very nice property that the rearranged columns of the $t + 1$ abacus are a staircase partition of base $t$:  

\[
\begin{array}{cccccccc}
\times & \bullet & \bullet & \bullet & \bullet & \bullet & \times \\
\times & \bullet & \bullet & \bullet & \bullet & \times & \times \\
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\end{array}
\]
This means that we can write down a recurrence for
\[ \sum_{\lambda \triangleleft \pi} q^{CH_{t,t+1}(\lambda)} \]. Call this \( \rho_t(n) \). The partitions contributing to \( \rho_t(n) \) are either contained in the \( t - 1/t \) triangle one size smaller, or have a largest part which touches the diagonal at part \( t - k \) for some \( k \):

\[
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\bullet & 
\end{array}
\]
The largest part touching the diagonal contributes $k$ parts of size $t - k$; parts above these cannot touch the diagonal. The $k \times (t - k)$ rectangle contributes a fixed amount to the sizes of co-hooks in each of the two triangles. Working it out, we get

$$\rho_t(n) = \rho_{t-1}^*(n) + q^{t(t-1)+2(t+1)(n-(t-1))} \rho_{t-1}(n - (t - 1))$$
$$+ q^{2t(t-2)+2(t+1)(2(n-2(t-2))+(t-2))} \rho_{t-2}(n - 2(t - 2))$$
$$+ \sum_{k=3}^{t-1} q^{kt(t-k)+2(t+1)(k(n-k(t-k))+(t-k))}$$
$$\sum_{c \leq \binom{k-1}{2}} q^{c(t^2-2kt-k)} \rho_{k-1}(c) \rho_{t-k}(n - k(t - k) - c) .$$

The asterisk denotes a $\rho$ with $t$ augmented; the recursion is thus doubled, which makes it somewhat cumbersome to use.
What about extending this more generally, to partitions $s$-core for all $s \in S$ for some set $S$ such that $gcd(S) = 1$? We know that this set must be finite. Unfortunately, just calculating the largest possible hook – the Frobenius number of this set – is a hard problem.

For instance, the average value of the Frobenius number for a semigroup of three generators $a$, $b$ and $c$ is $\frac{8}{\pi} \sqrt{abc}$, but explicit calculation is difficult.
Since the $t/t + 1$ cores are of independent interest, it might be useful to press the calculations further in this direction, and hope that some generatingfunctionology yields up a more compact form of the polynomial.

I am curious as to whether congruences might exist for $3/t$-cores and $4/t$-cores in some sort of localization for properties known to hold for 3-cores and 4-cores.
Thank you!


Tripathi, A. On the largest size of a partition that is both $s$-core and $t$-core. J. Num. Thy. 129 (2009) 1805-1811