3.2 Properties of Determinants

**Theorem.** Let $A$ be a square matrix.

1. If a multiple of one row of $A$ is added to another row of $A$ to produce a new matrix $B$, then $\det B = \det A$.

2. If two rows of $A$ are interchanged to produce $B$, then $\det B = -\det A$.

3. If one row of $A$ is multiplied by $k$ to produce $B$, then $\det B = k \det A$. 
\[
\begin{array}{ccc}
a & b & c \\
d & e & f \\
g & h & i \\
\end{array}
\begin{array}{ccc}
2a & 2b & 2c \\
3d & 3e & 3f \\
3g + 2a & 3h + 2b & 3i + 2c \\
\end{array}
\]

If \( \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} = 2 \), what is \( \begin{array}{ccc} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 3g + 2a & 3h + 2b & 3i + 2c \end{array} \)?

1. 12
2. 36
3. 72
**Example.** Compute \[
\begin{array}{ccc}
2 & 4 & 6 \\
5 & 6 & 7 \\
7 & 6 & 10
\end{array}
\] using a combination of row reduction and cofactor expansion.

**Solution.**
Suppose $A$ has been reduced to $U = \begin{bmatrix} u_{11} & * & * & \cdots & * \\ 0 & u_{22} & * & \cdots & * \\ 0 & 0 & u_{33} & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & u_{nn} \end{bmatrix}$ by row replacements and interchanges.

Then

$$\det A = \begin{cases} , & \text{if } A \text{ is invertible} \\ , & \text{if } A \text{ not invertible.} \end{cases}$$
Theorem. A square matrix $A$ is invertible if and only if $\det A \neq 0$. 
More Properties of the Determinant:

Theorem. \( \det A = \det A^T \)

Theorem. (Multiplicative Property)
\[ \det(AB) = \det A \det B \]
More Properties of the Determinant:

Theorem.  \( \det A = \det A^T \)

Theorem. (Multiplicative Property)

\[ \det(AB) = (\det A)(\det B) \]

If \( A \) is square, what is \( \det A^k \) in terms of \( \det A \)?
If $A$ is invertible, what is $\det A^{-1}$ in terms of $\det A$?
Example. If $B^2 = I$, what are the possible values of $\det B$?
Example. Using determinants, show that if $AB$ is invertible, then so are $A$ and $B$. (Assume all matrices are square.)
Example. Compute $\det B^4$ if

$$B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
3 & 2 & 0 & 0 \\
9 & 7 & -1 & 0 \\
8 & -2 & 1 & -2
\end{bmatrix}.$$
For any square matrix $A$, \[ \det A^T A \geq 0. \]

1. True
2. False
3.3 Cramer’s Rule

Notation:
If $A, n \times n$ and $b$ in $\mathbb{R}^n$:

$$A_i(b) = [a_1 \cdots a_{i-1} \ b \ a_{i+1} \cdots a_n]$$

Theorem. (Cramer’s Rule)
Let $A$ be an invertible $n \times n$ matrix. Then for $b$ in $\mathbb{R}^n$, the unique solution $x$ to $Ax = b$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}, \quad i = 1, \cdots, n.$$
Proof.
Example. Use Cramer’s rule to solve:

\[ -3x_1 + 5x_2 = 9 \]
\[ 7x_1 - 4x_2 = 2 \]
Using the Laplace transform to solve an initial value problem:

**Example.** Consider the following system of D.E.’s

\[
\begin{align*}
\frac{dx}{dt} &= 5x - 4y \\
\frac{dy}{dt} &= -2x + 3y
\end{align*}
\]

with initial conditions \( x(0) = 1, \) \( y(0) = 0 \).

**Recall:** The Laplace transform of \( f(t) \) is:

\[
\mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st} \, dt
\]

and \( \mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - f^{n-1}(0) \).
Procedure:
1. Set $\mathcal{L}\{x(t)\} = X(s)$, $\mathcal{L}\{y(t)\} = Y(s)$ and take the transforms of the two differential equations.
2. Get algebraic equations in $X(s), Y(s)$.
3. Solve for $X(s), Y(s)$ using matrix methods.
4. Find $x(t), y(t)$ using inverse transforms.
A Formula for $A^{-1}$

Let $x$ be the $j$th column of $A^{-1}$. Then $Ax = e_j$.

By Cramer’s rule,
A Formula for $A^{-1}$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix}
C_{11} & C_{21} & \cdots & C_{n1} \\
C_{12} & C_{22} & \cdots & C_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
C_{1n} & C_{2n} & \cdots & C_{nn}
\end{bmatrix}$$

where $C_{ij} = (i, j)$-cofactor of $A$. 
Graphs and Adjacency Matrices:
(see MATLAB Project # 2)

A graph is a set of points (called vertices or nodes) and a set of lines (called edges or paths of length one) connecting some pairs of nodes.

Two nodes connected by an edge are said to be adjacent.

The **Adjacency Matrix** for a graph with \( n \) nodes is an \( n \times n \) matrix \( A = (a_{ij}) \), where

\[
a_{ij} = \begin{cases} 
1 & \text{, if } i \text{ and } j \text{ are connected by an edge} \\
0 & \text{, otherwise.}
\end{cases}
\]
How many paths of length $k$ are there from node $i$ to node $j$?