2.3 Characterization of Invertible Matrices

Theorem. (The Invertible Matrix Theorem)

Let $A$ be an $n \times n$ matrix. Then the following are equivalent:

a. $A$ is an invertible matrix.

b. $A$ is row equivalent to $I_n$.

c. $A$ has $n$ pivot positions.

d. The equation $Ax = 0$ has only the trivial solution.

e. The columns of $A$ are linearly independent.

f. The equation $Ax = b$ has at least one solution for each $b$ in $\mathbb{R}^n$.

g. The columns of $A$ span $\mathbb{R}^n$. 
2.3 Characterization of Invertible Matrices

Theorem. (The Invertible Matrix Theorem) (cont’d)

h. There is an $n \times n$ matrix $C$ such that $CA = I_n$.
i. There is an $n \times n$ matrix $D$ such that $AD = I_n$.
j. $A^T$ is an invertible matrix.
Suppose that $A$ and $B$ are square matrices. If $AB$ is invertible, then $A$ and $B$ are both invertible.

1. True
2. False
Example. Use the Invertible Matrix Theorem to determine if $A$ is invertible. Use as few calculations as possible.

(1) $A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix}$
Example. (cont’d)

(2) \[ A = \begin{bmatrix} 
2 & 1 & 5 \\
0 & 3 & 9 \\
0 & 0 & 4 
\end{bmatrix} \]

(3) \[ A = \begin{bmatrix} 
3 & 0 & 0 \\
6 & 1 & 0 \\
0 & 3 & 2 
\end{bmatrix} \]
Example. (cont’d)

\[
(2) \quad A = \begin{bmatrix}
2 & 1 & 5 \\
0 & 3 & 9 \\
0 & 0 & 4
\end{bmatrix}
\]

\(A\) is an example of an *upper triangular* matrix.

\[
(3) \quad A = \begin{bmatrix}
3 & 0 & 0 \\
6 & 1 & 0 \\
0 & 3 & 2
\end{bmatrix}
\]

\(A\) is an example of a *lower triangular* matrix.
Example. Suppose $H$ is a $5 \times 5$ matrix and suppose there is a vector $v$ in $\mathbb{R}^5$ which is not a linear combination of the columns of $H$. What can you say about the number of solutions to $Hx = 0$?

Solution.
Suppose that $A$ is an $80 \times 80$ matrix and $A$ has 75 pivots. Then the columns of $A^T$ ________________.

1. are linearly independent.
2. are linearly dependent.
3. contain the zero vector.
Invertible Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix $A$ (that is, $T(x) = Ax$, $x$ in $\mathbb{R}^n$). Then $T$ is called an invertible transformation if its standard matrix $A$ is invertible. Its inverse transformation is given by

$$T^{-1} : \mathbb{R}^m \to \mathbb{R}^n,$$

where

$$T^{-1}(x) = A^{-1}x, \quad x \text{ in } \mathbb{R}^m.$$
2.4 Partitioned Matrices

Example.

\[
A = \begin{bmatrix}
1 & 0 & 2 & 5 & 4 \\
6 & -3 & 4 & 0 & 1 \\
1 & 2 & 6 & -3 & 2 \\
0 & 4 & 0 & 1 & 3 \\
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22} \\
\end{bmatrix},
\]

\[
A_{11} = \begin{bmatrix}
1 & 0 \\
6 & -3 \\
\end{bmatrix}, \quad A_{12} = \begin{bmatrix}
2 & 5 & 4 \\
4 & 0 & 1 \\
\end{bmatrix},
\]

\[
A_{21} = \begin{bmatrix}
1 & 2 \\
0 & 4 \\
\end{bmatrix}, \quad A_{22} = \begin{bmatrix}
6 & -3 & 2 \\
0 & 1 & 3 \\
\end{bmatrix}.
\]
2.4 Partitioned Matrices

Example.

\[ A = \begin{bmatrix}
1 & 0 & 2 & 5 & 4 \\
6 & -3 & 4 & 0 & 1 \\
1 & 2 & 6 & -3 & 2 \\
0 & 4 & 0 & 1 & 3
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & 5 & 4 \\
-3 & 0 & 2 \\
0 & 4 & -1 \\
1 & 3 & 5 \\
2 & -1 & 2
\end{bmatrix} \]

Let \( C = AB \). Partition \( C \) as

\[ C = \begin{bmatrix}
\begin{array}{c|c}
C_{11} & C_{12} \\
\hline
C_{21} & C_{22}
\end{array}
\end{bmatrix}, \quad \text{where } C_{11} \text{ is } 2 \times 2. \]
Let \( A = \begin{bmatrix} \frac{A_{11}}{A_{21}} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{B_{11}}{B_{21}} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \)

Then \( C = AB = \begin{bmatrix} \frac{C_{11}}{C_{21}} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \)

where
\[
\begin{align*}
C_{11} &= \\
C_{12} &= \\
C_{21} &= \\
C_{22} &= \\
\end{align*}
\]
Example (Cont’d)

\[
A = \begin{bmatrix}
1 & 0 & 2 & 5 & 4 \\
6 & -3 & 4 & 0 & 1 \\
1 & 2 & 6 & -3 & 2 \\
0 & 4 & 0 & 1 & 3 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & 5 & 4 \\
-3 & 0 & 2 \\
0 & 4 & -1 \\
1 & 3 & 5 \\
2 & -1 & 2 \\
\end{bmatrix}
\]

\[
C = AB = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22} \\
\end{bmatrix}
\]
Example. Let \( A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \), where \( B, C \) are invertible (square) matrices. Is \( A \) invertible? If so, what is its inverse?

Solution.
Example. Problem 6 from Review Sheet.
2.5 Matrix Factorizations

Lower Triangular matrix:  \( L = \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix} \)

Upper Triangular matrix:  \( U = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \)

LU-Factorization of an \( m \times n \) matrix \( A \):

\[
A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}
\]

\( m \times m \) \( m \times n \)
Why an LU-Factorization is useful:

Problem: (*) Solve $Ax = b$ for $x$.

If $A = LU$, can break up (*) into two “simpler” problems:
Why an LU-Factorization is useful:

**Problem:** \((\ast)\) Solve \(Ax = b\) for \(x\).

If \(A = LU\), can break up \((\ast)\) into two “simpler” problems:

1. Solve \(Ly = b\) for \(y\).
2. Solve \(Ux = y\) for \(x\).
Algorithm for finding an LU-factorization:
1. Reduce $A$ to an echelon form $U$ (upper triangular) by a sequence of row replacement operations, if possible.

$$(E_p \cdots E_2 E_1) A = U.$$
Algorithm for finding an LU-factorization:

1. Reduce $A$ to an echelon form $U$ (upper triangular) by a sequence of row replacement operations, if possible.

\[
(E_p \cdots E_2 E_1) A = U.
\]

2. $L = E_1^{-1} E_2^{-1} \cdots E_p^{-1}$
Example. Find an LU-factorization of

\[ A = \begin{bmatrix}
-5 & 3 & 4 \\
10 & -8 & -9 \\
15 & 1 & 2
\end{bmatrix}. \]

Solution.