1. Let $A$ be an $(n \times n)$ matrix and a subspace $E \subset \mathbb{R}^n$ be invariant with respect to $T(x) = Ax$ (i.e. $T(x) \in E$ whenever $x \in E$).
   (a) Show that $E$ is invariant with respect to $e^T$.
   (b) Let $x(t)$ denote a solution of the initial value problem
       \[ \dot{x} = Ax, \, x(0) = x_0. \]
       Show that $x(t) \in E$ if $x_0 \in E$.

2. Find the general solution and draw the phase portrait for
   \( a \) \[ \begin{align*}
   \dot{x}_1 &= x_1, \\
   \dot{x}_2 &= -x_1 + 2x_2,
   \end{align*} \]
   \( b \) \[ \begin{align*}
   \dot{x}_1 &= 3x_1 - 2x_2, \\
   \dot{x}_2 &= x_1 + x_2.
   \end{align*} \]

3. Use the variation of constant formula to solve
   \( a \) \[ \begin{align*}
   \dot{x} &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} t \\ 1 \end{pmatrix},
   \end{align*} \]
   \( b \) \[ \begin{align*}
   \dot{x} &= \begin{pmatrix} -2 \cos^2 t & -1 - \sin 2t \\ 1 - \sin 2t & -2 \sin^2 t \end{pmatrix} x + \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix},
   \end{align*} \]
   for $x(0) = (1, 0)^T$.
   For (b) first check that
   \[ X(t) = \begin{pmatrix} e^{-2t} \cos t & -\sin t \\ e^{-2t} \sin t & \cos t \end{pmatrix} \]
   is a fundamental matrix solution of the homogeneous equation and then use it to solve the nonhomogeneous problem above.

4. Consider a second-order ODE
   \[ \ddot{y} + p(t)\dot{y} + q(t)y = f(t) \quad (1) \]
   with $p, q$ and $f$ continuous on $[a, b]$. Let real-valued functions $\phi_1(t)$ and $\phi_2(t)$ be linearly independent solutions of the homogeneous equation on $(a, b)$ (i.e., (1) with $f \equiv 0$)
   \( a \) Rewrite (1) as a system of two first-order ODEs and show that
   \[ \Phi(t) = \begin{pmatrix} \phi_1 & \phi_2 \\ \dot{\phi}_1 & \dot{\phi}_2 \end{pmatrix} \]
   is a fundamental matrix of the associated homogeneous system.
(b) Find a solution \( \psi(t) = (\psi_1, \psi_2)^T \) of the non-homogeneous system such that \( \psi(t_0) = (0, 0)^T \). In particular, show that \( \psi(t) = (\psi_1, \dot{\psi}_1)^T \) and

\[
\dot{\psi}_1(t) = \int_{t_0}^{t} \frac{\phi_2(t)\phi_1(s) - \phi_1(t)\phi_2(s)}{W(s)} f(s) ds,
\]

where \( W(t) = \det \Phi(t) \).

HINT: Use the variation of constants formula and the fact that

\[
A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \quad \text{for} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.
\]

(c) Assume that \( p(t) \) in (1) is continuously differentiable and write an ODE for \( z(t) \), where

\[
y(t) = z(t) \exp \left[ -\frac{1}{2} \int_{t_0}^{t} p(s) ds \right].
\]