1. For a system of ODEs
\[
\begin{align*}
\dot{x} &= -2x + y + x^3, \\
\dot{y} &= -x - 2y + 3x^5,
\end{align*}
\]
(a) Find all equilibria.
(b) Linearize (1) around the equilibria and sketch the phase portraits for the linearized equations.
(c) Determine stability of the fixed points.

2. Consider a 2D system
\[
\begin{align*}
\dot{x} &= -y + f(x, y), \\
\dot{y} &= \sin x,
\end{align*}
\]
where \(f(x, y)\) is a smooth nonlinear function. Give sufficient conditions for \(f(x, y)\) so that \((0, 0)\) is a stable equilibrium of (2).

3. Let \(V : \mathbb{R}^n \rightarrow \mathbb{R}\) is a smooth function. A system of the form
\[
\dot{x} = -\nabla V(x), \quad x \in \mathbb{R}^n, \quad \nabla V(x) = \left( \frac{\partial V(x)}{\partial x_1}, \frac{\partial V(x)}{\partial x_2}, \ldots, \frac{\partial V(x)}{\partial x_n} \right)^T,
\]
is called a gradient system.
(a) Show that if \(x_0\) is an isolated minimum of \(V\) then \(x_0\) is an asymptotically stable equilibrium of (3).
(b) Show that the matrix of a linearized system around any equilibrium of (3) has only real eigenvalues.
(c) Show that (3) can not have periodic orbits.

4. Show that the following systems of ODEs
\[
a) \begin{cases} 
\dot{x} = x, \\
\dot{y} = \frac{1}{3}y,
\end{cases} \quad \text{b) } \begin{cases} 
\dot{x} = x - y, \\
\dot{y} = x + y,
\end{cases}
\]
are topologically equivalent to
\[
\begin{cases} 
\dot{x} = x, \\
\dot{y} = \frac{1}{3}y,
\end{cases}
\]
5. Consider a damped Duffing equation:
\[ \ddot{x} + \delta \dot{x} - x + x^3 = 0, \quad \delta \geq 0. \] (4)

a. Find the equilibria of (4) and determine their stability via linearization.

b. Note that if \( \delta = 0 \) Equation (4) describes a conservative system of one degree of freedom. Find the potential function for this system and use it to plot the phase portrait of (4) for \( \delta = 0 \). On the phase portrait identify the stable and unstable manifolds for a hyperbolic fixed point.

c. Use b. to plot the phase portrait for (4) for small values of \( \delta > 0 \). Identify the stable and unstable manifolds of the equilibria.