Homework 3. Due May 3 in class.

1. Consider a damped Duffing equation:

\[ \ddot{x} + \delta \dot{x} - x + x^3 = 0, \quad \delta \geq 0. \quad (1) \]

a. Find the equilibria of (1) and determine their stability via linearization.

b. Note that if \( \delta = 0 \) Equation (1) describes a conservative system of one degree of freedom. Find the potential function for this system and use it to plot the phase portrait of (1) for \( \delta = 0 \). On the phase portrait identify the stable and unstable manifolds for a hyperbolic fixed point.

c. Use b. to plot the phase portrait for (1) for small values of \( \delta > 0 \). Identify the stable and unstable manifolds of the equilibria.

2. Consider

\[
\begin{align*}
\dot{x} &= \alpha x^2 - xy, \\
\dot{y} &= -y + x^2.
\end{align*}
\quad (2)
\]

Use Anzats \( y = ax^2 + bx^3 + \ldots \) to find an approximation for the center manifold of the equilibrium at the origin. Determine stability of the equilibrium for different values of \( \alpha \).

3. Consider

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x - yw, \\
\dot{w} &= -w - x^2.
\end{align*}
\quad (3)
\]

Use Anzats \( w = ax^2 + byx + cy^2 + \ldots \) to find an approximation for the center manifold of the equilibrium at the origin. Write the reduced system on the center manifold. Use Lyapunov function

\[ V(x, y) = \frac{x^2 + y^2}{2} \]

to analyze stability of the equilibrium of the reduced system.