# 19 / 8.9  Find the unit normal to the surface \( z = \sqrt{x^2 + y^2} \) at \( P(6, 8, 10) \).

Let \( F(x, y, z) = \sqrt{x^2 + y^2} - z \). We need to find gradient of \( F(x, y, z) \), which is the normal to the surface \( z = \sqrt{x^2 + y^2} \).

\[
\nabla F = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1 \right) = \mathbf{n}.
\]

Then we find the normal vector at \( (6, 8, 10) \):

\[
\nabla F(6, 8, 10) = \left( \frac{6}{\sqrt{6^2 + 8^2}}, \frac{8}{\sqrt{6^2 + 8^2}}, -1 \right) = \left( \frac{6}{10}, \frac{8}{10}, -1 \right) = \left( \frac{3}{5}, \frac{4}{5}, -1 \right) \text{ ( Normalize).}
\]

# 33 / 8.9  Find the directional derivative of \( f = xyz \) at \( P = (-1, 1, 3) \), \( \mathbf{a} = i - 2j + 2k \)

\[
D_{\mathbf{a}} f = \frac{1}{||\mathbf{a}||} \nabla f \cdot \mathbf{a}
\]

\[
\nabla f = \left( yz, xz, xy \right) \quad \text{and} \quad ||\mathbf{a}|| = \sqrt{1^2 + (-2)^2 + 2^2} = 3
\]

Then:

\[
D_{\mathbf{a}} f = \frac{1}{3} \left( yz, xz, xy \right) \cdot (1, -2, 2) = \frac{1}{3} \left[ yz - 2xz + 2yx \right].
\]
Evaluating \( f \) at \((-1,1,3)\), we get the directional derivative of \( f \) at \( P \) in the direction of \( \mathbf{a} \):

\[
\nabla f = \frac{1}{3} \left( 3 \mathbf{i} + 0 \mathbf{j} - 2 \mathbf{k} \right) = \frac{\mathbf{r}}{3}
\]

8.10 Consider the flow with the velocity vector \( \mathbf{v} = y \mathbf{i} \). Show that this flow is incompressible. Show that the particles that at time \( t = 0 \) are in a cube bounded by the plane \( x = 1, x = 0, y = 1, y = 0, z = 1, z = 0 \) occupy at time \( t = 1 \) the volume !.

To show it is incompressible we check \( \text{div} \, \mathbf{v} \):

\[
\nabla \cdot (y, 0, 0) = \frac{\partial y}{\partial x} = 0 \quad \text{so it is incompressible.}
\]

For part 2: If \( \mathbf{v}(t) = (y, 0, 0) \) then \( \mathbf{r}(t) = (x(t), y(t), z(t)) \), and the position vector \( \mathbf{r}(t) \) can be found by:

\[
\mathbf{r}(t) = (y, 0, 0)
\]

As can be seen, \( z(t) \) is constant and the position \( (x, y, z) \) of a particle changes w.r.t. time as follow:

1. \( z(t) = c_3 \) never changes
2. \( y(t) = c_2 \) constant but the \( y \) coordinate of the particle affects the \( x \)-coordinate
(3) $x$ depends on the $y$ coord. of its location.

Since there is a flow, the shape will deviate from a cube w.r.t. time. We'll find how it changes & what the new shape will be to compute its volume.

Observation: Since $y$ is independent & is constant, the height (z-values) doesn't change, they remain bounded by $z=0$ & $z=1$ planes. So we'll be concerned with the $x$ and $y$ coordinates of the particles at $t=1$.

We'll find the new locations of the bottom corners of the initial cube.

<table>
<thead>
<tr>
<th>Position @ $t=0$</th>
<th>Position @ $t=1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0,1,0)$</td>
<td>$(c_1, c_2, c_3)$</td>
</tr>
<tr>
<td>$(0,0,0)$</td>
<td>$(c_2t+c_1, c_2, c_3)$</td>
</tr>
</tbody>
</table>

So the new locations of the corners are:

Therefore the base is a parallelogram with area = 1 and the height is still 1. So the new volume the particles form is equal to 1.
For the velocity vector given as \( \mathbf{v} = x^3 \mathbf{k} \). Is the flow rotational? Incompressible? Find paths of particles.

(i) We'll check \( \text{curl} \mathbf{v} \) : 
\[
\nabla \times \mathbf{v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & 0 & x^3
\end{vmatrix} = -\hat{j} (3x^2) \neq 0 \quad \text{for } x \neq 0
\]

Hence it is rotational for \( x \neq 0 \).

(ii) We'll check \( \text{div} \mathbf{v} \) : 
\[
\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x} x^3 = 0 \quad \text{incompressible.}
\]

(c) \( \mathbf{v} = (0, 0, x^3) \). Recall: \( \mathbf{v} = r'(t) = (x'(t), y'(t), z'(t)) \) for \( r(t) = (x(t), y(t), z(t)) \).
\[
\begin{align*}
\mathbf{r}_0 : & \quad x'(t) = 0 \quad \Rightarrow \quad x(t) = c_1 \\
y'(t) = 0 \quad \Rightarrow \quad y(t) = c_2 \\
z'(t) = x^3 \quad \Rightarrow \quad z(t) = c_1 x^3 t + c_3
\end{align*}
\]