Show that the differential form under integral sign is exact & evaluate the integral.

\[ \int (y \cdot \sinh xz \, dx + \cosh xz \, dy + xy \cdot \sinh xz \, dz) \]

This differential form is exact if it is the differential of a differentiable function \( f \), that is \( \nabla f \)

\[ df = F_1 \, dx + F_2 \, dy + F_3 \, dz \]

which implies:

the differential form is exact if and only if \( \nabla f = (F_1, F_2, F_3) \)

\[ \frac{\partial f}{\partial x} = F_1, \quad \frac{\partial f}{\partial y} = F_2, \quad \frac{\partial f}{\partial z} = F_3 \]  \( \text{(x)} \)

We'll then find a function \( f \) s.t. \( \text{(x)} \) holds

\[ F_1 = y \cdot \sinh xz, \quad \frac{\partial f}{\partial x} = y \cdot \sinh xz \]  \( \text{(0)} \)

\[ f(x, y, z) = \int y \cdot \sinh xz \, dx = \overbrace{y \cdot \cosh xz + h(y, z)}^z \]

\[ f(x, y, z) = y \cdot \cosh xz + h(y, z) \]

\[ \frac{\partial f}{\partial y} = F_2 \]  \[ \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (\cosh xz + h(y, z)) = \cosh xz + \frac{\partial h}{\partial y} \]

\[ = \frac{\partial h}{\partial y} = 0 \]  \[ \Rightarrow h(y, z) = g(z) \quad \text{then} \]

\[ f(x, y, z) = y \cdot \cosh xz + g(z) \]

\[ \frac{\partial f}{\partial z} = F_3 \]  \[ \frac{\partial f}{\partial z} = \frac{\partial}{\partial z} (y \cdot \cosh xz + g(z)) = xy \cdot \sinh xz + g'(z) \]

\[ g'(z) = 0 \]

Then

\[ f(x, y, z) = y \cdot \cosh xz \]
So \( \int (y \sinhx + dx \coshx + dy + y \sinhx - dz) \)

\[
(0,1,3) = f(41,1) - f(0,1,3) = \cosh 1 - 2 \cosh 0
\]

\( = \cosh 1 - 2 \)

Check for path independence, if it is path independent, integrate from \((0,0,0)\) to \((a,b,c)\).

\[
e^x dx + 2y dy + x e^z dz
\]

Check for exactness:

\[
\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = 0 = 0 \quad \checkmark
\]

\[
\frac{\partial F_3}{\partial z} = 2y = 0 \quad \checkmark
\]

\[
\frac{\partial F_3}{\partial x} = 0 = 0
\]

Therefore, the differential form is exact and the integral will be path independent.

\[
C : \{ (ct) = (0,0,0) + t \cdot (a,b,c) = (at, bt, ct) \}
\]

\[
\int_{(0,0,0)}^{(a,b,c)} e^x dx + 2y dy + x e^z dz = \int_{(ct)}^{(ct, bt, ct + at)} \cdot (a,b,c) \cdot dt
\]

\[
= \int \left( ae + 2bt + act + c e^t \right) dt = \frac{a \cdot e + b^2 t^2 + ac(ce - 1)}{c} e^t
\]

\[
= \frac{a \cdot e + b^2 + a(e - e^c) - (a - a)}{c} (e - e^c)
\]

\[
= \left( \frac{a}{c} + a - a \right) e^c + b^2 = ae^c + b^2
\]
#10 \[ \int_{\frac{\pi}{4}}^{\pi/4} \int_{0}^{\infty} x^2 \sin y \, dx \, dy = \int_{0}^{\pi/4} \left( \frac{x^3}{3} \sin y \right) \, dy \]

\[ = \int_{0}^{\frac{\pi}{4}} \frac{\cos^2 y}{3} \, dy = -\frac{1}{3} \int_{0}^{\frac{\pi}{4}} \cos^4 y \, dy \]

\[ = -\frac{1}{12} \left( \frac{1}{4} - 1 \right) = -\frac{1}{12} \cdot \frac{-3}{4} = \frac{1}{16} \]

#12 Find the volume of the 1st octant region bounded by coordinate planes and surfaces \( z = 1 - x^2 \), \( y = 1 - x^2 \).

\[ V = \int_{0}^{1} \int_{0}^{1-x^2} (1-x^2) \, dy \, dx \]

\[ R: \]

Then:

\[ V = \int_{0}^{1} \int_{y}^{1} (1-x^2) \, dx \, dy \]

\[ U = \int_{0}^{1} \int_{y}^{1} (1-x^2) \, dx \, dy = \int_{0}^{1} (1-x^2)^2 \, dx \]

\[ = \int_{0}^{1} (1-2x^2+x^4) \, dx = \left. x - \frac{2}{3} x^3 + \frac{x^5}{5} \right|_{0}^{1} = \frac{1}{3} - \frac{2}{3} + \frac{1}{5} = \frac{8}{15} \]