STOCHASTIC STABILITY OF CONTINUOUS TIME CONSENSUS PROTOCOLS

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Abstract. A unified approach to studying convergence and stochastic stability of continuous time consensus protocols (CPs) is presented in this work. Our method applies to networks with directed information flow, both cooperative and noncooperative interactions, networks under weak stochastic forcing, and those whose topology and strength of connections may vary in time. The graph theoretic interpretation of the analytical results is emphasized. We show how the spectral properties, such as algebraic connectivity and total effective resistance, as well as the geometric properties, such as the dimension and the structure of the cycle subspace of the underlying graph, shape stability of the corresponding CPs. In addition, we explore certain implications of spectral graph theory to CP design. In particular, we point out that expanders, sparse highly connected graphs, generate CPs whose performance remains uniformly high when the size of the network grows unboundedly. Similarly, we highlight the benefits of using random versus regular network topologies for CP design. We illustrate these observations with numerical examples and refer to the relevant graph theoretic results.

Key words. consensus protocol, dynamical network, synchronization, robustness to noise, algebraic connectivity, effective resistance, expander, random graph

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1. Introduction. The theory of consensus protocols (CPs) is a framework for design and analysis of distributed algorithms for coordination of the groups of dynamic agents. In many control problems, agents in the group need to agree upon a certain quantity, whose interpretation depends on the problem at hand. The theory of CPs studies the convergence to a common value (consensus) in its general and, therefore, abstract form. It has been a subject of intense research due to diverse applications in applied science and engineering. The latter includes coordination of groups of unmanned vehicles [42, 37]; synchronization of power, sensor, and communication networks [10, 12]; and principles underlying collective behavior in social networks [36] and biological systems [4, 45].

From the mathematical point of view, analysis of continuous time CPs is a stability problem for systems of linear differential equations, possibly with additional features such as stochastic perturbations or time-delays. There are many effective techniques for studying stability of linear systems [11, 17]. The challenge of applying these methods to the analysis of CPs is twofold. First, one is interested in characterizing stability under a minimal number of practically relevant assumptions on the structure of the matrix of coefficients, which may depend on time. Second, it is important to identify the relation between the structure of the graph of interactions in the network to the dynamical performance of CPs. A successful solution of the second problem requires a compilation of dynamical systems and graph theoretic techniques. This naturally leads to spectral methods, which play important roles in

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both mathematical disciplines and are especially useful for problems on the interface between dynamics and graph theory [20]. A general idea for using spectral methods for analyzing CPs is that, on the one hand, stability of the continuous time CP is encoded in the eigenvalues (EVs) of the matrix of coefficients; on the other hand, EVs of the same matrix capture structural properties of the graph of the CP. Spectral graph theory offers many fine results relating the structural properties of graphs to the EVs of the adjacency matrix and the graph Laplacian [13, 1, 8, 9, 18, 35, 50]. This provides a link between the network topology and the dynamical properties of CPs.

In this paper, under fairly general assumptions on CPs, we study two problems: convergence of CPs and their stability in the presence of stochastic perturbations. The former is the problem of asymptotic stability of the consensus subspace, a one-dimensional invariant (center) subspace. The latter is a special, albeit representative, form of stability of the consensus subspace with respect to constantly acting perturbations [26]. The rate of convergence to the consensus subspace sets the timescale of the consensus formation (or synchronization) from arbitrary initial conditions or upon instantaneous perturbation. Therefore, the convergence rate is important in applications where the timing of the system’s responses matters (e.g., in decision making algorithms, neuronal networks, etc). Stochastic stability, on the other hand, characterizes robustness of the consensus to noise. This form of stability is important when the consensus needs to be maintained in a noisy environment over large periods of time (e.g., communication networks, control of unmanned vehicles, etc). We believe that our quantitative description of these two forms of stability elucidates two important aspects of the performance of CPs.

The questions investigated in this paper have been studied before under various hypotheses on CPs: constant weights [38, 37, 42], time-dependent interactions [33, 41, 42], and CPs with time-delays [24, 38, 47]. Optimization problems arising in the context of CP design were studied in [46, 48, 49]. There is a body of related work on discrete time CPs [48, 37]. Robustness of CPs to noise was studied in [51, 23]. In this paper, we offer a unified approach to studying convergence and stochastic stability of CPs. Our method applies to networks with directed information flow, both cooperative and noncooperative interactions, networks under weak stochastic forcing, and those whose topology and strength of connections may vary in time. We derive sufficient conditions guaranteeing convergence of time-dependent CPs and present estimates characterizing their stochastic stability. For CPs on undirected graphs, we show that the rate of convergence and degree of stability to random perturbations are captured by the generalized algebraic connectivity and the total effective resistance of the underlying graphs. Previously, these results were available only for CPs on graphs with positive weights [33, 51]. To further elucidate the role that network topology plays in shaping the dynamical properties of CPs, we further develop our results for CPs on simple networks (see text for the definition of a simple network). Our analysis of simple networks reveals the role of the geometric properties of the cycle subspace associated with the graph of the network (such as the first Betti number of the graph and the length and the mutual position of the independent cycles) in the stability of CPs to random perturbations. In addition, we explore several implications of the results of spectral graph theory in CP design. First, we show that expanders, sparse highly connected graphs [18, 43], generate CPs with the rate of convergence bounded from zero uniformly when the size of the network tends to infinity. In particular, CPs based on expanders are effective for coordinating large networks. Second, we point out
that CPs with random connections have nearly optimal convergence rate. In contrast, the convergence of CPs on regular lattice-like graphs slows down rapidly as the size of the network grows. We illustrate these observations with numerical examples and refer to the relevant graph theoretic results.

The mathematical analysis of CPs in this paper uses the method which we recently developed for studying synchronization in systems of coupled nonlinear oscillators [29, 28] and reliability of neuronal networks [30]. We further develop this method in several ways. First, we relate the key properties of the algebraic transformation of the coupling operator used in [29, 28, 30] for studying synchronization to general properties of a certain class of pseudosimilarity transformations. Second, we strengthen the graph theoretic interpretation of the stability analysis. We believe that our method will be useful for design and analysis of CPs and for studying synchronization in a large class of models.

The outline of the paper is as follows. In section 2, we study the properties of a pseudosimilarity transformation, which is used in the analysis of CPs in the remainder of the paper. Section 3 is devoted to the convergence analysis of CPs. After formulating the problem and introducing necessary terminology in section 3.1, we study convergence of CPs with constant and time-dependent coefficients in sections 3.2 and 3.3, respectively. Section 4 presents estimates characterizing stochastic stability of stationary and time-dependent CPs. These results are applied to the study of CPs on undirected weighted graphs in section 5. In section 6, we discuss the relation between the connectivity of the graph and dynamical performance of CPs. The results of this paper are summarized in section 7.

2. Pseudosimilarity transformation. The analysis of CPs in the sections that follow relies on certain properties of a pseudosimilarity transformation, which we study first.

**Definition 2.1.** Matrix \( \hat{D} \in \mathbb{R}^{(n-p) \times (n-p)} \) (1 \( \leq p \leq n-1 \)) is pseudosimilar to \( D \in \mathbb{R}^{n \times n} \) via \( S \in \mathbb{R}^{(n-p) \times n} \) if

\[
SD = \hat{D}S.
\]

Equation (2.1) is equivalent to the property

\[
Sq(D) = q(\hat{D})S
\]

for any polynomial \( q(t) \).

To study the existence and the properties of pseudosimilar matrices, we recall the definition of the Moore–Penrose pseudoinverse of a rectangular matrix (cf. [15]).

**Definition 2.2.** \( A^+ \in \mathbb{R}^{n \times m} \) is called a pseudoinverse of \( A \in \mathbb{R}^{m \times n} \) if

\[
AA^+A = A \quad \text{and} \quad A^+AA^+ = A^+.
\]

Throughout this section, we use the following assumption.

**Assumption 2.3.** Let \( D \in \mathbb{R}^{n \times n} \) and \( S \in \mathbb{R}^{(n-p) \times n} \) (1 \( \leq p \leq n-1 \)) such that

\[
\begin{align*}
\text{rank } S &= n - p, \\
\ker S &\subset \ker D.
\end{align*}
\]

Condition (2.3) implies that

\[
S^+ = S^T(SS^T)^{-1},
\]
and, therefore,

\begin{equation}
S^+ S = P_{R(S^\top)} = P_{(\ker S)^\perp} \quad \text{and} \quad SS^+ = I_{n-p}.
\end{equation}

Here, \( P_{R(S^\top)} \) and \( I_{n-p} \) denote the projection matrix onto the column space of \( S^\top \) and the \((n-p) \times (n-p)\) identity matrix.

The combination of (2.3) and (2.4) guarantees the existence and uniqueness of the pseudosimilar matrix for \( D \) via \( S \).

**Lemma 2.4.** Let \( S \in \mathbb{R}^{(n-p) \times n} \) and \( D \in \mathbb{R}^{n \times n} \) satisfy Assumption 2.3. Then

\begin{equation}
\hat{D} = SDS^+
\end{equation}

is a unique pseudosimilar matrix to \( D \) via \( S \).

**Proof.** By the first identity in (2.6),

\[ DS^+ S = D. \]

Therefore, (2.1) is solvable with respect to \( \hat{D} \). By multiplying both sides of (2.1) by \( S^+ \) from the right and using the second property in (2.6), we obtain (2.7). \( \square \)

**Corollary 2.5.**

\begin{equation}
\exp\{t\hat{D}\} = S \exp\{tD\} S^+, \quad t \in \mathbb{R}.
\end{equation}

**Proof.** Equation (2.8) follows from the second identity in (2.6) and the series representation of \( \exp\{tD\} \).

The next lemma relates the spectral properties of \( D \) and \( \hat{D} \).

**Lemma 2.6.** Suppose \( D \) and \( S \) satisfy Assumption 2.3 and \( \hat{D} \) is the pseudosimilar matrix to \( D \) via \( S \).

(A) If \( \lambda \in \mathbb{C} \) is a nonzero EV of \( D \), then \( \lambda \) is an EV of \( \hat{D} \) of the same algebraic and geometric multiplicity. Moreover, \( S \) carries out a bijection from the generalized \( \lambda \)-eigenspace of \( D \) onto that of \( \hat{D} \) preserving the Jordan block structure.

(B) \( \lambda = 0 \) is an EV of \( \hat{D} \) if and only if the algebraic multiplicity of 0 as an EV of \( D \) exceeds \( p \). In this case, the algebraic multiplicity of 0 as an EV of \( \hat{D} \) is diminished by \( p \). \( S \) maps the generalized 0-eigenspace of \( D \) onto that of \( \hat{D} \).

(C) \( S \) maps a Jordan basis of \( D \) onto that of \( \hat{D} \).

**Proof.**

(A) \( S \) restricted to the direct sum of generalized eigenspaces of \( D \) corresponding to nonzero eigenvalues is injective.

Let \( \lambda \) be a nonzero EV of \( D \). Since for any \( m \in \mathbb{N} \), \((\hat{D} - \lambda I_{n-p})^m S = S(D - \lambda I_n)^m \) (cf. (2.2)), \( S \) maps the generalized \( \lambda \)-eigenvector of \( \hat{D} \) of index \( m \) if and only if \( v \) is a generalized \( \lambda \)-eigenvector of \( D \) of index \( m \). Therefore, \( S \) bijectively maps the generalized \( \lambda \)-eigenspace of \( D \) onto that of \( \hat{D} \). The associated Jordan block structures are the same.

(B) If the generalized 0-eigenspace of \( D \) is larger than \( \ker S \), then \( \ker \hat{D} \) is non-trivial. Choose a Jordan basis for \( D \) restricted to its generalized 0-eigenspace

\[ v^{(1)}_1, v^{(1)}_2, \ldots, v^{(1)}_{k_1}, (\ker D^m \ominus \ker D^{m-1}), \]

\[ Dv^{(1)}_1, Dv^{(1)}_2, \ldots, Dv^{(1)}_{k_1}, v^{(2)}_1, v^{(2)}_2, \ldots, v^{(2)}_{k_2}, (\ker D^{m-1} \ominus \ker D^{m-2}), \]

\[ \ldots \]

\[ D^{(m-1)}v^{(1)}_1, D^{(m-1)}v^{(1)}_2, \ldots, D^{(m-1)}v^{(1)}_{k_1}, v^{(m)}_1, v^{(m)}_2, \ldots, v^{(m)}_{k_m} (\ker D). \]
The image of this basis under \( S \) consists of the vectors forming a Jordan basis of \( \hat{D} \) restricted to its generalized 0-eigenspace and \( p \) zero vectors. Under the action of \( S \), each cyclic subspace of \( D \)

\[
\text{span} \left( v_j^{(i)}, Dv_j^{(i)}, \ldots, D^{m-i}v_j^{(i)} \right)
\]

loses a unit in dimension if and only if \( D^{m-i}v_j^{(i)} \in \ker S \).

(C) The statement in (C) follows by applying the argument in (B) to a Jordan basis of \( D \) restricted to the generalized eigenspace corresponding to a nonzero eigenvalue.

Next, we apply Lemmas 2.4 and 2.6 to the situation, which will be used in the analysis of CPs below.

**Corollary 2.7.** Denote \( e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n \) and \( 1 = \text{span} \{ e \} \). Let \( D \in \mathbb{R}^{(n-1)\times n} \) and \( S \in \mathbb{R}^{(n-1)\times n} \) be such that

\[
D \in \mathcal{K} = \{ M \in \mathbb{R}^{n\times n} : Me = 0 \} \quad \text{and} \quad \ker S = 1.
\]

By Lemmas 2.4 and 2.6, we have the following:

1. (2.9) \( D \in \mathcal{K} \) and ker \( S = 1 \).

3. Convergence analysis of CPs. In section 3.1, we introduce a continuous time CP, a differential equation model that will be studied in the remainder of this paper. Convergence of CPs with constant and time-dependent coefficients is analyzed in sections 3.2 and 3.3, respectively.
3.1. The formulation of the problem. By a continuous time CP with constant coefficients we call the following system of ordinary differential equations (ODEs):

\[
\dot{x}^{(i)} = \sum_{j=1}^{n} a_{ij}(x^{(j)} - x^{(i)}), \quad i \in [n] := \{1, 2, \ldots, n\}.
\]

Unknown functions \(x^{(i)}(t), i \in [n]\), are interpreted as the states of \(n\) agents. The right-hand side of (3.1) models the information exchange between agents in the network. Coefficient \(a_{ij}\) is interpreted as the weight that agent \(i\) attributes to the information from agent \(j\). Positive weights promote synchronization between the states of the corresponding agents. Negative weights have the opposite effect and can be used to model noncooperative interactions between the agents. For more background and motivation for considering (3.1) and related models we refer the interested reader to [33, 36, 38, 42, 51].

An important problem in the analysis of CPs is identifying the conditions under which the states of the agents in the network converge to the same value.

**Definition 3.1.** We say that CP (3.1) reaches a consensus from initial state \(x(0) \in \mathbb{R}^n\) if

\[
\lim_{t \to \infty} |x^{(j)}(t) - x^{(i)}(t)| = 0 \quad \forall (i, j) \in [n]^2.
\]

If CP (3.1) reaches a consensus from any initial condition, then it is called convergent.

The second problem considered in this paper is that of stability of the consensus subspace \(1\) to instantaneous and constantly acting random perturbations. An important aspect of the analysis of convergence and stability of CPs is elucidating the relation between the structural properties of the network (e.g., connectivity and weight distribution) and the degree of stability of the corresponding CPs. To study this problem, we need the following graph theoretic interpretation of (3.1).

Let \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) be a matrix of coefficients of (3.1). Since (3.1) is independent from the choice of diagonal elements of \(A\), we set \(a_{ii} = 0, i \in [n]\). Using the terminology from the electrical networks theory [7], we call \(A\) a conductance matrix. Next, we associate with (3.1) a directed graph \(G = (V, E)\), where the vertex set \(V = [n]\) lists all agents and a directed edge \((i, j) \in [n]^2\) belongs to \(E\) if \(a_{ij} \neq 0\).

By the network we call \(N = (V, E, a) = (G, a)\), where function \(a : E \to \mathbb{R}\) assigns conductance \(a_{ij} \in \mathbb{R}\) to each edge \((i, j) \in E\). If conductance matrix \(A\) is symmetric, \(G\) can be viewed as an undirected graph. If, in addition, \(a_{ij} \in \{0, 1\}\), \(N\) is called simple.

3.2. Stationary CPs. The convergence analysis of CPs with constant and time-dependent coefficients relies on standard results of the theory of differential equations (see, e.g., [17]). It is included for completeness and to introduce the method that will be used later for studying stochastic stability of CPs.

We rewrite (3.1) in matrix form

\[
\dot{x} = Dx.
\]

The matrix of coefficients

\[
D = A - \text{diag}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_N), \quad \bar{a}_i = \sum_{j=1}^{N} a_{ij}
\]
is called a coupling matrix.

Let \( S \in \mathbb{R}^{(n-1) \times n} \) be a matrix with one-dimensional null space, \( \ker \tilde{S} = 1 \) (see Example 2.8 for a possible choice of \( \tilde{S} \)). The analysis in this section does not depend on the choice of \( \tilde{S} \). Suppose \( \tilde{S} \) has been fixed and define

\[
S = (\tilde{S}\tilde{S}^T)^{-\frac{1}{2}}\tilde{S}.
\]

Note that \( S \) has orthogonal rows

\[
SS^T = I_{n-1} \quad \text{and} \quad S^T S = \tilde{S}^+ \tilde{S} = P_{1\perp},
\]

where \( P_{1\perp} \) stands for the orthogonal projection onto \( 1\perp \). By definition, \( D \) and \( S \) the satisfy the conditions of Corollary 2.7. Therefore, there exists a unique \( S \)-reduced matrix

\[
\hat{D} = SDS^+ = (\tilde{S}\tilde{S}^T)^{-\frac{1}{2}}\tilde{S}D\tilde{S}^T (\tilde{S}\tilde{S}^T)^{-\frac{1}{2}} = SDS^T,
\]

whose properties are listed in Corollary 2.7. In addition, using normalized matrix \( S \) (cf. (3.5)) affords the following property.

**Lemma 3.2.** Let \( D \in \mathbb{K} \) and \( S \) be as defined in (3.5). Then \( \hat{D} \), the pseudosimilar matrix to \( D \) via \( S \), is normal (symmetric) if \( D \) is normal (symmetric).

**Proof.** If \( D \) is symmetric, then so is \( \hat{D} \) by (3.7).

Suppose \( D \) is normal. Then there exist an orthogonal matrix \( V = (e, v_2, v_3, \ldots, v_n) \) and diagonal matrix

\[
\Lambda = \text{diag} \left( 0, \alpha_2, \ldots, \lambda_{k_1}, \begin{pmatrix} \alpha_{k_1+1} & -\beta_{k_1+1} \\ \beta_{k_1+1} & \alpha_{k_1+1} \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_{k_l} & -\beta_{k_l} \\ \beta_{k_l} & \alpha_{k_l} \end{pmatrix} \right)
\]

such that \( D = V\Lambda V^T \). By Lemma 2.6, \( D = U\hat{\Lambda}U^T \) with \( U = (v_2, Sv_3, \ldots, Sv_n) \) and

\[
\hat{\Lambda} = \text{diag} \left( \lambda_2, \ldots, \lambda_{k_1}, \begin{pmatrix} \alpha_{k_1+1} & -\beta_{k_1+1} \\ \beta_{k_1+1} & \alpha_{k_1+1} \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_{k_l} & -\beta_{k_l} \\ \beta_{k_l} & \alpha_{k_l} \end{pmatrix} \right).
\]

Denote the columns of \( U \) by \( u_i, i \in [n]/\{1\} \). Since

\[
u_i^T u_i = v_j^T S^T Sv_i = v_j P_{1\perp} v_i = v_j^T v_i, \quad i, j \in \{2, 3, \ldots, n\},
\]

\( U \) is an orthogonal matrix. Therefore, \( \hat{D} \) is normal. \( \Box \)

By multiplying both sides of (3.3) by \( S \) and using (2.1), we obtain the reduced equation for \( y = Sx \in \mathbb{R}^{n-1} \):

\[
\dot{y} = \hat{D}y.
\]

Under \( S \), the consensus subspace of (3.3) is mapped to the origin of the phase space of the reduced system. Furthermore, because \( \hat{D} \) inherits its spectrum from \( D \) (cf. Lemma 2.6), there is a direct relation between the transverse stability of the consensus subspace \( 1 \), as an invariant center subspace of the original problem (3.1), and that of the equilibrium at the origin of the reduced system (3.8). This relation is described in the following theorem.

**Theorem 3.3.** CP (3.3) is convergent if and only if

\[
D \in \mathbb{D} := \{ M \in \mathbb{K} \text{ and } \hat{M} \text{ is a stable matrix} \},
\]
where \( \hat{M} \) is the pseudosimilar matrix to \( M \) via \( S \) (cf. (3.5)). If \( D \in D \), the rate of convergence to the consensus subspace is set by the nonzero EV of \( D \) with the largest real part. Specifically, let the EVs of \( D \) be arranged as in Corollary 2.7 and

\[
\alpha = -\max_{i \geq 2} \Re \lambda_i.
\]

Then there exists \( C_1 > 0 \) such that for any initial condition \( x(0) \in \mathbb{R}^n \) and any \( \epsilon > 0 \),

\[
|P_{1\perp}x(t)| \leq C_1|P_{1\perp}x(0)|\exp\{(-\alpha + \epsilon)t\},
\]

where \( |\cdot| \) stands for the Euclidean norm in \( \mathbb{R}^{n-1} \).

Proof. Let \( x(t) \) be a solution of (3.3). Denote the projection of \( x(t) \) onto \( 1^\perp \) by

\[
z(t) = P_{1\perp}x(t) = STSx(t).
\]

By multiplying both parts of (3.3) by \( ST \) and using (3.6), we derive an ODE for \( z(t) \)

\[
\dot{z} = ST\hat{D}S z.
\]

On the other hand,

\[
y(t) = Sx(t) = SS^TSx(t) = Sz(t)
\]

satisfies the reduced equation \( \dot{y} = \hat{D}y \). Therefore, \( S : \mathbb{R}^n \to \mathbb{R}^{n-1} \cong 1^\perp \) provides a one-to-one correspondence between the trajectories of the reduced system and the projections of the trajectories of (3.3) onto the orthogonal complement of the consensus subspace, \( 1^\perp \). In addition, \( S \) maps the consensus subspace to the origin of phase space of the reduced system. Therefore, transverse asymptotic stability of \( 1 \) as an invariant linear subspace of (3.3) is equivalent to the asymptotic stability of the fixed point at the origin of the reduced system. The necessary and sufficient condition for the latter is that \( \hat{D} \) is a stable matrix, i.e., \( D \in D \).

If \( D \in D \), then \( \lambda_1 = 0 \) is a simple eigenvalue and the real parts of the remaining EVs are negative. By standard results of the theory of ODEs, there exists a positive constant \( C_1 \) such that for any initial condition \( y(0) = Sx(0) \in \mathbb{R}^{n-1} \) and any \( \epsilon > 0 \), the solution of the reduced system satisfies

\[
|y(t)| \leq C_1|y(0)|\exp\{(-\alpha + \epsilon)t\}.
\]

Therefore,

\[
|P_{1\perp}x(t)| = |STy(t)| = |y(t)| \leq C_1|y(0)|\exp\{(-\alpha + \epsilon)t\}.
\]

For CPs with nonnegative weights \( a_{ij} \geq 0 \) there is a simple sufficient condition of convergence: nonnegative CP (3.3) is convergent if the corresponding digraph is strongly connected [38]. This condition does not hold in general if there are negative weights. Edges with positive weights help synchronization. In contrast, negative weight \( a_{ij} < 0 \) indicates that agent \( i \) does not cooperate with agent \( j \) in reaching consensus. The lack of cooperation by some agents in the network can be compensated by the cooperative interactions between other agents. The consensus can be reached even if many of the weights are negative, as shown in the following example.
Example 3.4. The following example of a random matrix $D \in \mathcal{D}$ was constructed in [29]:

$$D = \begin{pmatrix}
-1.0251 & 2.2043 & -1.6032 & 0.5044 & -0.0804 \\
-0.1264 & 0.2772 & -0.3006 & 0.2060 & -0.0562 \\
-1.1549 & 2.5819 & -1.9613 & 0.5210 & 0.0133 \\
-0.8807 & 1.9231 & -1.0823 & 0.0333 & 0.0066 \\
-0.9049 & 1.8778 & -1.0060 & 0.3772 & -0.3441
\end{pmatrix}. $$

(3.14)

About half of the entries of $D$ are negative; i.e., there are as many noncooperative interactions as cooperative ones. Nonetheless, the resultant CP is convergent.

3.3. Time-dependent CPs. In realistic networks, the strength and even topology of connections between the agents may depend on time. To account for a greater variety of possible modeling situations, in this section we use only very mild assumptions on the regularity of conductances $a_{ij}(t)$ as functions of time: $a_{ij}(t), (i,j) \in [n]^2$, are measurable locally bounded real-valued functions. Under these assumptions, we formulate two general sufficient conditions for convergence of time-dependent CPs.

By a time-dependent CP, we call the ODE

$$\dot{x} = D(t)x, \quad D(t) = (d_{ij}(t)) \in \mathbb{R}^{n \times n},$$

where the coupling matrix $D(t)$ is defined as before as

$$D(t) = A(t) - \text{diag} (\bar{a}_1(t), \bar{a}_2(t), \ldots, \bar{a}_n(t)), \quad \bar{a}_i(t) = \sum_{j=1}^{n} a_{ij}(t).$$

(3.16)

Under our assumptions on $a_{ij}(t)$, the solutions of (3.15) (interpreted as solutions of the corresponding integral equation) are well defined (cf. [17]).

By construction, coupling matrix $D$ satisfies the condition (see (3.4))

$$D(t) \in \mathcal{K} \quad \forall t \geq 0.$$

For convenience, we reserve a special notation for the class of admissible matrices of coefficients:

$$D(t) \in \mathcal{K}_1 := \{ M(t) \in \mathbb{R}^{n \times n} : m_{ij}(t) \in L^\infty_{\text{loc}}(\mathbb{R}^+) \text{ and } M(t) \in \mathcal{K} \forall t \geq 0 \}. $$

(3.17)

Note that for any $t > 0$ there exists a unique pseudosimilar matrix to $D(t)$ via $\tilde{S}$, $\tilde{D}(t) = \tilde{S} D(t) \tilde{S}^T$, provided $\tilde{S}$ satisfies Assumption 2.3. Below, we present two classes of convergent time-dependent CPs. The first class is motivated by the convergence analysis of CPs with constant coefficients.

Definition 3.5 (see [29]). Matrix-valued function $D(t) \in \mathcal{K}_1$ is called uniformly dissipative with parameter $\alpha$ if there exists $\alpha > 0$ such that

$$y^T \tilde{D}(t)y \leq -\alpha y^Ty \quad \forall y \in \mathbb{R}^{n-1} \text{ and } \forall t \geq 0,$$

(3.18)

where $\tilde{D}(t)$ is the pseudosimilar matrix to $D(t)$ via $\tilde{S}$. The class of uniformly dissipative matrices is denoted by $\mathcal{D}_\alpha$.

The convergence of uniformly dissipative CPs is given in the following theorem.

Theorem 3.6. Let $D(t) \in \mathcal{D}_\alpha$, $\alpha > 0$. CP (3.15) is convergent with the rate of convergence at least $\alpha$. 
Proof. It is sufficient to show that \( y(t) \equiv 0 \) is an asymptotically stable solution of the reduced system for \( y = \tilde{S}x \)

\[
\begin{equation}
\dot{y} = \tilde{D}(t).
\end{equation}
\]

For solution \( y(t) \) of the reduced system, we have

\[
\frac{d}{dt} |y|^2 = 2y^T \tilde{D}(t)y \leq 2\alpha |y|^2.
\]

Thus,

\[
|y(t)| \leq |y(0)| \exp\{\alpha t\}.
\]

In conclusion, we prove convergence for a more general class of CPs.

**Definition 3.7.** The coupling matrix \( D(t) \) is called asymptotically dissipative if

\[
\begin{equation}
D \in \tilde{D} = \left\{ M(t) \in K_1 : \limsup_{t \to \infty} t^{-1} \int_0^t \sup_{|y|=1} y^T M(u)ydu < 0 \right\}.
\end{equation}
\]

**Theorem 3.8.** If \( D(t) \in \tilde{D} \), then CP (3.15) is convergent.

Proof. Let \( y(t) \) be a solution of the reduced system (3.19). Then

\[
\frac{d}{dt} |y|^2 = 2y^T \tilde{D}(t)y \leq 2\gamma(t) |y|^2,
\]

where \( \gamma(t) := \sup_{|y|=1} y^T \tilde{D}(t)y \).

By Gronwall’s inequality,

\[
|y(t)| \leq |y(0)| \exp \left\{ \int_0^t \gamma(u)du \right\}.
\]

Since \( D(t) \in \tilde{D} \),

\[
\exists \alpha > 0 \text{ and } T > 0 : (t \geq T) \Rightarrow \int_0^t \gamma(u)du \leq -\alpha t.
\]

Thus,

\[
|y(t)| \leq |y(0)| \exp\{-\alpha t\} < -\alpha t, \quad t \geq T.
\]

**4. Stochastic stability.** In this section, we study stochastic stability of CPs. Specifically, we consider

\[
\begin{equation}
\dot{x} = D(t)x + \sigma U(t)\dot{w}, \quad x(t) \in \mathbb{R}^n,
\end{equation}
\]

where \( \dot{w} \) is a white noise process in \( \mathbb{R}^n \), \( D(t) \in K_1 \) (cf. (3.17)), and \( U(t) = (u_{ij}(t)) \in \mathbb{R}^{n \times n}, u_{ij}(t) \in L_{loc}^\infty(\mathbb{R}^+) \). The consensus subspace, \( 1 \), forms an invariant center subspace of the deterministic system obtained from (4.1) by setting \( \sigma = 0 \).

Since the transverse stability of the consensus subspace is equivalent to the stability of the equilibrium of the corresponding reduced equation, along with (4.1), we consider the corresponding equation for \( y = Sx \),

\[
\begin{equation}
\dot{y} = \tilde{D}(t)y + \sigma SU(t)\dot{w},
\end{equation}
\]
where $S$ is defined as in (3.5). The solution of (4.2) with deterministic initial condition $y(0) = y_0 \in \mathbb{R}^{n-1}$ is a Gaussian random process. The mean vector and the covariance matrix functions of stochastic process $y(t)$,

$$m(t) := \mathbb{E}y(t) \quad \text{and} \quad V(t) := \mathbb{E}\left[(y(t) - m(t))(y(t) - m(t))^T\right],$$

satisfy the linear equations (cf. [21])

$$\dot{m} = \dot{D}m \quad \text{and} \quad \dot{V} = \dot{D}V + V\dot{D}^T + \sigma^2SU(t)U(t)^TST.$$

The trivial solution $y(t) \equiv 0$ of the reduced equation (3.8) is not a solution of the perturbed equation (4.2). Nonetheless, if the origin is an asymptotically stable equilibrium of the deterministic reduced equation obtained from (4.2) by setting $\sigma = 0$, the trajectories of (4.1) exhibit stable behavior. In particular, if (3.15) is a convergent CP, for small $\sigma > 0$, the trajectories of the perturbed system (4.1) remain in the $O(\sigma)$ vicinity of the consensus subspace on finite time intervals of time with high probability. We use the following form of stability to describe this situation formally.

**Definition 4.1.** CP (3.15) is stable to random perturbations (cf. (4.1)) if for any initial condition $x(0) \in \mathbb{R}^n$ and $T > 0$

$$\lim_{t \to \infty} \mathbb{E}P_{1\perp}x(t) = 0 \quad \text{and} \quad \mathbb{E}|P_{1\perp}x(t)|^2 = O(\sigma^2), \quad t \in [0,T].$$

**Theorem 4.2.** Let $D \in \mathcal{D}_a$ be a uniformly dissipative matrix with parameter $\alpha$ (cf. Definition 3.5). Then CP (3.15) is stable to random perturbations. In particular, the solution of the initial value problem for (4.1) with deterministic initial condition $x(0) \in \mathbb{R}^n$ satisfies the estimate

$$\mathbb{E}|P_{1\perp}x(t)|^2 \leq \frac{\sigma^2n}{2\alpha} \sup_{u \in [0,t]} \|U(u)U^T(u)\|, \quad t > 0,$$

where $\| \cdot \|$ stands for the operator matrix norm induced by the Euclidean norm.

**Remark 4.3.** Suppose the strength of interactions between the agents in the network can be controlled by an additional parameter $g$,

$$\dot{x} = gD(t)x + \sigma U(t)\dot{w}.$$  

Here, the larger values of $g > 0$ correspond to stronger coupling, i.e., to faster information exchange in the network. By applying estimate (4.6) to (4.7), we have

$$\mathbb{E}|P_{1\perp}x(t)|^2 \leq \frac{\sigma^2n}{2\gamma \alpha} \sup_{u \in [0,t]} \|U(u)U^T(u)\|, \quad t > 0.$$  

Note that the variance of $|P_{1\perp}x(t)|$ can be effectively controlled by $g$. In particular, the accuracy of the consensus can be enhanced to any desired degree by increasing the rate of information exchange between the agents in the network. For the applications of this observations to neuronal networks, we refer the reader to [30, 31].

**Remark 4.4.** Since $P_{1\perp}x(t) = S^Ty(t)$ and $(P_{1\perp}x(t))^TP_{1\perp}x(t) = y(t)^Ty(t)$, it is sufficient to prove (4.6) with $P_{1\perp}x(t)$ replaced by the solution of the reduced problem (4.2), $y(t) = Sx(t)$.

**Proof.** Let $\Phi(t)$ denote the principal matrix solution of the homogeneous equation (4.2) with $\sigma = 0$. The solution of the initial value problem for (4.2) is a Gaussian
random process whose expected value and covariance matrix are given by

\begin{align}
(4.9) & \quad \mathbb{E} y_t = \Phi(t)y_0, \\
(4.10) & \quad \text{cov } y_t = \sigma^2 \Phi(t) \int_0^t \Phi^{-1}(u)SU(u)U(u)^T S^T(\Phi^{-1}(u)) du \Phi(t)^T.
\end{align}

Since \( D(t) \in \mathcal{D}_\alpha \), we have

\begin{equation}
(4.11) \quad y^T \hat{D}(t)y \leq -\alpha y^T y \quad \forall y \in \mathbb{R}^{n-1}, \ t \geq 0.
\end{equation}

This has the following implication. For all \( t \geq 0 \), \(-\hat{D}^*(t)\) is positive definite and, therefore, \((-\hat{D}^*(t))^T \) is well defined. Here and throughout this paper, \( M^* := 2^{-1}(M + M^T) \) stands for the symmetric part of square matrix \( M \).

Using this observation, we rewrite the integrand in (4.10) as\(^1\)

\begin{align}
(4.12) & \quad \Phi^{-1}(u)(-\hat{D}^*(u)) \hat{\Phi}(-\hat{D}^*(u)) \Phi(u) U(u)^T S^T(-\hat{D}^*(u)) \Phi^{-1}(u)^T \\
& \quad \leq -\| F(u) \| \Phi(u)^{-1} \hat{D}^*(u) \Phi(u)^{-1})^T = \frac{1}{2} \| F(u) \| \frac{d}{du} \{ \Phi(u)^{-1} \Phi(u)^{-1})^T \},
\end{align}

where

\[ F(u) := (-\hat{D}^*(u)) \Phi(u) U(u)^T S^T(-\hat{D}^*(u)) \Phi(u)^{-1} \]

By taking into account (4.12), from (4.10) we have

\begin{equation}
(4.13) \quad \text{Tr } \text{cov } y_t \leq \frac{\sigma^2}{2} \sup_{u \in [0, t]} \{ \| F(u) \| \} \text{Tr } \{ I - \Phi(t)^T \Phi(t) \} \leq \frac{\sigma^2 n}{2} \sup_{u \in [0, t]} \| F(u) \|.
\end{equation}

Further,

\begin{align}
(4.14) & \quad \| F(u) \| = \| (-\hat{D}^*(u)) \Phi(u) U(u)^T S^T(-\hat{D}^*(u)) \Phi(u)^{-1} \| \\
& \quad \leq \| U(u) U^T(u) \| \| S S^T \| \| (\hat{D}^*(u))^{-1} \| \\
& \quad \leq \alpha^{-1} \| U(u) U^T(u) \|,
\end{align}

since \( \| S S^T \| = 1 \) (cf. (3.6)). Estimate (4.6) follows from (4.9), (4.13), and (4.14).

Theorem 4.2 describes a class of stochastically stable time-dependent CPs. Because much of the previous work focused on CPs with constant coefficients, we study them separately. To this end, we consider

\begin{equation}
(4.15) \quad \dot{x} = Dx + \sigma \dot{w}
\end{equation}

and the corresponding reduced system for \( y = Sx \),

\begin{equation}
(4.16) \quad \dot{y} = \hat{D}y + \sigma S \dot{w}.
\end{equation}

In (4.15), we set \( U(t) = I_n \) to simplify notation.

\(^1\)The derivations of the estimates (4.12) and (4.14) use the following matrix inequality: \( A(B - \| B \| I_n) A^T \leq 0 \), which obviously holds for any nonnegative definite matrix \( B \in \mathbb{R}^{n \times n} \) and any \( A \in \mathbb{R}^{n \times n} \).
THEOREM 4.5. Suppose that $D \in \mathcal{D}$ and $\alpha$ has the same meaning as in (3.10). Then for any $0 < \epsilon < \alpha$, there exists a positive constant $C_2$ such that

$$\lim_{t \to \infty} \mathbb{E}|P_{1\perp} x(t)| = 0 \quad \text{and} \quad \mathbb{E}|P_{1\perp} x(t)|^2 \leq \frac{C_2 \sigma^2}{\alpha - \epsilon}. \quad (4.17)$$

Proof. Since $D \in \mathcal{D}$, for any $\epsilon \in (0, \alpha)$, there exists a change of coordinates in $\mathbb{R}^n$, $x = Q_\epsilon \hat{x}$, $Q_\epsilon \in \mathbb{R}^{n \times n}$, such that $\hat{D}_\epsilon = Q_\epsilon^{-1} D Q_\epsilon \in D_{n-\epsilon}$ (cf. [3]). By Theorem 4.2, solutions of

$$\dot{\hat{x}} = \hat{D}_\epsilon \hat{x} + \sigma Q_\epsilon^{-1} \hat{w}$$

satisfy (4.6). Thus, (4.17) holds for some $C_2 > 0$ possibly depending on $\epsilon$. \square

The estimate of $\mathbb{E}|P_{1\perp} x(t)|^2$ in (4.17) characterizes the dispersion of the trajectories of the stochastically forced CP (4.15) around the consensus subspace. $\mathbb{E}|P_{1\perp} x(t)|^2$ can be viewed as a measure of stochastic stability of consensus subspace. In (4.17), the upper bound for $\mathbb{E}|P_{1\perp} x(t)|^2$ is given in terms of the leading nonzero eigenvalue of $D$. If $D$ is normal, then precise asymptotic values for $\text{cov } S x(t)$ and $\mathbb{E}|P_{1\perp} x(t)|^2$ are available. The stability analysis of (4.15) with normal $D$ is important for understanding the properties of CPs on undirected graphs, which we study in the next section.

THEOREM 4.6. Suppose $D \in \mathcal{D}$ is normal. Denote the EVs of $D$ by $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $\lambda_1 = 0$ is a simple EV. Let $\hat{D}$ be a pseudosimilar matrix to $D$ via $S$ (cf. (3.5)). Then for any deterministic initial condition $x(0) \in \mathbb{R}^n$, the trajectory of (4.15) is a Gaussian random process with the following asymptotic properties:

$$\lim_{t \to \infty} |\mathbb{E} P_{1\perp} x(t)| = 0, \quad (4.18)$$
$$\lim_{t \to \infty} \text{cov } S x(t) = 2^{-1} \sigma^2 \hat{D}^{-1}, \quad (4.19)$$
$$\lim_{t \to \infty} \mathbb{E}|P_{1\perp} x(t)|^2 = 2^{-1} \sigma^2 \sum_{i=2}^{n} (\text{Re} \lambda_i)^{-1}. \quad (4.20)$$

Proof. By the observation in Remark 4.4, it is sufficient to prove the relations (4.18) and (4.20) with $P_{1\perp} x(t)$ replaced by $y(t) = S x(t)$.

The solution of the reduced equation (4.16) with a deterministic initial condition is a Gaussian process (cf. [21])

$$y(t) = e^{t \hat{D}} y_0 + \sigma \int_0^t e^{(t-u) \hat{D}} S \mathbb{w}(u). \quad (4.21)$$

From (4.4) specialized to solutions of (4.16), we have

$$\mathbb{E} y(t) = e^{t \hat{D}} y_0 \to 0 \text{ as } t \to \infty, \quad (4.22)$$
$$V(t) = \text{cov } y(t) = \sigma^2 \int_0^t e^{(t-u) \hat{D}} S S^T e^{(t-u) \hat{D}^T} du. \quad (4.23)$$

Since $SS^T = I_{n-1}$ and $\hat{D}$ is a stable normal matrix, from (4.23) we have

$$V(t) = \int_0^t e^{2u \hat{D}^s} du \to \frac{\sigma^2}{2} (\hat{D}^s)^{-1}, \quad t \to \infty, \quad (4.24)$$
where $\hat{D}^s = 2^{-1}(\hat{D} + \hat{D}^T)$ stands for the symmetric part of $\hat{D}$. By taking into account (4.22), we have

$$\lim_{t \to \infty} E|y(t)|^2 = \lim_{t \to \infty} \text{Tr} V(t) = \frac{\sigma^2}{2} \text{Tr} \hat{D}^s = \frac{\sigma^2}{2} \sum_{i=2}^{n} (\text{Re}\lambda_i)^{-1}.$$  

Remark 4.7. Estimate (4.20) was derived in [51] for CPs with positive weights. A similar estimate was obtained in [30] in the context of analysis of a neuronal network.

5. CPs on undirected graphs. In this section, we apply the results of the previous section to CPs on undirected graphs. The analysis reveals the contribution of the network topology to stability of CPs. In particular, we show that the dimension and the structure of the cycle subspace associated with the graph of the network are important for stability. The former quantity, the first Betti number of the graph, is a topological invariant of the graph of the network.

5.1. Graph theoretic preliminaries. We start by reviewing certain basic algebraic constructions used in the analysis of graphs (cf. [5]). Throughout this subsection, we assume that $G = (V(G), E(G))$ is a connected graph with $n$ vertices and $m$ edges:

$$V(G) = \{v_1, v_2, \ldots, v_n\} \quad \text{and} \quad E(G) = \{e_1, e_2, \ldots, e_m\}.$$  

The vertex space, $C_0(G)$, and the edge space, $C_1(G)$, are the finite-dimensional vector spaces of real-valued functions on $V(G)$ and $E(G)$, respectively.

We fix an orientation on $G$ by assigning positive and negative ends for each edge in $E(G)$. The matrix of the coboundary operator $H : C_1(G) \to C_0(G)$ with respect to the standard bases in $C_0(G)$ and $C_1(G)$ is defined by

$$H = (h_{ij}) \in \mathbb{R}^{m \times n}, \quad h_{ij} = \begin{cases} 1, & v_j \text{ is a positive end of } e_i, \\ -1, & v_j \text{ is a negative end of } e_i, \\ 0, & \text{otherwise.} \end{cases}$$  

The Laplacian of $G$ is expressed in terms of the coboundary matrix

$$L = H^T H.$$  

By $\tilde{G} = (V(\tilde{G}), E(\tilde{G})) \subset G$ we denote a spanning tree of $G$, a connected subgraph of $G$ such that

$$|V(\tilde{G})| = n \quad \text{and} \quad |E(\tilde{G})| = n - 1.$$  

$\tilde{G}$ contains no cycles. Without loss of generality, we assume that

$$E(\tilde{G}) = \{e_1, e_2, \ldots, e_{n-1}\}.$$  

A cycle $O$ of length $|O| = k$ is a cyclic subgraph of $G$:

$$O = (V(O), E(O)) \subset G: \quad V(O) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\},$$

$$E(O) = \{(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_k}, v_{i_1})\}.$$  

for some $k$ distinct integers $(i_1, i_2, \ldots, i_k) \in [n]^k$. Two cyclic orderings of the vertices of $O$ induce two possible orientations. Suppose the orientation of $O$ has been fixed.
We will refer to it as the cycle orientation. For each \( e_i \in E(O) \), we thus have two orientations: one induced by the orientation of \( G \) and the other induced by the cycle orientation. We define \( \xi(O) = (\xi_1, \xi_2, \ldots, \xi_m)^T \in C_1(G) \) such that
\[
\xi_i = \begin{cases} 
1 & \text{if } e_i \in E(O) \text{ and the two orientations of } e_i \text{ coincide,} \\
-1 & \text{if } e_i \in E(O) \text{ and the two orientations of } e_i \text{ differ,} \\
0 & \text{otherwise.} 
\end{cases}
\] (5.4)

To each cycle \( O \) of \( G \) there corresponds \( \xi(O) \in C_1(G) \). All such vectors span the cycle subspace of \( G \), \( \text{Cyc}(G) \subset C_1(G) \). The cycle subspace coincides with the kernel of the incidence mapping \( H^T \).
\[
\text{Cyc}(G) = \ker \, H^T, 
\] (5.5)
and its dimension is equal to the corank of \( G \),
\[
c = m - n + 1. 
\] (5.6)

To each edge \( e_{n-1+k}, k \in [c] \), not belonging to the spanning tree \( \tilde{G} \), there corresponds a unique cycle \( O_k \) such that
\[
e_{n-1+k} \in E(O_k) \subset E(\tilde{G}) \cup \{e_k\}. 
\]
We orient cycles \( O_k, k \in [c] \), in such a way that the orientation of \( e_{n-1+k} \) as an edge of \( G \) and that of \( O_k \) coincide. The vectors
\[
\xi^k = \xi(O_k), \quad k \in [c], 
\] (5.7)
form a basis in \( \text{Cyc}(G) \). We will refer to cycles \( O_k, k \in [c] \), as the fundamental cycles of \( G \).

Define a \( c \times m \) matrix
\[
Z = \begin{pmatrix} 
\xi_1^T \\
\xi_2^T \\
\vdots \\
\xi_c^T 
\end{pmatrix},
\]
By construction, \( Z \) has the block structure
\[
Z = (Q \, I_c), 
\] (5.8)
where \( I_c \) is the \( c \times c \) identity matrix. Matrix \( Q = (q_{kl}) \in \mathbb{R}^{c \times (n-1)} \) contains the coefficients of expansions of \( \vec{e}_{n-1+k}, k \in [c] \) with respect to \( \{\vec{e}_i, i \in [n-1]\} \),
\[
\vec{e}_{n-1+k} = - \sum_{l=1}^{n-1} q_{kl} \vec{e}_l, \quad q_{kl} \in \{0, \pm 1\}. 
\] (5.9)
Here, \( \vec{e}_i, i \in [m] \), denote the oriented edges of \( G \). This motivates the following definition.

**Definition 5.1.**

1. Matrix \( Q \in \mathbb{R}^{c \times (n-1)} \) is called a cycle incidence matrix of \( G \).
2. Matrix \( L_c(G) = QQ^T \in \mathbb{R}^{c \times c} \) is called a cycle Laplacian of \( G \).
The following properties of the cycle Laplacian follow from its definition:

(A) The spectrum of $L_c(G)$ does not depend on the choice of orientation used in the construction of $Q$. Indeed, if $Q_1$ and $Q_2$ are two cycle incidence matrices of $G$ corresponding to two distinct orientations, then $Q_2 = P_c Q_1 P_t$, where we have diagonal matrices

$$P_t = \text{diag}\{p_1, p_2, \ldots, p_{n-1}\} \in \mathbb{R}^{(n-1) \times (n-1)},$$
$$P_c = \text{diag}\{p_n, p_{n+1}, \ldots, p_m\} \in \mathbb{R}^{c \times c}.$$

If the two orientations of $G$ yield the same orientation for edge $e_i \in E(G), i \in [m]$, then $p_{i} = 1$, and $p_{i} = -1$ otherwise. Thus,

$$Q_2 Q_2^T = P_c Q_1 P_t P_t^T Q_1^T P_c^T = P_c Q_1 Q_1^T P_c^T.$$

The spectra of $Q_2 Q_2^T$ and $Q_1 Q_1^T$ coincide, because $P_c$ is an orthogonal matrix.

(B)

(5.10) $(L_c(G))_{ij} = \langle \text{Row}_i(L_c(G)), \text{Row}_j(L_c(G)) \rangle = \begin{cases} |O_i| - 1, & i = j, \\ \pm |O_i \cap O_j|, & i \neq j. \end{cases}$

(C) If the cycles are disjoint, then assuming that $O_k, k \in [c]$, are ordered by their sizes, we have

(5.11) $\lambda_k(I_c + QQ^T) = |O_k|, \quad k \in [c].$

The cycle incidence matrix provides a convenient partition of the coboundary matrix.

**Lemma 5.2.** Let $G = (V(G), E(G))$ be a connected graph of positive corank $c$. Then using the notation of this section, we have the coboundary matrix

(5.12) $H = \begin{pmatrix} I_{n-1} & \hat{H} \\ -Q \end{pmatrix},$

where $\hat{H}$ is the coboundary matrix of a spanning tree $\hat{G}$, and $Q$ is the corresponding cycle incidence matrix.

**Proof.** Since $\text{rank}H = \text{rank} \hat{H} = n - 1$, there is a unique $B \in \mathbb{R}^{c \times (n-1)}$ such that

(5.13) $H = \begin{pmatrix} \hat{H} \\ B \hat{H} \end{pmatrix}.$

Using (5.13) and (5.8), we obtain

$$ZH = 0 \Rightarrow Q \hat{H} + B \hat{H} = 0 \Rightarrow B = -Q. \quad \Box$$

**5.2. Stability analysis.** We are now in a position to apply the results of section 4 to study CPs on undirected graphs. Let $G = (V(G), E(G))$ be a connected undirected graph. Since the interactions between the agents are symmetric, the coupling matrix in (4.15) can be rewritten in terms of the coboundary matrix of $G$ and the conductance matrix $C = \text{diag}(c_1, c_2, \ldots, c_m),

(5.14) D = -H^T C H.$

The conductances $c_i \in \mathbb{R}, i \in [m]$, are assigned to all edges of $G$. If the network is simple, $C = I_m$. 
Let $\tilde{G}$ be a spanning tree of $G$. We continue to assume that the edges of $G$ are ordered so that (5.3) holds. Let $\tilde{H}$, $Q$, and $C = \text{diag}(C_1, C_2)$, $C_1 \in \mathbb{R}^{(n-1) \times (n-1)}$, $C_2 \in \mathbb{R}^{c \times c}$, be the coboundary, cycle incidence matrix, and conductance matrices corresponding to the spanning tree $\tilde{G}$, respectively. Using (5.12), we recast the coupling matrix as

\begin{equation}
D = -\tilde{H}^T(C_1 + Q^T C_2 Q) \tilde{H}.
\end{equation}

To form the reduced equation, we let

$S = (\tilde{H} \tilde{H}^T)^{-\frac{1}{2}} \tilde{H}$ and $y = Sx$.

Then

\begin{equation}
\dot{y} = \hat{D} y + \sigma S \dot{w},
\end{equation}

where

\begin{equation}
\hat{D} = -(\tilde{H} \tilde{H}^T)^{\frac{1}{2}}(C_1 + Q^T C_2 Q)(\tilde{H} \tilde{H}^T)^{\frac{1}{2}}.
\end{equation}

Both $D$ and $\hat{D}$ are symmetric matrices. By Lemma 2.6, the eigenspaces of $\hat{D}$ and $D$ are related via $S$. The EVs of $\hat{D}$ are the same as those of $D$ except for a simple zero EV $\lambda_1 = 0$ corresponding to the constant eigenvector $e$.

A CP with positive weights $c_i$ is always convergent, as can be easily seen from (5.17). For a general case, we have the following necessary and sufficient condition for convergence of the CP on an undirected graph.

**Theorem 5.3.** The CP (3.3) with matrix (5.14) is convergent if and only if matrix $C_1 + Q^T C_2 Q$ is positive definite for some spanning tree $\tilde{G}$.

**Proof.** By (5.17), $D$ is a stable matrix if and only if $C_1 + Q^T C_2 Q$ is positive definite. \qed

If $D \in \mathcal{D}$, stochastic stability of the CP (4.15) and (5.14) is guaranteed by Theorem 4.6. In particular, (4.20) characterizes the dispersion of trajectories around the consensus subspace. Theorems 3.3 and 4.6 provide explicit formulas for the rate of convergence and the degree of stability of convergent CPs on undirected graphs. Specifically, let $\mathcal{N} = (G, C)$ be a network corresponding to (4.15) with $D \in \mathcal{D}$ (cf. (5.15)). Let

$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$

denote the EVs of $-D$, and define

\begin{equation}
\alpha(\mathcal{N}) = \lambda_2 \quad \text{and} \quad \rho(\mathcal{N}) = \sum_{i=2}^{n} \frac{1}{\lambda_i}.
\end{equation}

Formulas in (5.18) generalize algebraic connectivity and (up to a scaling factor) total effective resistance of a simple graph to weighted networks corresponding to convergent CPs on undirected graphs. By replacing the EVs of $D$ by those of $D'$ in (5.18), the definitions of $\alpha(\mathcal{N})$ and $\rho(\mathcal{N})$ can be extended to convergent CPs with normal coupling matrices. For simple networks, there are many results relating algebraic connectivity and total effective resistance and the structure of the graph (cf. [13, 1, 16, 8, 9, 50, 18, 35]). Theorems 3.3 and 4.6 link structural properties of the network and dynamical performance of the CPs.
To conclude this section, we explore some implications of (5.17) for stability of (4.15). To make the role of the network topology in shaping stability properties of the system more transparent, in the remainder of this paper, we consider simple networks, i.e., $C = I_m$. In the context of stability of CPs, the lower bounds for $\alpha(N)$ and the upper bounds for $\rho(N)$ are important.

Lemma 5.4. Let $\tilde{H}$ stand for the coboundary matrix of a spanning tree $\tilde{G}$ of undirected graph $G$ and let $Q$ be the corresponding cycle incidence matrix. Then

1. \( \alpha(G) \geq \alpha(\tilde{G}) \lambda_1 \left( I_{n-1} + Q^T Q \right) \),

2. \( \rho(N) \leq \min \left\{ \frac{n-1}{\lambda_1 \left( I_{n-1} + Q^T Q \right)^{-1}} \frac{\text{Tr} \left( I_{n-1} + Q^T Q \right)^{-1}}{\alpha(G)} \right\} \leq (n-1) \min \left\{ 1, \frac{1}{\alpha(G)} \right\} \),

where \( \alpha(G) \) stands for the algebraic connectivity of $G$, and \( \lambda_1 \left( I_{n-1} + Q^T Q \right) \) denotes the smallest EV of the positive definite matrix $I_{n-1} + Q^T Q$.

Proof. Since $N$ is a simple network, the coupling matrix $D$ taken with a negative sign is the Laplacian of $G$, $L = H^T H$. Likewise, $\tilde{L} = \tilde{H}^T \tilde{H}$ is a Laplacian of $\tilde{G}$. Let

\[ 0 = \lambda_1(G) < \lambda_2(G) \leq \cdots \leq \lambda_n(G) \]

denote the EVs of $L$. Below we use the same notation to denote the EVs of other positive definite matrices, e.g., $\tilde{L}$ and $I_{n-1} + Q^T Q$.

By Lemma 2.6, the second EV of $G$, $\alpha(G)$, coincides with the smallest EV of

\[ \tilde{L} = -\tilde{D} = (\tilde{H} \tilde{H}^T)^\dagger (I_{n-1} + Q^T Q)(\tilde{H} \tilde{H}^T)^\dagger. \]

We will use the following observations:

1. The sets of nonzero EVs of two symmetric matrices $\tilde{H} \tilde{H}^T$ and $\tilde{L} = \tilde{H}^T \tilde{H}$ coincide. Since the former is a full rank matrix, the spectrum of $\tilde{H} \tilde{H}^T$ consists of nonzero EVs of $\tilde{G}$. In particular,

\[ \lambda_1(\tilde{H} \tilde{H}^T) = \alpha(\tilde{G}). \]

2. The EVs of $\tilde{L}$ and those of $(\tilde{H} \tilde{H}^T)(I_{n-1} + Q^T Q)$ coincide.

Using the variational characterization of the EVs of symmetric matrices (cf. [19]) and the observations (a) and (b) above, we have

\[ \lambda_1(\tilde{L}) = \lambda_1 \left( (\tilde{H} \tilde{H}^T)(I_{n-1} + Q^T Q) \right) \geq \lambda_1(\tilde{H} \tilde{H}^T) \lambda_1(I_{n-1} + Q^T Q) = \alpha(\tilde{G}) \lambda_1(\tilde{H} \tilde{H}^T) \lambda_1(I_{n-1} + Q^T Q). \]

Hence,

\[ \alpha(G) = \lambda_1(\tilde{L}) \geq \alpha(\tilde{G}) \lambda_1(I_{n-1} + Q^T Q). \]

Likewise,\(^2\)

\[ \rho(G) = \text{Tr} \left( -\tilde{D}^{-1} \right) = \text{Tr} \left\{ (\tilde{H} \tilde{H}^T)^{-1}(I_{n-1} + Q^T Q)^{-1} \right\} \]

\[ \leq \frac{\text{Tr} \left\{ (\tilde{H} \tilde{H}^T)^{-1} \right\}}{\lambda_1(I_{n-1} + Q^T Q)} \leq \frac{\rho(\tilde{G})}{\lambda_1(I_{n-1} + Q^T Q)} = \frac{n-1}{\lambda_1(I_{n-1} + Q^T Q)}. \]

\(^2\)Estimate (5.20) uses the following inequality: $\lambda_1(A) \text{Tr} \ B \leq \text{Tr} \ (AB) \leq \lambda_n(A) \text{Tr} \ B$, which holds for any symmetric $A \in \mathbb{R}^{n \times n}$ and nonnegative definite $B \in \mathbb{R}^{n \times n}$ (cf. [22]). If $A = \Lambda$ is diagonal, the double inequality above is obvious. The case of symmetric $A = U \Lambda U^T$ is reduced to the previous one via the identity $\text{Tr} \ U^T \{ U \Lambda U^T B \} = \text{Tr} \ {\Lambda U^T BU}$.
A symmetric argument yields
\[ \rho(G) \leq \frac{\text{Tr} \left( (I_n - 1 + Q^T Q)^{-1} \right)}{\lambda_1(\tilde{H} \tilde{H}^T)} = \frac{\text{Tr} \left( (I_n - 1 + Q^T Q)^{-1} \right)}{\alpha(\tilde{G})}. \]

Lemma 5.4 shows respective contributions of the spectral properties of the spanning tree \( \tilde{G} \) and those of the cycle subspace to the algebraic connectivity and effective resistance of \( G \). In this respect, it is of interest to study the spectral properties of \( I_n - 1 + Q^T Q \), in particular, its smallest EV and the trace. Another motivation for studying \( I_n - 1 + Q^T Q \) comes from the following lemma.

**Lemma 5.5.** Under the assumptions of Lemma 5.4, the solutions of CP (4.15) satisfy

\[ \lim_{t \to \infty} E|\tilde{H}x(t)|^2 = \sigma^2 \kappa(G, \tilde{G}) = \text{Tr} \left( (I_n - 1 + Q^T Q)^{-1} \right). \]

**Proof.** The reduced equation for \( y = \tilde{H}x \) has the form

\[ \dot{y} = \hat{D}y + \sigma \tilde{H}\dot{w}, \]

where

\[ \hat{D} = \tilde{H} \tilde{D} \tilde{H}^+. \]

Using \( D = -H^TH \) and (5.12), we rewrite (5.23) as

\[ \hat{D} = \tilde{H} \tilde{D} \tilde{H}^+ = -\tilde{H} \tilde{H}^T \tilde{H} \tilde{H}^T (\tilde{H} \tilde{H}^T)^{-1} = -\tilde{H} \tilde{H}^T (I_n - 1 + Q^T Q). \]

By applying the argument used in the proof of Theorem 4.6 to the reduced equation (5.22), we obtain

\[ \lim_{t \to \infty} E|\tilde{H}x(t)|^2 = \frac{\sigma^2}{2} \text{Tr} \left\{ \tilde{H} \tilde{H}^T (\hat{D})^{-1} \right\} = \frac{\sigma^2}{2} \kappa(G, \tilde{G}). \]

The combination of (5.25) and (5.23) yields (5.21). □

The following lemma provides the graph theoretic interpretation of \( \kappa(G, \tilde{G}) \).

**Lemma 5.6.** Let \( G = (V(G), E(G)) \), \( |V(G)| = n \), be a connected graph, and let \( \tilde{G} \subset G \) be a spanning tree of \( G \).

(A) If \( G \) is a tree, then

\[ \kappa(G, \tilde{G}) = n - 1. \]

(B) Otherwise, denote the corank of \( G \) by \( c > 0 \), and let \( \{O_k\}_{k=1}^c \) be the system of fundamental cycles corresponding to \( G \).

(B.1) Denote

\[ \mu = \frac{1}{n - 1} \sum_{k=1}^c (|O_k| - 1). \]

Then

\[ \frac{1}{1 + \mu} \leq \frac{\kappa(G, \tilde{G})}{n - 1} \leq 1. \]
(B.2) If $0 < c < n - 1$, then

\[(5.29) \quad 1 - \frac{c}{n - 1} \left(1 - \frac{1}{\delta}\right) \leq \frac{\kappa(G, \tilde{G})}{n - 1} \leq 1,\]

where

\[(5.30) \quad \delta = \max_{k \in [c]} \left\{|O_k| + \sum_{l \neq k} |O_k \cap O_l|\right\},\]

(B.3) If $O_k, k \in [c]$, are disjoint, then

\[(5.31) \quad \frac{\kappa(G, \tilde{G})}{n - 1} = 1 - \frac{c}{n - 1} \left(1 - \frac{c}{1 - \min_{k \in [c]} |O_k|}\right).\]

In particular,

\[\kappa(G, \tilde{G}) \leq 1 - \frac{c}{n - 1} \left(1 - \frac{1}{\min_{k \in [c]} |O_k|}\right)\]

and

\[\kappa(G, \tilde{G}) \geq 1 - \frac{c}{n - 1} \left(1 - \frac{c}{\sum_{k \in [c]} |O_k|}\right) \geq 1 - \frac{c}{n - 1} \left(1 - \frac{1}{\max_{k \in [c]} |O_k|}\right).\]

Proof.

(A) If $G$ is a tree, then $Q = 0$ and $\kappa(G, \tilde{G}) = \text{Tr} I_{n - 1} = n - 1$.

(B.1) Suppose $c > 0$. Let $\lambda_i, i \in [n]$, denote the EVs of $I_{n - 1} + Q^TQ$:

\[1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n - 1}.\]

By the arithmetic-harmonic means inequality, we have

\[(5.32) \quad \left(\frac{1}{n - 1} \sum_{i=1}^{n-1} \lambda_i\right)^{-1} \leq \frac{1}{n - 1} \sum_{i=1}^{n-1} \frac{1}{\lambda_i} \leq \frac{1}{\min_{i \in [n - 1]} \lambda_i} \leq 1.\]

The double inequality in (5.28) follows from (5.32) by noting that

\[\kappa(G, \tilde{G}) = \sum_{i=1}^{n-1} \frac{1}{\lambda_i}\]

and

\[\sum_{i=1}^{n-1} \lambda_i = \text{Tr} (I_{n - 1} + Q^TQ) = n - 1 + \text{Tr} QQ^T = n - 1 + \sum_{k=1}^{c} (|O_k| - 1).\]

(B.2) Since rank $Q^TQ = c < n - 1$, by the interlacing theorem (cf. Theorem 4.3.4 of [19]), we have

\[(5.33) \quad 1 \leq \lambda_k (I_{n - 1} + Q^TQ) \leq \lambda_{k+c} (I_{n - 1}) = 1, \quad k \in [n - 1 - c].\]
For $k > n - 1 - c$, we use Weyl’s theorem to obtain
\[(5.34) \quad 1 \leq \lambda_k(I_{n-1} + Q^TQ) \leq 1 + \lambda_{n-1}(Q^TQ) = 1 + \lambda_c(QQ^T).\]

Using (5.10), by Gershgorin’s theorem, we further have
\[(5.35) \quad 1 + \lambda_c(QQ^T) \leq \max_{k \in [c]} \left\{|O_k| + \sum_{l \neq k} |O_k \cap O_l|\right\}.\]

The combination of (5.34) and (5.35) yields
\[(5.36) \quad \kappa(G, \hat{G}) = \sum_{k=1}^{n-1} \frac{1}{\lambda_k} \geq n - 1 + \frac{c}{\delta}.\]

(B.3) Since each cycle $O_k$ contains at least two edges from the spanning tree $\hat{G}$, then the number of disjoint cycles $c$ cannot exceed the integer part of $0.5(n - 1)$. In particular, $c < n - 1$.

By (5.10),
\[QQ^T = \text{diag}(|O_1| - 1, |O_2| - 1, \ldots, |O_c| - 1)\]

because the cycles are disjoint. Further, the nonzero eigenvalues of $Q^TQ$ and $QQ^T$ coincide. Thus,
\[(5.37) \quad \lambda_k ((I_{n-1} + Q^TQ)^{-1}) = \begin{cases} 1, & k \in [n - 1 - c], \\ |O_{k+c+1-n}|^{-1}, & n - c \leq k \leq n - 1. \end{cases}\]

By plugging (5.36) in (5.21), we obtain (5.31).

Remark 5.7. Estimates of $\kappa(G, \hat{G})$ in Lemma 5.6, combined with the estimates in Lemmas 5.4 and 5.5, show how stochastic stability of CPs depends on the geometric properties of the cycle subspace associated with the graph, such as the first Betti number (cf. (5.29)) and the length and the mutual position of the fundamental cycles (cf. (5.27), (5.28), (5.30), (5.31)). In particular, from (5.21) and the estimates in Lemma 5.6 one can see how the changes of the graph of the network, which do not affect a spanning tree, impact the stochastic stability of the CP. Likewise, by combining the statements in Lemma 5.6 with the estimate of the total effective resistance (cf. Lemma 5.4),
\[(5.37) \quad \rho(N) \leq \alpha(\hat{G}) \kappa(G, \hat{G}),\]

one can see how the properties of the spanning tree and the corresponding fundamental cycles contribute to the stochastic stability of the CP.

6. Network connectivity and performance of CPs. In the previous section, we derived several quantitative estimates characterizing convergence and stochastic stability of CPs. In this section, we discuss two examples illustrating how different structural features of the graph shape the dynamical properties of CPs. In the first pair of examples, we consider graphs of extreme degrees: 2 versus $n$. In the second example, we take two networks of equal degree but with disparate connectivity patterns: random versus symmetric. These examples show that both the degree of the network and its connectivity are important.
Example 6.1. Consider two simple networks supported by a path, $P_n$, and by a complete graph, $K_n$ (Figure 1(a),(b)). The coupling matrices of the corresponding CPs are given by

$$D_p = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 & 0 \\ 1 & -2 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \end{pmatrix}, \quad D_c = \begin{pmatrix} -n+1 & 1 & \ldots & 1 \\ 1 & -n+1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & -n+1 \end{pmatrix}. $$

The nonzero EVs of $P_n$ and $K_n$ are given by

$$\lambda_{i+1}(P_n) = 4\sin^2\left(\frac{\pi i}{2n}\right) \quad \text{and} \quad \lambda_{i+1}(K_n) = n, \quad i = 1, 2, \ldots, n-1. $$

Thus,

$$\alpha(P_n) = 4\sin^2\left(\frac{\pi}{2n}\right), \quad \alpha(K_n) = n, \quad \text{and} \quad \rho(K_n) = 1 - n^{-1}. $$

To compute $\rho(P_n)$, we use the formula for the total effective resistance of a tree (cf. equation (5) in [16]),

$$\rho(P_n) = n^{-1} \sum_{i=1}^{n-1} i(n - i) = \frac{1}{6}(n^2 - 1). $$

Equation (6.3) shows that for $n \gg 1$, the convergence rate of the CP based on the complete graph is much larger than that based on the path:

$$\alpha(K_n) = n \gg O(n^{-2}) = \alpha(P_n). $$

One may be inclined to attribute the disparity in the convergence rates to the fact that the degrees of the underlying graphs (and, therefore, the total number of edges) differ substantially. To see to what extent the difference of the total number of the edges, or, in electrical terms, the amount of wire used in the corresponding electrical circuits, can account for the mismatch in the rates of convergence, we scale the coupling matrices in Example 6.1 by the degrees of the corresponding graphs:

$$\tilde{D}_p = \frac{1}{2}D_p \quad \text{and} \quad \tilde{D}_c = \frac{1}{n-1}D_c. $$

Fig. 1. Examples of graphs used in the text: (a) a path, (b) a complete graph (see Example 6.1 for discussion of the properties of graphs in (a) and (b)), and (c) a star.
The algebraic connectivities of the rescaled networks are still far apart:

\[ \lambda_2(\tilde{D}_c) = 1 + O(n^{-1}) \gg O(n^{-2}) = \lambda_2(\tilde{D}_p). \]

This shows that the different values of \( \alpha \) in (6.5) reflect the distinct patterns of connectivity of these networks.

**Remark 6.2.** Explicit formulas for the EVs of the graph Laplacian similar to those that we used for the complete graph and the path are available for a few other canonical coupling architectures such as a cycle, a star, and an \( m \)-dimensional lattice (see, e.g., section 4.4 in [13]). Explicit examples of graphs with known EVs can be used for developing intuition on how the structural properties of the graphs translate to the dynamical properties of the corresponding CPs.

Equation (6.5) shows that the rate of convergence of CPs based on local nearest-neighbor interactions decreases rapidly when the network size grows. Therefore, this network architecture is very inefficient for coordination of large groups of agents. The following estimate shows that very slow convergence of the CPs based on a path for \( n \gg 1 \) is not specific to this particular network topology, but is typical for networks with regular connectivity. For graph \( G_n \) of degree \( d \) on \( n \) vertices, the following inequality holds [2]:

\[ \alpha(G_n) \leq 2d \left[ \frac{2 \log_2 n}{\text{diam}(G_n)} \right]^2. \]

This means that if the diameter of \( G_n \) grows faster than \( \log_2 n \) (as in the case of a path or any lattice), the algebraic connectivity, and, therefore, the convergence rate of the CP, goes to zero as \( n \to \infty \). Therefore, regular network topologies such as lattices result in poor performance of CPs. In contrast, below we show that a random network with high probability has a much better (in fact, nearly optimal) rate of convergence.

The algebraic connectivity of the (rescaled) complete graph does not decrease as the size of the graph goes to infinity (cf. (6.6)). There is another canonical network architecture, a star (see Figure 1(c)), whose algebraic connectivity remains unaffected by increasing the size of the network:

\[ \lambda_2(D_s) = 1. \]

However, both the complete graph and the star have disadvantages from the CP design viewpoint. CPs based on the complete graph are expensive because they require \( O(n^2) \) interconnections. The star uses only \( n - 1 \) edges, but the performance of the entire network critically depends on a single node, a hub, that connects to all other nodes. In addition, updating the information state of the hub requires simultaneous knowledge of the states of all other agents in the network. Therefore, neither the complete graph nor the star can be used for distributed consensus algorithms.

Ideally, one would like to have a family of sparse graphs that behaves like that of complete graphs in the sense that the algebraic connectivity remains bounded from zero uniformly:

\[ \alpha(G_n) \geq \bar{\alpha} > 0, \quad n \in \mathbb{N}. \]

Moreover, the greater the value of \( \bar{\alpha} \), the better the convergence of the corresponding CPs. Such graphs are called (spectral) expanders [18, 43]. Expanders can be used for producing CPs with a guaranteed rate of convergence regardless of the size of the network. There are several known explicit constructions of expanders, including the
celebrated Ramanujan graphs [27, 25] (see [18] for an excellent review of the theory and applications of expanders). In addition, random graphs are very good expanders. To explain this important property of random graphs, let us consider a family of graphs \( \{G_n\} \) on \( n \) vertices of fixed degree \( d \geq 3 \); i.e., \( G_n \) is an \((n,d)\)-graph. The following theorem due to Alon and Boppana yields an (asymptotic in \( n \)) upper bound on \( \alpha(G_n) \).

**Theorem 6.3** (see [35]). For any \((n,d)\)-graph \( G_n, d \geq 3, \) and \( n \gg 1 \),

\[
\alpha(G_n) \leq g(d) + o_n(1), \quad g(d) := d - 2\sqrt{d-1} > 0.
\]

Therefore, for large \( n \), \( \alpha(G_n) \) cannot exceed \( g(d) \) by more than a small margin. The following theorem of Friedman shows that for a random \((n,d)\)-graph \( G_n, \) \( \alpha(G_n) \) is nearly optimal with high probability.

**Theorem 6.4** (see [14]). For every \( \epsilon > 0 \),

\[
\text{Prob} \{ \alpha(G_n) \geq g(d) - \epsilon \} = 1 - o_n(1),
\]

where \( \{G_n\} \) is a family of random \((n,d)\)-graphs.

Theorem 6.4 implies that CPs based on random graphs exhibit fast convergence even when the number of dynamic agents grows unboundedly. Note that for \( n \gg 1 \), an \((n,d)\)-graph is sparse. Nonetheless, the CP based on a random \((n,d)\)-graph possesses the convergence speed that is practically as good as that of the normalized complete graph (cf. (6.6)). Therefore, random graphs provide a simple, practical way to design CPs that are efficient for coordinating large networks.

**Example 6.5.** In this example, we compare the performance of two CPs with regular and random connectivity.

(a) The former is a cycle on \( n \) vertices, \( C_n \). Each vertex of \( C_n \) is connected to \( d/2 \) (\( d \) is even) of its nearest neighbors from each side (see Figure 2(a)).

(b) The latter is a bipartite graph on \( 2m \) vertices, \( B_{2m} \). The edges are generated using the following algorithm:

1. Let \( p : [m] \to [m] \) be a random permutation. In our numerical experiments, we used MATLAB function \texttt{randperm} to generate random permutations. For \( i \in [m] \), add edge \((i, m + p(i))\).
2. Repeat step 1 \( d - 1 \) times.

In Figure 3, we present numerical simulations for CPs based on graphs in Example 6.5 for \( d = 4 \) and \( n \in \{100, 200, 400\} \). The rates of convergence for these CPs are summarized in Table 6.

The CP based on regular graphs \( C_n \) already has a very small rate of convergence for \( n = 100 \). As \( n \to \infty \), \( \alpha(C_n) \) tends to zero. In contrast, random graphs yield rates of convergence with very mild dependence on the size of the network. For the values of \( n \) used in this experiment, \( \alpha(B_n) \) \((n = \{1, 2, 4\} \times 10^3)\) are close to the optimal limiting
Numerical simulations of the CPs based on degree 4 graphs with regular and random connectivity (see Example 6.5 for the definitions of the graphs). Plots (a) and (b) show convergence of CPs for two networks. For each network architecture three network sizes were used: 100 (dashed line), 200 (dash-dotted line), and 400 (solid line). The graphs show the Euclidean norm of the trajectory of the reduced system $y$. The CP based on a random graph in (b) shows a much better convergence rate compared to that based on a regular graph in (a). Plots in (c) and (d) show the corresponding results for randomly perturbed CPs.

Table 1

<table>
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<th>$\alpha$ (\backslash) n</th>
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<th>200</th>
<th>400</th>
<th>$\infty$</th>
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<td>.005</td>
<td>.001</td>
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<td>$\alpha(B_\infty)$</td>
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<td>.554</td>
<td>.547</td>
<td>$4 - 2\sqrt{3} \approx 0.536$</td>
</tr>
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</table>

rate $\alpha(B_\infty) = 4 - 2\sqrt{3} \approx 0.536$. The difference in the convergence rates is clearly seen in the numerical simulations of the corresponding CPs (see Figure 3(a),(b)). The trajectories generated by CPs with random connections converge to the consensus subspace faster. We also conducted numerical experiments with the randomly perturbed equation (4.1) to compare the robustness to noise of the random and regular CPs. The CP on the random graph is more stable to random perturbations than the one on the regular graph (see Figure 3(c),(d)).

7. Conclusions. In this paper, we presented a unified approach to studying convergence and stochastic stability of a large class of CPs, including CPs on weighted directed graphs, CPs with both positive and negative conductances, time-dependent CPs, and those under stochastic forcing. We derived analytical estimates characterizing convergence of CPs and their stability to random perturbations. Our analysis shows how spectral and structural properties of the graph of the network contribute to stability of the corresponding CP. In particular, it suggests that the geometry of the cycle subspace associated with the graph of the CP plays an important role in shaping its stability. Further, we highlighted the advantages of using expanders, and, in particular, random graphs, for CP design.
The results of this paper elucidate the link between the structural properties of the graphs and dynamical performance of CPs. The theory of CPs is closely related to the theory of synchronization [6, 34, 39, 44]. With minimal modifications, the results of the present study carry over to a variety of coupled one-dimensional dynamical systems ranging from the models of power networks [12] to neuronal networks [30, 31, 32] and drift-diffusion models of decision making [40]. Moreover, the method of this paper naturally fits into a general scheme of analysis of synchronization in coupled systems of multidimensional nonlinear oscillators. The interested reader is referred to [28, 29, 31, 32] for related techniques and applications.

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