Statistics of irregular bursting

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Abstract

We study dynamical mechanisms and statistical properties of irregular bursting oscillations arising in a class of neuronal models of Hodgkin-Huxley type with and without noise. Specifically, we consider a phenomenological model of a square-wave bursting neuron in the regime close to the transition from tonic spiking to bursting. We identify two distinct mechanisms for generating irregular bursting oscillations. The first mechanism is based on chaotic spiking dynamics arising near the transition to bursting, while bursting oscillations generated by the second mechanism are induced by small random perturbations. For each case, we present a (randomly perturbed) Poincare map and analyze statistical properties of the trajectories of the discrete system. Our mathematical analysis suggests that the number of spikes within one burst are distributed approximately geometrically. However, the geometric distributions are determined by different factors depending on the underlying dynamical mechanism. In particular, the random perturbations have different effects when applied to the fast or the slow subsystem of the differential equation model.

1. The deterministic model

The $I_{NaP} + I_K + I_{KM}$ model [1]:

$$\begin{aligned} C\dot v &= F(v,n,w),\\ \tau_n^{-1}\dot n &= n_\infty(v) - n,\\ \tau_w^{-1}\dot w &= w_\infty(v) - w, \end{aligned}$$

where

$$\begin{aligned} F(v,n,w) = &- g_{NaP}m_\infty(v)(v-E_{NaP}) - g_K n(v-E_K)\\ &- \gamma w(v-E_K) - g_L(v-E_L) + I \end{aligned}$$

Control parameter: $\gamma := g_{KM}$

The fast subsystem:

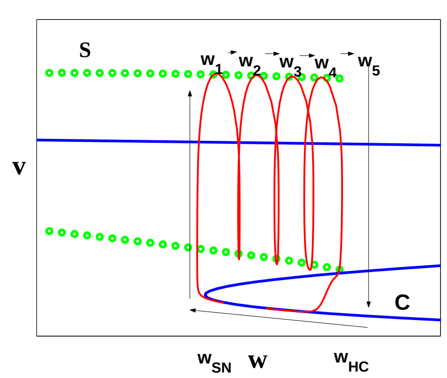
In the limit: $\epsilon := \tau_w^{-1} \to 0$

$$\begin{aligned} C\dot v &= F(v,n;w),\\ \tau_n^{-1}\dot n &= n_\infty(v) - n \end{aligned}$$

The slow subsystem:

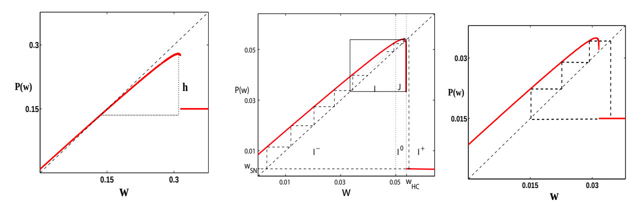
$$\dot w = \epsilon\,(w_\infty(v) - w)$$

2. Reduction to a 1D map [2]



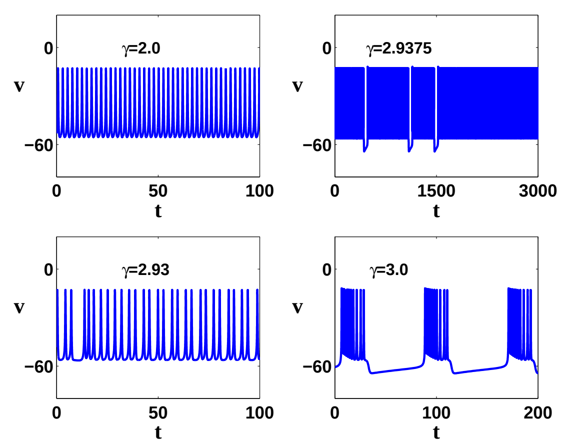
$$w_{n+1} = P_\gamma(w_n),\ n = 1, 2, \ldots.$$

3. Transition from tonic spiking to bursting

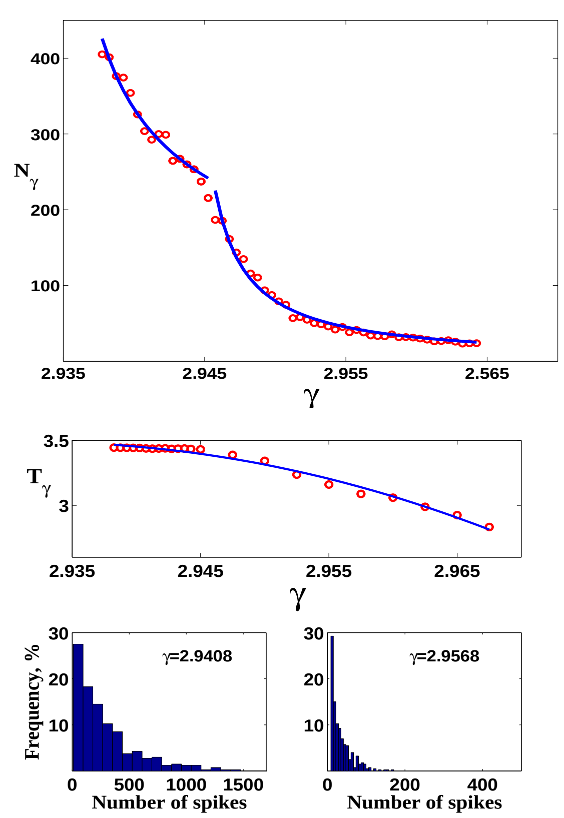


The first return maps corresponding to different regimes: tonic spiking (left, $\gamma = 0.5$), transition to bursting (center, $\gamma = 2.9$), bursting (right, $\gamma = 5.0$)

Irregular bursting



Statistics of irregular bursting [3]



N_γ and T_γ are the mean values of the number of spikes and the ISIs respectively

4. The randomly perturbed system

$$\begin{aligned} x &= (x_1, x_2) := (v, n) \quad \text{(fast variables)}\\ y &:= w \quad \text{(slow variable)} \end{aligned}$$

Model I (forcing via slow subsystem)

$$\begin{aligned} \dot x_t &= f(x_t, y_t)\\ \dot y_t &= \epsilon\left(g(x_t) - y_t + \sigma \dot W_t\right). \end{aligned}$$

Model II (forcing via fast subsystem)

$$\begin{aligned} \dot x_t &= f(x_t, y_t) + \sigma p \dot W_t,\\ \dot y_t &= \epsilon\,(g(x_t) - y_t),\ p = (p_1, p_2)^T, p_1 \in \{0,1\}, p_2 = 1 - p_1. \end{aligned}$$

5. Analysis

Model I

The randomly perturbed map:

$$\tilde P(y) = P(y) + \epsilon\sigma r, \quad \text{where } r = N(0, \sigma_r), \sigma_r = O(1)$$

The linear approximation of the $1D$ map:

$$\eta_n = (1 - O(\epsilon))\eta_{n-1} + \epsilon\sigma r_n, \quad n = 1, 2, 3, \ldots$$

The number of spikes within one burst is approximated by

$$\tau = \inf\{k \ge 1:\ \eta_k > h\}.$$

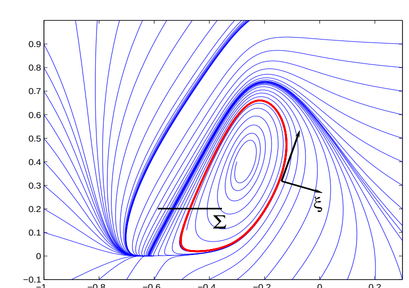
Statement: τ can be approximated by a geometric random variable with the parameter

$$p_\eta \sim \frac{C_1\sqrt{\epsilon}\sigma}{h} e^{\frac{-C_2 h^2}{\epsilon\sigma^2}}.$$

Model II

On finite time intervals, we have

$$\begin{aligned} x_t &= f(x_t, y_0) + \sigma p \dot W_t + O(\epsilon)\\ y_t &= y_0 + O(\epsilon) \end{aligned}$$



The linear approximation of the first-return map:

$$\begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix} = \begin{pmatrix} 1 - O(\epsilon) & O(\epsilon) \\ 0 & \mu\left(1 + \sigma r^{(1)}_{n-1}\right) \end{pmatrix} \begin{pmatrix} \eta_{n-1} \\ \xi_{n-1} \end{pmatrix} + \begin{pmatrix} \epsilon\sigma r^{(2)}_{n-1} \\ \sigma r^{(3)}_{n-1} \end{pmatrix}$$

Random variables

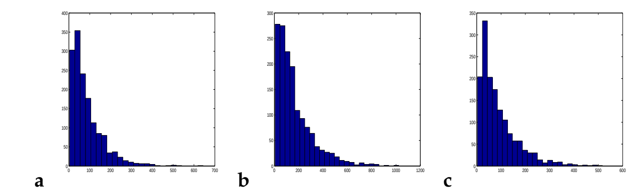
$$\tau_\xi = \inf\{k \ge 1:\ \xi_k > h_\xi\} \quad \text{and} \quad \tau_\eta = \inf\{k \ge 1:\ \eta_k > h_\eta\}.$$

are approximately geometric with parameters

$$p_\xi \sim \frac{C_1\sigma}{h_\xi} e^{\frac{-C_2 h_\xi^2}{\sigma^2}} \text{ and } p_\eta \sim \frac{C_1\sqrt{\epsilon}\sigma}{h_\eta} e^{\frac{-C_2 h_\eta^2}{\epsilon\sigma^2}}.$$

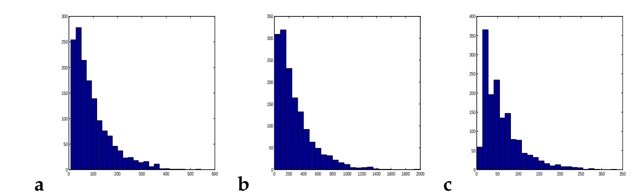
6. Histograms

Model I



Histograms for the number of spikes within one burst : (a) $\gamma = 0.8$, $\sigma = 0.01$, (b) $\gamma = 0.5$, $\sigma = 0.01$, (c) $\gamma = 0.5$, $\sigma = 0.0251$.

Model II



Histograms for the number of spikes within one burst : (a) $\gamma = 0.8$, $\sigma = 0.01$, (b) $\gamma = 0.5$, $\sigma = 0.01$, (c) $\gamma = 0.5$, $\sigma = 0.0251$.

Acknowledgments

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References

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