CONVERGENCE OF THE BORN AND INVERSE BORN SERIES FOR ELECTROMAGNETIC SCATTERING

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Abstract. We consider the Born and Inverse Born series that arise in the theory of electromagnetic scattering and related inverse problems. We establish sufficient conditions for convergence of both series.

1. Introduction

Scattering theory is a topic of fundamental interest and considerable applied importance [2]. Integral equation methods have played a key role in the development of the subject, especially in the physics literature, where they have been used to construct series solutions to the problem of scattering from inhomogeneous media. In this paper, we consider the scattering of time-harmonic electromagnetic waves in a nonmagnetic medium. The electric field E obeys the wave equation

$$\nabla \times \nabla \times E - k^2 \epsilon(x) E = 0, \tag{1}$$

where k is the free-space wavenumber and ϵ is the dielectric permittivity. The total field E can be decomposed into its incident and scattered components E^i and E^s:

$$E = E^i + E^s, \tag{2}$$

where E^i obeys (1) with $\epsilon = 1$. Evidently, E^s obeys the equation

$$\nabla \times \nabla \times E^s - k^2 E^s = k^2 \eta(x) E, \tag{3}$$

where $\eta = \epsilon - 1$. Suppose that η is supported in B_a, the ball of radius a centered at the origin. Then it is known, see [6] that (3), combined with the Silver-Müller radiation condition

$$\lim_{|x|\to\infty} |x| \left[(\nabla \times E^s) \times \frac{x}{|x|} - ikE^s \right] = 0, \tag{4}$$

is equivalent to the integral equation

$$E^s(x) = (k^2 + \nabla\nabla\cdot) \int_{B_a} G(x,y)\eta(y)E(y)dy. \tag{5}$$

Here G is the fundamental solution for the Helmholtz equation:

$$G(x,y) = \frac{\exp(ik|x-y|)}{4\pi|x-y|}. \tag{6}$$

The field E thus obeys the integral equation

$$E(x) = E^i(x) + (k^2 + \nabla\nabla\cdot) \int_{B_a} G(x,y)\eta(y)E(y)dy. \tag{7}$$

If we apply fixed point iteration to (7) beginning with $E = E^i$, we obtain an infinite series for E of the form

$$E = E^i + E^{(1)} + E^{(2)} + \cdots, \tag{8}$$

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where

$$E^{(j+1)}(x) = \int_{B_a} G(x,y)\eta(y)E^{(j)}(y)dy. \tag{9}$$

We will refer to (8) as the Born series and the approximation that results from retaining only the linear term in the series as the Born approximation.

The main result of this paper is the following theorem, which gives a sufficient condition for the convergence of the Born series.

Theorem 1. *Let $K \Subset \mathbb{R}^3 \setminus \overline{B}_a$ and suppose that the smallness condition $\|\eta\|_{L^\infty(B_a)} < 1/\mu$ is satisfied. Then the Born series (8) converges in $[L^2(K)]^3$ and the error estimate*

$$\left\| E^s - \sum_{j=1}^{N} E^{(j)} \right\|_{[L^2(K)]^3} \leq \frac{\nu}{\mu} \frac{\left(\mu\|\eta\|_{L^\infty(B_a)}\right)^{N+1}}{\left(1-\mu\|\eta\|_{L^\infty(B_a)}\right)}$$

holds, where

$$\mu = \frac{17}{2}(ka)^2 + 2\sqrt{74}ka + 105,$$

$$\nu = |B_a|^{1/2} \left\|E^i\right\|_{[L^2(B_a)]^3} \sup_{x\in B_a} \left\|(k^2 + \nabla_{x_1}\nabla_{x_1}\cdot)G(\cdot,x)I\right\|_{[L^2(K)]^3}.$$

The inverse scattering problem is to recover the susceptibility η from measurements of the scattered field E^s. The inverse Born series is a reconstruction method that has been applied to a variety of inverse scattering problems [7, 9, 10, 4, 5, 1], including the case of the Maxwell equations [7]. Here, as an application of the method we develop to analyze the Born series, we obtain a sufficient condition for convergence of the inverse Born series.

This paper is organized as follows. In Section 2, the proof of Theorem 1 on the convergence of the Born series is presented. In Section 3 we discuss the convergence of the inverse Born series for electromagnetic scattering.

2. Born Series

It will prove useful to rewrite the Born series as a power series in tensor powers of η of the form

$$E^s = K_1\eta + K_2\eta\otimes\eta + K_3\eta\otimes\eta\otimes\eta + \cdots. \tag{10}$$

Here the operators forward operators K_n are defined by

$$(K_1\eta)(x) = (k^2 + \nabla\nabla\cdot)\int_{B_a} G(x,y)\eta(y)E^i(y)dy, \tag{11}$$

$$(K_2\eta\otimes\eta)(x) = (k^2 + \nabla_x\nabla_x\cdot)\int_{B_a} \eta(y_1)G(x,y_1) \cdot \left[(k^2 + \nabla_{y_1}\nabla_{y_1}\cdot)\int_{B_a} \eta(y_2)E^i(y_2)G(y_1,y_2)dy_2\right] dy_1, \tag{12}$$

and, in general, K_n is the multilinear form

$$(K_n\eta\otimes\ldots\otimes\eta)(x) =(k^2+\nabla\nabla\cdot)\int\limits_{B_a} G(x,y_1)\eta(y_1)(k^2+\nabla_{y_1}\nabla_{y_1}\cdot) \cdot\int\limits_{B_a}\eta(y_2)G(y_1,y_2)\cdots(k^2+\nabla_{y_{n-1}}\nabla\cdot_{y_{n-1}}) \cdot\int\limits_{B_a}\eta(y_n)G(y_{n-1},y_n)E^i(y_n)dy_1\cdots dy_n. \quad (13)$$

In the above, the point x will be taken to belong to a compact set which lies outside the support of η.

We now consider the convergence of the series (10). To this end, we represent the multilinear form K_n as compositions of linear operators. We define the operator $S:[L^2(B_a)]^3\to[L^2(\mathbb{R}^3)]^3$ by

$$S(v)=(k^2+\nabla\nabla\cdot)\int\limits_{B_a} G(x,y)v(y)dy. \quad (14)$$

We also define the operator $A_\eta:[L^2(B_a)]^3\to[L^2(B_a)]^3$ by

$$A_\eta(v)=(k^2+\nabla\nabla\cdot)\int\limits_{B_a} G(x,y)\eta(y)v(y)dy. \quad (15)$$

Then, we note that

$$K_1\eta=S(\eta E^i)$$
$$K_2\eta\otimes\eta=S(\eta(A_\eta E^i)),$$

and in general,

$$K_n\eta\otimes\cdots\otimes\eta=S(\eta(A_\eta^{n-1}E^i)). \quad (16)$$

Remark 1. *While K_n is most naturally defined as a multilinear form with n input functions of a single variable, it is easily extended to operate on a single function of n variables. That is, if we replace the product $\eta(y_1)\cdots\eta(y_n)$ in (13) by a function $f(y_1,\ldots,y_n)$ in the innermost integral, we can view K_n as an operator on $L^\infty(B_a\times\cdots\times B_a)$.*

By making use of (16), we can obtain a bound on the operator K_n by estimating the norm of A_η. We begin by defining the operator V:

$$(Vf)(x)=\int\limits_{B_a} G(x,y)f(y)dy. \quad (17)$$

Note that

$$A_\eta v=(k^2+\nabla\nabla\cdot)V(\eta v). \quad (18)$$

The following result gives an estimate on the norm of V.

Lemma 1. *The operator $V:[L^2(B_a)]^3\to[L^2(B_a)]^3$ defined by (17) is bounded and*

$$\|Vf\|_{[L^2(B_a)]^3}\le\frac{a^2}{2}\|f\|_{[L^2(B_a)]^3}.$$

Proof. From Young's inequality for convolutions, we have

$$\|Vf\|_{[L^2(B_a)]^3} \leq \|G\|_{L^1(B_a)} \|f\|_{[L^2(B_a)]^3}.$$

In addition, we have

$$\|G\|_{L^1(B_a)} \leq \frac{a^2}{2},$$

from which the result follows. □

Since V is two orders smoothing, the operator A_η is bounded. An estimate on the norm of A_η is provided in the following proposition.

Proposition 1. *Suppose that $\eta \in L^\infty(B_a)$. The operator $A_\eta : [L^2(B_a)]^3 \to [L^2(B_a)]^3$ defined by (18) is bounded and*

$$\|A_\eta v\|_{[L^2(B_a)]^3} \leq \mu \|\eta\|_{L^\infty(B_a)} \|v\|_{[L^2(B_a)]^3},$$

where

$$\mu = \frac{17}{2}(ka)^2 + 2\sqrt{74}ka + 105. \tag{19}$$

Proof. Recall that

$$\begin{aligned}\|A_\eta v\|_{[L^2(B_a)]^3} &= \left\|(k^2 + \nabla\nabla\cdot)V(\eta v)\right\|_{[L^2(B_a)]^3} \\ &\leq k^2 \|V(\eta v)\|_{[L^2(B_a)]^3} + \|\nabla\nabla \cdot V(\eta v)\|_{[L^2(B_a)]^3}\end{aligned}$$

Making use of Lemma 1 we have

$$\|A_\eta v\|_{[L^2(B_a)]^3} \leq \frac{(ka)^2}{2} \|\eta v\|_{[L^2(B_a)]^3} + \|\nabla\nabla \cdot V(\eta v)\|_{[L^2(B_a)]^3}. \tag{20}$$

Furthermore, since

$$\|\eta v\|_{[L^2(B_a)]^3} \leq \|\eta\|_{L^\infty(B_a)} \|v\|_{[L^2(B_a)]^3}, \tag{21}$$

we can combine (20) with (21) to obtain

$$\|A_\eta v\|_{[L^2(B_a)]^3} \leq \frac{(ka)^2}{2} \|\eta\|_{L^\infty(B_a)} \|v\|_{[L^2(B_a)]^3} + \|\nabla\nabla \cdot V(\eta v)\|_{[L^2(B_a)]^3}. \tag{22}$$

Next, we require an estimate of the form

$$\|\nabla\nabla \cdot Vf\|_{[L^2(B_a)]^3} \leq C \|f\|_{[L^2(B_a)]^3},$$

which establishes the boundedness of the operator $\nabla\nabla \cdot V$ on $[L^2(B_a)]^3$. To proceed, we follow the technique used in [8] of smoothing V by truncating it in Fourier space. We will use the following definitions of the Fourier transform $\hat{u}(\xi) = \mathcal{F}u$ and the inverse Fourier transform $u = \mathcal{F}^*\hat{u}$:

$$\hat{u}(\xi) = \mathcal{F}u(x) = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x} u(x)dx$$

and

$$u(x) = \mathcal{F}^*\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i\xi\cdot x} \hat{u}(\xi)d\xi.$$

We first note that the Fourier symbol for $\nabla\nabla\cdot$ is given by the matrix D with entries $D_{ij} = 4\pi^2\xi_i\xi_j$. We note that the matrix norm of D is given by

$$\|D\| = 4\pi^2 |\xi|^2. \tag{23}$$

Recall that for a differential operator $\mathcal{P}$ a parametrix $\mathcal{G}$ is defined to be a linear operator such that

$$\mathcal{P}\mathcal{G}u = u - \mathcal{K}_1 u, \tag{24}$$
$$\mathcal{G}\mathcal{P}u = u - \mathcal{K}_2 u, \tag{25}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are compact operators. That is, $\mathcal{G}$ is the inverse of the differential operator $\mathcal{P}$ modulo a smoothing operator. In our case, $\mathcal{P}$ is given by the Helmholtz operator $\Delta + k^2$. Then, the Fourier symbol for $\mathcal{P}$, denoted by P, is given by

$$P(\xi) = 4\pi^2 |\xi|^2 - k^2.$$

Since $1/P(\xi)$ blows up when $|\xi| = k/2\pi$, we introduce a Fourier cutoff. Note that for $|\xi| > k/\pi$,

$$\left|P(\xi)^{-1}\right| = \left|\frac{1}{4\pi^2 |\xi|^2 - k^2}\right| \leq \frac{1}{3\pi^2 |\xi|^2}.$$

Using this fact, we introduce the cutoff function χ:

$$\chi(\xi) = 1 \quad \text{for} \quad |\xi| < \frac{k}{\pi}, \quad \chi(\xi) = 0 \text{ otherwise }. \tag{26}$$

We then define

$$\mathcal{G}u(x) = \mathcal{F}^* \left[(1 - \chi(\xi))P(\xi)^{-1}\hat{u}(\xi)\right]. \tag{27}$$

It is easily verified that $\mathcal{G}$ defines a parametrix for $\mathcal{P}$, that is, it satisfies (24) and (25) with

$$\mathcal{K}_1 u = \mathcal{K}_2 u = \mathcal{F}^* (\chi \hat{u}).$$

We will therefore put

$$\mathcal{K}u = \mathcal{F}^* (\chi \hat{u}). \tag{28}$$

In addition to a frequency cutoff, we will also need a spatial cutoff function. Given $f \in [L^2(B_a)]^3$, we choose a ball B_{2a} and extend f so that it vanishes on B_{2a}. We also extend the operator V, as defined by (17), accordingly. Thus

$$u = Vf \in H^2(B_{2a}), \tag{29}$$

satisfies

$$\mathcal{P}u = f \text{ on } B_{2a}. \tag{30}$$

We also introduce a third ball $B_{(3a)/2}$. Evidently,

$$\overline{B_a} \subset B_{(3a)/2} \quad \text{and} \quad \overline{B_{(3a)/2}} \subseteq B_{2a}.$$

We denote by A the outer annulus

$$A = B_{2a} \setminus \overline{B_{(3a)/2}}.$$

We then define the spatial cutoff function $\chi_1 \in H_0^2(B_{2a})$ by $\chi_1 = 1$ on $B_{(3a)/2}$ and $0 \leq \chi_1 \leq 1$. Note that we can construct χ_1 explicitly as a radially varying piecewise C^1 cubic function:

$$\chi_1 = 1 \text{ in } B_{(3a)/2}, \quad \chi_1 = (\frac{4\rho}{a} - 5)(-\frac{2\rho}{a} + 4)^2 \text{ in } A, \tag{31}$$

and $\chi_1 = 0$ elsewhere. Here $\rho = |x|$ is the distance from the origin. We then find that

$$|\nabla \chi_1| \leq \frac{3}{a}, \quad |\Delta \chi_1| \leq \frac{24}{a^2}. \tag{32}$$

Note that from (24), (25) and (30) we have

$$\begin{aligned}\chi_1 u - \mathcal{K}(\chi_1 u) &= \mathcal{G}\mathcal{P}(\chi_1 u)\\ &= \mathcal{G}(\chi_1 \mathcal{P}u) + \mathcal{G}[\mathcal{P}\chi_1 u - \chi_1 \mathcal{P}u]\\ &= \mathcal{G}(f) + \mathcal{G}[\mathcal{P}\chi_1 u - \chi_1 \mathcal{P}u],\end{aligned}$$

since $\mathcal{P}u = f$ and $\chi_1 = 1$ on the support of f. We thus find that

$$\begin{aligned}
\|\nabla\nabla\cdot u\|_{[L^2(B_a)]^3} &= \|\nabla\nabla\cdot\chi_1 u\|_{[L^2(B_a)]^3} \\
&= \|\nabla\nabla\cdot(\mathcal{K}(\chi_1 u)+\mathcal{G}(f)+\mathcal{G}[\mathcal{P}\chi_1 u-\chi_1\mathcal{P}u])\|_{[L^2(B_a)]^3}\,. \qquad (33)
\end{aligned}$$

We now examine each term of (33) individually. First, we have by (28) that

$$\|\nabla\nabla\cdot\mathcal{K}(\chi_1 u)\|_{[L^2(B_a)]^3} = \|\nabla\nabla\cdot\mathcal{F}^*(\chi\widehat{\chi_1 u})\|_{[L^2(B_a)]^3}\,,$$

and

$$\begin{aligned}
\|\nabla\nabla\cdot\mathcal{F}^*(\chi\widehat{\chi_1 u})\|^2_{[L^2(B_a)]^3} &= \int_{\mathbb{R}^3}|D(\xi)\chi\widehat{\chi_1 u}|^2 d\xi \\
&\le \int_{|\xi|<k/\pi}(4\pi^2|\xi|^2)^2|\widehat{\chi_1 u}|^2 d\xi,
\end{aligned}$$

which follows from the definition of the Fourier cutoff (26) and the matrix norm of the Fourier multiplier D (23). Therefore, we have

$$\begin{aligned}
\|\nabla\nabla\cdot\mathcal{F}^*(\chi\widehat{\chi_1 u})\|^2_{[L^2(B_a)]^3} &\le (4k^2)^2\,\|\widehat{\chi_1 u}\|^2_{[L^2(\mathbb{R}^3)]^3} \\
&= (4k^2)^2\,\|\chi_1 u\|^2_{[L^2(\mathbb{R}^3)]^3} \\
&\le (4k^2)^2\,\|u\|^2_{[L^2(B_a)]^3}\,.
\end{aligned}$$

The above shows that

$$\begin{aligned}
\|\nabla\nabla\cdot\mathcal{K}(\chi_1 u)\|_{[L^2(B_a)]^3} &\le 4k^2\,\|u\|_{[L^2(B_{2a})]^3} \qquad (34)\\
&\le 2(k2a)^2\,\|f\|_{[L^2(B_a)]^3} \qquad (35)\\
&= 8(ka)^2\,\|f\|_{[L^2(B_a)]^3}\,, \qquad (36)
\end{aligned}$$

where we have used Lemma 1 on the ball B_{2a} and the fact that f is supported on B_a. Next, we consider the second term of (33). We calculate that

$$\begin{aligned}
\|\nabla\nabla\cdot\mathcal{G}f\|^2_{[L^2(B_a)]^3} &= \int_{\mathbb{R}^3}|(1-\chi(\xi))|P(\xi)^{-1}D(\xi)\widehat{f}(\xi)|^2 d\xi \\
&\le \int_{|\xi|\ge k/\pi}(4\pi^2|\xi|^2)^2|P(\xi)^{-1}\widehat{f}(\xi)|^2 d\xi \\
&\le \int_{|\xi|\ge k/\pi}\frac{(4\pi^2|\xi|^2)^2}{(3\pi^2|\xi|^2)^2}|\widehat{f}(\xi)|^2 d\xi \\
&\le \frac{16}{9}\,\|f\|^2_{[L^2(\mathbb{R}^3)]^3} \\
&= \frac{16}{9}\,\|f\|^2_{[L^2(B_a)]^3} \qquad (37)
\end{aligned}$$

where we have used (2) and the fact that the support of f lies in B_a. Finally, we consider the last term in (33), for which we require the following lemma.

Lemma 2. *Let $\mathcal{P} = \Delta + k^2$, $\mathcal{G}$ the parametrix defined by (27), χ_1 the spatial cutoff given by (31), and $u = Vf$. We then have*

$$\|\nabla\nabla\cdot\mathcal{G}(\mathcal{P}\chi_1 u-\chi_1\mathcal{P}u)\|_{[L^2(B_a)]^3} \le 2\sqrt{74}(ka+6)\|f\|_{[L^2(B_a)]^3}. \qquad (38)$$

Proof. By the same argument as in (37), we have that

$$\begin{aligned}
\|\nabla\nabla\cdot\mathcal{G}(\mathcal{P}\chi_1 u-\chi_1\mathcal{P}u)\|_{[L^2(B_a)]^3} &\le 2\|\mathcal{P}\chi_1 u-\chi_1\mathcal{P}u\|_{L^2(\mathbb{R}^3)} \\
&= 2\|\mathcal{P}\chi_1 u-\chi_1\mathcal{P}u\|_{L^2(A)} \qquad (39)
\end{aligned}$$

since χ_1 only varies in the region A. We calculate directly that

$$\mathcal{P}\chi_1 u - \chi_1 \mathcal{P} u = 2\nabla\chi_1 \cdot \nabla u + \Delta\chi_1 u. \tag{40}$$

Using (32), we obtain

$$\begin{aligned}
\|2\nabla\chi_1 \cdot \nabla u + \Delta\chi_1 u\|_{L^2(A)} &\leq \frac{24}{a^2}\|u\|_{L^2(A)} + \frac{6}{a}\|\nabla u\|_{L^2(A)} \\
&\leq \frac{24}{a^2}|A|^{1/2}\|u\|_{L^\infty(A)} + \frac{6}{a}|A|^{1/2}\|\nabla u\|_{L^\infty(A)} \\
&\leq \left[\frac{24}{a^2}|A|^{1/2}|B_a|^{1/2}\|G\|_\infty + \frac{6}{a}|A|^{1/2}|B_a|^{1/2}\|\nabla G\|_\infty\right]\|f\|_{L^2(B_a)}. \qquad (41)
\end{aligned}$$

Here

$$\begin{aligned}
\|G\|_\infty &= \sup_{x\in A}\sup_{y\in B_a}|G(x,y)|\ , \\
\|\nabla G\|_\infty &= \sup_{x\in A}\sup_{y\in B_a}|\nabla_x G(x,y)|\ .
\end{aligned}$$

Recall that $A = B_{2a} \setminus \overline{B_{(3a)/2}}$. Hence

$$|A|^{1/2} = \left(\frac{37\pi}{6}\right)^{1/2} a^{3/2}.$$

We also have

$$\|G\|_\infty \leq \left|\frac{1}{4\pi \mathrm{dist}(A, B_a)}\right| = \frac{1}{2\pi a},$$

and

$$\begin{aligned}
\|\nabla G\|_\infty &\leq \left|\frac{k}{4\pi\mathrm{dist}(A,B_a)} + \frac{1}{4\pi\mathrm{dist}(A,B_a)^2}\right| \\
&\leq \frac{k}{4\pi(1/2)a} + \frac{1}{4\pi(1/4)a^2} \\
&\leq \frac{1}{\pi}\left(\frac{k}{2a} + \frac{1}{a^2}\right).
\end{aligned}$$

Substituting the above into (41), we see that

$$\begin{aligned}
\|2\nabla\chi_1 \cdot \nabla u + \Delta\chi_1 u\|_{L^2(A)} &\leq \left[24a\pi\left(\frac{74}{9}\right)^{1/2}\frac{1}{2\pi a} + 6a^2\pi\left(\frac{74}{9}\right)^{1/2}\left(\frac{1}{\pi}\left(\frac{k}{2a} + \frac{1}{a^2}\right)\right)\right]\|f\|_{L^2(B_a)} \\
&\leq \sqrt{74}(ka+6)\,\|f\|_{L^2(B_a)}\,.
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
\|\nabla\nabla\cdot\mathcal{G}(\mathcal{P}\chi_1 u - \chi_1\mathcal{P}u)\|_{L^2(B_a)^3} &\leq 2\|2\nabla\chi_1\cdot\nabla u + \Delta\chi_1 u\|_{L^2(A)} \\
&\leq 2\sqrt{74}(ka+6)\,\|f\|_{L^2(B_a)}\,.
\end{aligned}$$

□

Combining (34), (37) and Lemmas 1 and 2, (33) becomes

$$\begin{aligned}
\|\nabla\nabla\cdot Vf\|_{[L^2(B_a)]^3} &= \|\nabla\nabla\cdot u\|_{[L^2(B_a)]^3} \\
&\leq \|\nabla\nabla\cdot\mathcal{K}(\chi_1 u)\|_{[L^2(B_a)]^3} + \|\nabla\nabla\cdot\mathcal{G}(\chi_1 f)\|_{[L^2(B_a)]^3} \\
&\quad + \|\nabla\nabla\cdot\mathcal{G}[\mathcal{P}\chi_1 u - \chi_1\mathcal{P}u]\|_{[L^2(B_a)]^3} \\
&\leq (8(ka)^2 + 4/3 + 2\sqrt{74}(ka+6)\|f\|_{[L^2(B_a)]^3} \\
&= \left(8(ka)^2 + 2\sqrt{74}ka + 105\right)\|f\|_{[L^2(B_a)]^3}\,. \qquad (42)
\end{aligned}$$

Using the above estimate and (22), we obtain the result stated in Proposition 1. ☐

We are now prepared to derive a bound on the norm of the forward operator K_n.

Proposition 2. *Let $K \Subset \mathbb{R}^3 \setminus \overline{B}_a$. Then the operator*

$$K_n : L^\infty(B_a \times \cdots \times B_a) \longrightarrow [L^2(K)]^3$$

defined by (13) is bounded and

$$\|K_n \eta_1 \otimes \cdots \otimes \eta_n\|_{[L^2(K)]^3} \leq \nu\mu^{n-1}\,\|\eta_1\|_{L^\infty(B_a)}\cdots\|\eta_n\|_{L^\infty(B_a)},$$

where

$$\nu = |B_a|^{1/2}\left\|E^i\right\|_{[L^2(B_a)]^3}\sup_{x\in B_a}\left\|(k^2+\nabla_{x_1}\nabla_{x_1}\cdot)G(\cdot,x)I\right\|_{[L^2(K)]^3} \qquad (43)$$

and μ is defined by (19).

Remark 2. *The L^2 norm of the above matrix valued function is defined to be the L^2 norm of the 3×3 Euclidean (or spectral) matrix norm.*

Proof. We begin by bounding K_1. From (14), we have that

$$K_1\eta = S(\eta E^i)$$

Note that

$$\begin{aligned}
|S(\eta E^i)(x_1)| &= \left|(k^2+\nabla_{x_1}\nabla_{x_1}\cdot)\int_{B_a}\eta(y_1)E^i(y_1)G(x_1,y_1)dy_1\right| \\
&\leq \left\|E^i\right\|_{[L^2(B_a)]^3}\left(\int_{B_a}\|\eta(y_1)(k^2+\nabla_{x_1}\nabla_{x_1}\cdot)G(x_1,y_1)I\|^2 dy_1\right)^{1/2},
\end{aligned}$$

where I is the 3×3 identity matrix, the divergence of the matrix GI is taken column wise, the norm inside the integral is the Euclidean 3×3 matrix norm, and we have used the Cauchy-Schwarz inequality. We then have

$$\begin{aligned}
\left\|S(\eta E^i)\right\|^2_{[L^2(B_a)]^3} &\leq \left\|E^i\right\|^2_{[L^2(K)]^3}\int_K\left(\int_{B_a}\|\eta(y_1)(k^2+\nabla_{x_1}\nabla_{x_1}\cdot)G(x_1,y_1)I)\|^2 dy_1\right)dx_1 \\
&\leq \left\|E^i\right\|^2_{[L^2(B_a)]^3}|B_a|\,\|\eta\|^2_{L^\infty(B_a)}\sup_{x\in B_a}\left\|\|(k^2+\nabla_{x_1}\nabla_{x_1}\cdot)G(\cdot,x)I\|\right\|^2_{[L^2(K)]^3} \\
&= \left\|E^i\right\|^2_{[L^2(B_a)]^3}|B_a|\,\|\eta\|^2_{L^\infty(B_a)}\sup_{x\in B_a}\left\|(k^2+\nabla_{x_1}\nabla_{x_1}\cdot)G(\cdot,x)I\right\|^2_{[L^2(K)]^3},
\end{aligned}$$

where for convenience, we have not indicated the matrix norm in the last step. We thus find that

$$\begin{aligned}
\|K_1\eta\| &= \left\|S(\eta E^i)\right\|_{[L^2(K)]^3} \\
&\leq \nu\,\|\eta\|_{L^\infty(B_a)}\,.
\end{aligned}$$

From Proposition 1 we obtain

$$\|A_\eta v\|_{[L^2(B_a)]^3} \leq \mu\,\|\eta\|_{L^\infty(B_a)}\,\|v\|_{[L^2(B_a)]^3}\,. \qquad (44)$$

By the same argument, we find that

$$
\begin{aligned}
\|K_2\eta\otimes\eta\|_{[L^2(K)]^3)} &= \left\|S(\eta A_\eta(E^i))\right\|_{[L^2(K)]^3} \\
&\le |B_a|^{1/2}\left\|A_\eta(E^i)\right\|_{[L^2(B_a)]^3}\|\eta\|_{L^\infty(B_a)}\sup_{x\in B_a}\left\|(k^2+\nabla_{x_1}\nabla_{x_1}\cdot)G(\cdot,x)I\right\|_{[L^2(K)]^3)} \\
&\le \mu\|\eta\|_{L^\infty(B_a)}\|E_i\|_{[L^2(B_a)]^3}|B_a|^{1/2}\|\eta\|_{L^\infty(B_a)}\sup_{x\in B_a}\left\|(k^2+\nabla_{x_1}\nabla_{x_1}\cdot)G(\cdot,x)I\right\|_{[L^2(K)]^3} \\
&\le \mu\nu\|\eta\|^2_{L^\infty(B_a)}
\end{aligned}
$$

Applying this bound repeatedly, we obtain

$$
\begin{aligned}
\|K_n\eta\otimes\ldots\otimes\eta\|_{[L^2(K)]^3)} &= \left\|S(\eta(A_\eta^{n-1}(E^i)))\right\|_{[L^2(K)]^3} \\
&\le \nu\mu^{n-1}\|\eta\|^n_{L^\infty(B_a)}.
\end{aligned}
$$

□

Remark 3. *The bounds presented in Proposition 2 are most likely not optimal. In the above proof it was necessary to construct two cutoff functions; one in space and the other in Fourier space. The choice made here for these functions was somewhat arbitrary and leads to bounds which are not likely to be sharp.*

We can now apply Proposition 2 to prove Theorem 1.

Proof of Theorem 1.

To show convergence of the Born series (10), we majorize the sum

$$
\sum_j \|(K_j\eta\otimes\ldots\otimes\eta)\|_{[L^2(K)]^3}
$$

by a geometric series

$$
\sum_j \|(K_j\eta\otimes\ldots\otimes\eta)\|_{[L^2(K)]^3} \le \frac{\nu}{\mu}\sum_j\left(\mu\|\eta\|_{L^\infty(B_a)}\right)^j,
$$

which converges if $\mu\|\eta\|_{[L^\infty(B_a)]^3} < 1$.

To derive the error estimate we note that

$$
\begin{aligned}
\left\|E^s-\sum_{j=1}^N K_j\eta\otimes\cdots\otimes\eta\right\|_{[L^2(K)]^3} &\le \sum_{j=N+1}^\infty \|K_j\eta\otimes\cdots\otimes\eta\|_{[L^2(K)]^3} \\
&\le \sum_{j=N+1}^\infty \nu\mu^{j-1}\|\eta\|^j_{L^p(B_a)} \\
&= \frac{\nu}{\mu}\frac{\left(\mu\|\eta\|_{L^\infty(B_a)}\right)^{N+1}}{1-\mu\|\eta\|_{L^\infty(B_a)}}.
\end{aligned}
$$

□

3. Inverse Born Series

The inverse scattering problem is to recover the susceptibility η from measurements of the scattered field E^s. The inverse Born series is a reconstruction method that has been applied to a variety of inverse scattering problems, including the case of the Maxwell equations [7]. As shown in [7], the solution to the inverse problem can be expressed as a power series of the form

$$\eta = \mathcal{K}_1 E^s + \mathcal{K}_2 E^s \otimes E^s + \mathcal{K}_3 E^s \otimes E^s \otimes E^s + \cdots , \tag{45}$$

where

$$\begin{aligned}
\mathcal{K}_1 &= K_1^+, \\
\mathcal{K}_2 &= -\mathcal{K}_1 K_2 \mathcal{K}_1 \otimes \mathcal{K}_1, \\
\mathcal{K}_3 &= -(\mathcal{K}_2 K_1 \otimes K_2 + \mathcal{K}_2 K_2 \otimes K_1 + \mathcal{K}_1 K_3)\mathcal{K}_1 \otimes \mathcal{K}_1 \otimes \mathcal{K}_1, \\
\mathcal{K}_j &= -\left(\sum_{m=1}^{j-1} \mathcal{K}_m \sum_{i_1+\ldots+i_m=j} K_{i_1} \otimes \cdots \otimes K_{i_m} \right) \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_1.
\end{aligned}$$

Here K_1^+ denotes the regularized pseudoinverse of K_1. In [7], the inverse Born series (45) was applied to the inverse problem for the Maxwell equations in the near-field. Numerical reconstructions were reported, but the convergence of the method was not analyzed.

By applying the results of [9], we obtain here a sufficient condition for convergence of the inverse Born series.

Theorem 2. *The inverse Born series (45) converges in the L^2 norm if $\|\mathcal{K}_1\| < 1/(\mu+\nu)$ and $\|\mathcal{K}_1 E^s\|_{L^\infty(B_a)} < 1/(\mu+\nu)$, where μ and ν are defined in (19) and (43), respectively. Furthermore, if $\tilde{\eta}$ is the limit of (45), the following error estimate holds:*

$$\left\| \tilde{\eta} - \sum_{j=1}^{N} \mathcal{K}_j E^s \otimes \cdots \otimes E^s \right\|_{L^2(B_a)} \leq C \frac{\Big((\mu+\nu) \, \|\mathcal{K}_1 E^s\|_{L^\infty(B_a)} \Big)^{N+1}}{1 - (\mu+\nu) \, \|\mathcal{K}_1 E^s\|_{L^\infty(B_a)}}, \tag{46}$$

where C depends on $\|\mathcal{K}_1\|$.

Here by $\|\mathcal{K}_1\|$ we mean the operator norm from $[L^2(K)]^3$ into $L^\infty(B_a)$.

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