Inverse Born Series for Diffuse Waves

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ABSTRACT. We consider the inverse scattering problem for diffuse waves. We analyze the convergence of the inverse Born series and study its use in numerical simulations for the case of a spherically-symmetric absorbing medium in three dimensions.

1. Introduction

Optical tomography is an emerging biomedical imaging modality that uses diffuse light to probe structural variations in the optical properties of tissue [1]. The associated inverse scattering problem for diffuse waves consists of recovering the spatially-varying absorption of the interior of a domain from boundary measurements. The standard approach to this problem is formulated in terms of the minimization of a nonlinear functional. Such an approach gives rise to image reconstruction algorithms that, at present, are not well understood mathematically. In previous work, we have shown that, to some extent, it is possible to fill this gap by employing methods which invert the Born series [3]. The resulting image reconstruction algorithms are fast, direct, and have analyzable convergence, stability and approximation error [4].

In this paper we extend the results of [4] in two regards. First, we obtain a stronger result on the convergence of the inverse Born series. Second, we study numerically the convergence of the method for a medium with spherical symmetry. In this setting, which is effectively one dimensional, it is possible to obtain the scattering data exactly.

The remainder of this paper is organized as follows. In Section 2, we develop the scattering theory of diffuse waves in an inhomogeneous medium—this corresponds to the forward problem of optical tomography. The inversion of the Born series is taken up in Section 3, where we also obtain our main result on the convergence of the inverse series. In Section 4, we consider the forward problem in the radial case, compute the scattering data for an annular inhomogeneity and present the results of numerical reconstructions.

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2. Forward Problem

2.1. Diffuse waves. We consider a bounded domain Ω in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$, in which a diffuse wave propagates. The energy density u of the wave is taken to satisfy the time-independent diffusion equation

$$-\nabla^2 u(x) + k^2(1 + \eta(x))u(x) = 0, \quad x \in \Omega, \quad (2.1)$$

where the diffuse wavenumber k is a positive constant and $\eta(x) \geq -1$ for all $x \in \Omega$. The function η is the spatially varying part of the absorption coefficient which is assumed to be supported in a closed ball B_a of radius a . The energy density is also taken to obey the boundary condition

$$u(x) + \ell \nu \cdot \nabla u(x) = 0, \quad x \in \partial\Omega, \quad (2.2)$$

where ν is the unit outward normal to $\partial\Omega$ and the extrapolation length ℓ is a nonnegative constant. Note that k and η are related to the absorption and reduced scattering coefficients μ_a and μ'_s by $k = \sqrt{3\bar{\mu}_a\mu'_s}$ and $\eta(x) = \delta\mu_a(x)/\bar{\mu}_a$, where $\delta\mu_a(x) = \bar{\mu}_a - \mu_a(x)$ and $\bar{\mu}_a$ is constant [2].

The forward problem of optical tomography is to determine the energy density u for a given absorption η . If the medium is illuminated by a point source, the solution to the forward problem is given by the integral equation

$$u(x) = u_i(x) - k^2 \int_{\Omega} G(x, y) u(y) \eta(y) dy, \quad x \in \Omega. \quad (2.3)$$

Here u_i is the energy density of the incident diffuse wave which obeys the equation

$$-\nabla^2 u_i(x) + k^2 u_i(x) = \delta(x - x_1), \quad x \in \Omega, \quad x_1 \in \partial\Omega \quad (2.4)$$

and G is the Green's function for the operator $-\nabla^2 + k^2$, where G obeys the boundary condition (2.2).

The integral equation (2.3) has a unique solution. If we apply fixed point iteration (beginning with u_i), we obtain an infinite series for u of the form

$$\begin{aligned} u(x) &= u_i(x) - k^2 \int_{\Omega} G(x, y) \eta(y) u_i(y) dy \\ &\quad + k^4 \int_{\Omega \times \Omega} G(x, y) \eta(y) G(y, y') \eta(y') u_i(y') dy dy' + \cdots \end{aligned} \quad (2.5)$$

We will refer to (2.5) as the Born series and the approximation to u that results from retaining only the linear term in η as the Born approximation.

It will prove useful to express the Born series as a formal power series in tensor powers of η of the form

$$\phi = K_1 \eta + K_2 \eta \otimes \eta + K_3 \eta \otimes \eta \otimes \eta + \cdots, \quad (2.6)$$

where $\phi = u_i - u$. Physically, the scattering data $\phi(x_1, x_2)$ is proportional to the change in intensity measured by a point detector at $x_2 \in \partial\Omega$ due to a point source at $x_1 \in \partial\Omega$ [3]. Each term in the series is multilinear in η and the operator K_j is

defined by

$$\begin{aligned} (K_j \eta \otimes \cdots \otimes \eta)(x_1, x_2) = \\ (-1)^{j+1} k^{2j} \int_{B_a \times \cdots \times B_a} G(x_1, y_1) G(y_1, y_2) \cdots G(y_{j-1}, y_j) G(y_j, x_2) \\ \cdot \eta(y_1) \cdots \eta(y_j) dy_1 \cdots dy_j, \end{aligned} \quad (2.7)$$

for $x_1, x_2 \in \partial\Omega$. It is shown in [4] that the operator ¹

$$K_j : L^2(B_a) \otimes \cdots \otimes L^2(B_a) \longrightarrow L^2(\partial\Omega \times \partial\Omega) \quad (2.8)$$

is bounded and

$$\|K_j\| \leq \nu \mu^{j-1}, \quad (2.9)$$

where

$$\mu = \sup_{x \in B_a} k^2 \|G(x, \cdot)\|_{L^2(B_a)}, \quad (2.10)$$

$$\nu = k^2 |B_a|^{1/2} \sup_{x \in B_a} \|G(x, \cdot)\|_{L^2(\partial\Omega)}^2. \quad (2.11)$$

3. Inverse Series

The inverse scattering problem is to determine the absorption coefficient η everywhere within Ω from measurements of the scattering data ϕ on $\partial\Omega \times \partial\Omega$. Towards this end, we formally express η as a series in tensor powers of ϕ of the form

$$\eta = \mathcal{K}_1 \phi + \mathcal{K}_2 \phi \otimes \phi + \mathcal{K}_3 \phi \otimes \phi \otimes \phi + \cdots, \quad (3.1)$$

where, as shown in [4], the operators $\mathcal{K}_j$ are given by

$$\mathcal{K}_1 = K_1^+, \quad (3.2)$$

$$\mathcal{K}_2 = -\mathcal{K}_1 K_2 \mathcal{K}_1 \otimes \mathcal{K}_1, \quad (3.3)$$

$$\mathcal{K}_3 = -(\mathcal{K}_2 K_1 \otimes K_2 + \mathcal{K}_2 K_2 \otimes K_1 + \mathcal{K}_1 K_3) \mathcal{K}_1 \otimes \mathcal{K}_1 \otimes \mathcal{K}_1, \quad (3.4)$$

$$\mathcal{K}_j = - \left(\sum_{m=1}^{j-1} \mathcal{K}_m \sum_{i_1 + \cdots + i_m = j} K_{i_1} \otimes \cdots \otimes K_{i_m} \right) \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_1. \quad (3.5)$$

We will refer to (3.1) as the inverse scattering series. Here we note several of its properties. First, K_1^+ is the regularized pseudoinverse of the operator K_1 . Since the operator $\mathcal{K}_1$ is unbounded, regularization of K_1^+ is required to control the ill-posedness of the inverse problem. Second, the coefficients in the inverse series have a recursive structure. Third, the operator $\mathcal{K}_j$ is determined by the coefficients of the Born series $K_1, K_2, \dots, K_j$. Finally, inversion of only the linear term in the Born series is required to compute the inverse series to all orders. Thus an ill-posed nonlinear inverse problem is reduced to an ill-posed linear inverse problem plus a well-posed nonlinear problem, namely the computation of the higher order terms in the series.

¹We note that in [4], Lemma 2.2 is formulated in terms of the space $L^2(B_a \times \cdots \times B_a)$ which contains $L^2(B_a) \otimes \cdots \otimes L^2(B_a)$ as a subspace.

We now proceed to study the convergence of the inverse series. We begin with an estimate on the norm of the operator $\mathcal{K}_j$.

LEMMA 3.1. *Let $(\mu + \nu)\|\mathcal{K}_1\| < 1$. Then the operator*

$$\mathcal{K}_j : L^2(\partial\Omega \times \partial\Omega) \otimes \cdots \otimes L^2(\partial\Omega \times \partial\Omega) \longrightarrow L^2(B_a)$$

defined by (3.5) is bounded and

$$\|\mathcal{K}_j \phi \otimes \cdots \otimes \phi\|_{L^2(B_a)} \leq C(\mu + \nu)^j \|\mathcal{K}_1 \phi\|_{L^2(B_a)}^j ,$$

where $C = C(\mu, \nu, \|\mathcal{K}_1\|)$ is independent of j .

PROOF. Using (3.5), we see that

$$\begin{aligned} \|\mathcal{K}_j \phi \otimes \cdots \otimes \phi\|_{L^2(B_a)} &\leq \sum_{m=1}^{j-1} \sum_{i_1 + \cdots + i_m = j} \|\mathcal{K}_m\|_p \|K_{i_1}\|_p \cdots \|K_{i_m}\|_p \|\mathcal{K}_1 \phi\|_{L^2(B_a)}^j \\ &\leq \|\mathcal{K}_1 \phi\|_{L^2(B_a)}^j \sum_{m=1}^{j-1} \sum_{i_1 + \cdots + i_m = j} \|\mathcal{K}_m\| \\ &\quad \cdot \nu \mu^{i_1-1} \cdots \nu \mu^{i_m-1} , \end{aligned} \tag{3.6}$$

where we have used (2.9) to obtain the second inequality. Next, we define $\Pi(j, m)$ to be the number of ordered partitions of the integer j into m parts. It can be seen that

$$\Pi(j, m) = \binom{j-1}{m-1} . \tag{3.7}$$

It follows that

$$\begin{aligned} \|\mathcal{K}_j \phi \otimes \cdots \otimes \phi\|_{L^2(B_a)} &\leq \|\mathcal{K}_1 \phi\|_{L^2(B_a)}^j \sum_{m=1}^{j-1} \|\mathcal{K}_m\| \Pi(j, m) \nu^m \mu^{j-m} \\ &\leq \|\mathcal{K}_1 \phi\|_{L^2(B_a)}^j \left(\sum_{m=1}^{j-1} \|\mathcal{K}_m\| \right) \left(\sum_{m=1}^{j-1} \Pi(j, m) \nu^m \mu^{j-m} \right) \\ &\leq \nu \|\mathcal{K}_1 \phi\|_{L^2(B_a)}^j \left(\sum_{m=1}^{j-1} \|\mathcal{K}_m\| \right) \left(\sum_{m=0}^{j-1} \binom{j-1}{m} \nu^m \mu^{j-1-m} \right) \\ &= \nu \|\mathcal{K}_1 \phi\|_{L^2(B_a)}^j (\mu + \nu)^{j-1} \sum_{m=1}^{j-1} \|\mathcal{K}_m\| . \end{aligned} \tag{3.8}$$

Thus

$$\|\mathcal{K}_j \phi \otimes \cdots \otimes \phi\|_{L^2(B_a)} \leq (\mu + \nu)^j \|\mathcal{K}_1 \phi\|_{L^2(B_a)}^j \sum_{m=1}^{j-1} \|\mathcal{K}_m\|_p . \tag{3.9}$$

In [4] it was shown that

$$\|\mathcal{K}_j\| \leq C[(\mu + \nu)\|\mathcal{K}_1\|]^j \|\mathcal{K}_1\| , \tag{3.10}$$

where C is independent of j . Thus

$$\|\mathcal{K}_j \phi \otimes \cdots \otimes \phi\|_{L^2(B_a)} \leq C(\mu + \nu)_2^j \|\mathcal{K}_1 \phi\|_{L^2(B_a)}^j \frac{\|\mathcal{K}_1\|}{1 - (\mu + \nu)\|\mathcal{K}_1\|} . \tag{3.11}$$

This completes the proof. $\square$

The following theorem establishes the convergence of the inverse series. A similar result with stronger hypotheses was given in [4].

THEOREM 3.2. *The inverse scattering series converges in the L^2 norm if $\|\mathcal{K}_1\| < 1/(\mu + \nu)$ and $\|\mathcal{K}_1\phi\|_{L^2(B_a)} < 1/(\mu + \nu)$.*

PROOF. The series $\sum_j \mathcal{K}_j \phi \otimes \cdots \otimes \phi$ converges in norm if

$$\sum_j \|\mathcal{K}_j \phi \otimes \cdots \otimes \phi\|_{L^p(B_a)} \leq C \sum_j [(\mu + \nu) \|\mathcal{K}_1 \phi\|_{L^2(B_a)}]^j, \quad (3.12)$$

converges, which follows from Lemma 3.1. The right hand side of (3.12) is a geometric series which converges when

$$(\mu + \nu) \|\mathcal{K}_1 \phi\|_{L^2(B_a)} < 1. \quad (3.13)$$

$\square$

4. Numerical Reconstructions

4.1. Radial problem. We now consider the forward problem for the case of three-dimensional media which vary only in the radial direction. In this case, Ω is assumed to be a sphere of radius R centered at the origin, and the coefficient $\eta = \eta(r)$, where r is the radial coordinate in spherical coordinates. The Green's function is then given by [2]

$$G(x, y) = \sum_{l=0}^{\infty} \sum_{m=-l}^l g_l(x, y) Y_{lm}(\hat{x}) Y_{lm}^*(\hat{y}). \quad (4.1)$$

Here Y_{lm} are spherical harmonics, i_l, k_l are modified spherical Bessel functions, $x = (|x|, \hat{x})$, $y = (|y|, \hat{y})$, $r_< = \min(|x|, |y|)$ and $r_> = \max(|x|, |y|)$. The radial Green's function is given by

$$g_l(x, y) = \frac{2k}{\pi} \left(k_l(kr_>) i_l(kr_<) - \frac{k_l(kR) + k\ell k'_l(kR)}{i_n(kR) + k\ell i'_l(kR)} i_l(k|x|) i_l(k|y|) \right). \quad (4.2)$$

Note that when $y \in \partial\Omega$, the functions g_l have the simplified form

$$\begin{aligned} g_l(x, y) &= g_l(x) \\ &= \frac{\ell}{R^2} \frac{i_l(k|x|)}{i_l(kR) + k\ell i'_l(kR)}. \end{aligned}$$

We can now calculate the first term in the Born series. Employing spherical coordinates on the boundary, with $x_1 = (R, \hat{x}_1)$ and $x_2 = (R, \hat{x}_2)$, we obtain

$$\phi^{(1)}(\hat{x}_1, \hat{x}_2) = k^2 \int_{\Omega} \sum_{\substack{l_1, m_1 \\ l_2, m_2}} Y_{l_1 m_1}(\hat{x}_1) Y_{l_2 m_2}^*(\hat{x}_2) g_{l_1}(x) g_{l_2}(x) \eta(x) dx. \quad (4.3)$$

From this we find the generalized Fourier coefficients

$$\begin{aligned}\phi_{l_1, m_1, l_2, m_2}^{(1)} &= k^2 \int_{S^2 \times S^2} Y_{l_1 m_1}^*(\hat{x}_1) Y_{l_2 m_2}(\hat{x}_2) \phi^{(1)}(\hat{x}_1, \hat{x}_2) d\hat{x}_1 d\hat{x}_2 \\ &= k^2 \delta_{l_1 l_2} \delta_{m_1 m_2} \int_0^R g_{l_1}(r) g_{l_2}(r) r^2 \eta(r) dr ,\end{aligned}\quad (4.4)$$

where we have used the orthogonality of the spherical harmonics and the radial dependence of η . Note that the above expression is independent of m_1 and m_2 , so that we can define a singly indexed family of coefficients

$$\begin{aligned}\phi_m^{(1)} &= \phi_{m, m_1, m, m_2}^{(1)} \\ &= k^2 \int_0^R (g_m(r))^2 r^2 \eta(r) dr .\end{aligned}\quad (4.5)$$

We now rescale the above coefficients and put

$$\psi_m^{(1)} = \left(\frac{R^2}{\ell} \right)^2 (i_m(kR) + k\ell i'_m(kR))^2 \phi_m^{(1)} .\quad (4.6)$$

We thus obtain

$$\psi_m^{(1)} = k^2 \int_0^R (i_m(kr))^2 r^2 \eta(r) dr .\quad (4.7)$$

Consider now a general term in the forward series,

$$\begin{aligned}\phi^{(n)}(x_1, x_2) &= (K_n \eta \otimes \cdots \otimes \eta)(x_1, x_2) \\ &= (-1)^{n+1} k^{2n} \int_{\Omega \times \cdots \times \Omega} G(x_1, y_1) \cdots G(y_n, x_2) \\ &\quad \cdot \eta(y_1) \cdots \eta(y_n) dy_1 \cdots dy_n .\end{aligned}\quad (4.8)$$

Using (4.1) and restricting to the boundary, we can calculate the generalized Fourier coefficients

$$\begin{aligned}\phi_{l_1, m_1, l_2, m_2}^{(n)} &= \int_{S^2 \times S^2} Y_{l_1 m_1}^*(\hat{x}_1) Y_{l_2 m_2}(\hat{x}_2) \phi^{(n)}(\hat{x}_1, \hat{x}_2) d\hat{x}_1 d\hat{x}_2 \\ &= (-1)^{n+1} k^{2n} \delta_{l_1, l_2} \delta_{m_1, m_2} \int_0^R \cdots \int_0^R g_{l_1}(r_1) g_{l_1}(r_1, r_2) \\ &\quad \cdots g_{l_1}(r_{n-1}, r_n) g_{l_2}(r_n) r_1^2 \eta(r_1) \cdots r_n^2 \eta(r_n) dr_1 \cdots dr_n .\end{aligned}\quad (4.9)$$

Once again, we singly index the coefficients

$$\phi_m^{(n)} = \phi_{m, m_1, m, m_2}^{(n)} ,\quad (4.10)$$

and using the same rescaling we set

$$\psi_m^{(n)} = \left(\frac{R^2}{\ell} \right)^2 (i_m(kR) + k\ell i'_m(kR))^2 \phi_m^{(n)} ,\quad (4.11)$$

so that we have the formula

$$\begin{aligned}\psi_m^{(n)} &= (-1)^{n+1} k^{2n} \int_0^R \cdots \int_0^R i_m(kr_1) g_m(r_1, r_2) \\ &\quad \cdots g_m(r_{n-1}, r_n) i_m(kr_n) r_1^2 \eta(r_1) \cdots r_n^2 \eta(r_n) dr_1 \cdots dr_n .\end{aligned}\quad (4.12)$$

REMARK 4.1. The above formula is equivalent to the n th term in the Born series (2.6) for radial η .

4.2. Scattering data. We now compute the forward scattering data for the annulus by separation of variables. To proceed, we put

$$\eta(r) = \begin{cases} 0 & 0 \leq r \leq R_1 \\ \eta_1 & R_1 < r \leq R_2 \\ 0 & R_2 < r \leq R \end{cases} . \quad (4.13)$$

The solution will be constructed in each subdomain of Ω . The first subdomain is the inner sphere

$$\Omega_1 = \{x : |x| \leq R_1\} , \quad (4.14)$$

the second is the inner annulus

$$\Omega_2 = \{x : R_1 < |x| \leq R_2\} , \quad (4.15)$$

and the third is the outer annulus

$$\Omega_3 = \{x : R_2 < |x| \leq R\} . \quad (4.16)$$

The diffusion equation (2.1) and the boundary condition (2.2) leads to the system of PDEs:

$$-\nabla^2 u_1 + k^2 u_1 = 0 \quad \text{in } \Omega_1 , \quad (4.17)$$

$$-\nabla^2 u_2 + k_1^2 u_2 = 0 \quad \text{in } \Omega_2 , \quad (4.18)$$

$$-\nabla^2 u_3 + k^2 u_3 = \delta(x - x_0) \quad \text{in } \Omega_3 , \quad (4.19)$$

$$u_1 = u_2 \quad \text{on } \partial\Omega_1 , \quad (4.20)$$

$$u_2 = u_3 \quad \text{on } \partial\Omega_2 , \quad (4.21)$$

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \quad \text{on } \partial\Omega_1 , \quad (4.22)$$

$$\frac{\partial u_2}{\partial \nu} = \frac{\partial u_3}{\partial \nu} \quad \text{on } \partial\Omega_2 , \quad (4.23)$$

$$u_3 + l \frac{\partial u_3}{\partial \nu} = 0 \quad \text{on } \partial\Omega_3 , \quad (4.24)$$

where we have imposed the condition that u and $\partial u / \partial \nu$ must be continuous across each interface. Here $x_0 \in \partial\Omega$ is a source and $k_1^2 = k^2(1 + \eta_1)$.

We can express the solution to the above system as

$$u_1(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_l i_l(kr) Y_{lm}(\hat{x}) Y_{lm}^*(\hat{x}_0) , \quad (4.25)$$

$$u_2(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^l (b_l k_l(k_1 r) + c_l i_l(k_1 r)) Y_{lm}(\hat{x}) Y_{lm}^*(\hat{x}_0) , \quad (4.26)$$

$$u_3(x) = G_0(x, x_0) + \sum_{l=0}^{\infty} \sum_{m=-l}^l (d_l k_l(kr) + e_l i_l(kr)) Y_{lm}(\hat{x}) Y_{lm}^*(\hat{x}_0) . \quad (4.27)$$

Here G_0 is the fundamental solution to the diffusion equation which is given by

$$G_0(x, x_0) = \frac{1}{4\pi} \frac{e^{-k|x-x_0|}}{|x-x_0|} \quad (4.28)$$

$$= \frac{2k}{\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^l i_l(kr) k_l(kR) Y_{lm}(\hat{x}) Y_{l,m}^*(\hat{x}_0), \quad (4.29)$$

where $x = (r, \hat{x})$ and $x_0 = (R, \hat{x}_0)$. To determine the coefficients a_l, b_l, c_l, d_l, e_l , we apply the interface and boundary conditions to arrive at the following system of linear equations:

$$\begin{bmatrix} i_l(kR_1) & -k_l(k_1R_1) & -i_l(k_1R_1) & 0 & 0 \\ k_l'(kR_1) & -k_1k_l'(k_1R_1) & -k_1i_l'(k_1R_1) & 0 & 0 \\ 0 & k_l(k_1R_2) & i_l(k_1R_2) & -k_l(kR_2) & -i_l(kR_2) \\ 0 & k_1k_l'(k_1R_2) & k_1i_l'(k_1R_2) & -kk_l'(kR_2) & -ki_l'(kR_2) \\ 0 & 0 & 0 & k_l(kR) + klk_l'(kR) & i_l(kR) + kli_l'(kR) \end{bmatrix} \begin{bmatrix} a_l \\ b_l \\ c_l \\ d_l \\ e_l \end{bmatrix} = \frac{2k}{\pi} \begin{bmatrix} 0 \\ 0 \\ i_l(kR_2)k_l(kR) \\ k_l'(kR_2)k_l(kR) \\ -i_l(kR)k_l(kR) - kli_l(kR)k_l'(kR) \end{bmatrix} \quad (4.30)$$

To calculate the scattering data ϕ , we rewrite the series expansion for the Green's function in the form

$$G(x, x_0) = G_0(x, x_0) + \sum_{m,l} f_l i_l(kr) Y_{lm}(\hat{x}) Y_{lm}^*(\hat{x}_0), \quad (4.31)$$

where

$$f_l = -\frac{2k}{\pi} \frac{i_l(kR)k_l(kR) + kl i_l(kR)k_l'(kR)}{i_l(kR) + kl i_l'(kR)}. \quad (4.32)$$

This yields the data function for $x \in \partial\Omega$,

$$\phi(\hat{x}, \hat{x}_0) = \sum_{m,l} ((f_l - e_l)i_l(kR) - d_l k_l(kR)) Y_{lm}^*(\hat{x}) Y_{lm}(\hat{x}_0). \quad (4.33)$$

Next, we compute the generalized Fourier coefficients

$$\begin{aligned} \phi_{l_1 m_1}^{l_2 m_2} &= \int_{S^2 \times S^2} Y_{l_1 m_1}(\hat{x}_1) Y_{l_2 m_2}^*(\hat{x}_2) \phi(\hat{x}_1, \hat{x}_2) d\hat{x}_1 d\hat{x}_2 \\ &= \delta_{l_1 l_2} \delta_{m_1 m_2} ((f_l - e_l)i_l(kR) - d_l k_l(kR)), \end{aligned} \quad (4.34)$$

and set

$$\begin{aligned} \phi_m &= \phi_{mm_1}^{mm_2} \\ &= (f_m - e_m)i_m(kR) - d_m k_m(kR). \end{aligned} \quad (4.35)$$

FIGURE 1. Illustrating the use of the inverse scattering series.

Finally, we rescale ϕ_m as in (4.6) and define

$$\psi_m = \left(\frac{R^2}{l} \right)^2 (i_m(kR) + k l i'_m(kR))^2 \phi_m \quad (4.36)$$

$$= \left(\frac{R^2}{l} \right)^2 (i_m(kR) + k l i'_m(kR))^2 ((f_m - e_m) i_m(kR) - d_m k_m(kR)) \quad (4.37)$$

The quantity ψ_m is the data we will use to reconstruct the coefficient η .

4.3. Numerical results. We now present the results of numerical reconstructions for the model system defined by (4.13). When computing the terms of the inverse series, we use recursion to implement the formula (3.5). The scattering data is computed from (4.37). The forward operators are implemented using the formula (4.12). We compute the pseudo-inverse $\mathcal{K}_1 = K_1^+$, by using MATLAB's built-in singular value decomposition. Since the singular values of K_1 are exponentially small, we set all but the largest $M = 13$ singular values to zero, and make use of only the first M .

We take the domain to be the ball of radius $R = 3$ cm, the extrapolation length $\ell = 0.3$ cm, and the background wavenumber to be $k = 1 \text{ cm}^{-1}$. When computing both (3.5) and (4.37), we use $m = 90$ modes and discretize the integral operators on

a spatial grid of 90 uniformly-spaced nodes in the radial direction. We found that increasing the number of modes did not significantly change the reconstructions.

Figure 1 show a series of reconstructions where $k_1 = 1.5$, $R_1 = 1.5$ and $R_2 = 1.3, 1.5, 1.7, 1.9$. In each case, five terms in the inverse series are computed. We also show the projection of η onto the subspace generated by the first M singular vectors, which gives a sense for what can be reconstructed at low frequencies, for a particular regularization. Note that for relatively small inhomogeneities, the series appears to converge quite rapidly to a reconstruction that is close to the projection. As the size of the inhomogeneity is increased, the higher order terms make significant improvements to the linear reconstructions. A precise comparison of the theory (convergence criteria and error estimates) and numerics will be presented elsewhere.

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