Abstract. We consider the inverse scattering problem for diffuse waves. We analyze the convergence of the inverse Born series and study its use in numerical simulations for the case of a spherically-symmetric absorbing medium in two and three dimensions.

1. Introduction

Optical tomography is a recently proposed imaging modality that uses diffuse light to probe structural variations in the optical properties of random media [1–3]. The associated inverse scattering problem for diffuse waves consists of recovering the spatially-varying absorption of the interior of a domain from boundary measurements. The standard approach to this problem is often framed in terms of nonlinear optimization. Such an approach gives rise to image reconstruction algorithms that, at present, are not well understood mathematically. In previous work, we have shown that methods which invert the Born series [4, 5] can, in principle, fill this gap. The resulting image reconstruction algorithms are fast, direct, and have analyzable convergence, stability and approximation error [5].

In this paper we extend our previous results [4,5] by studying numerically the convergence of the inverse Born series for a medium with radial symmetry. We make use of exact solutions to the forward problem to calculate the scattering data. Results in both two and three dimensions are presented and reconstructions are computed up to fifth order in the inverse series. We find that the series appears to converge quite rapidly for low contrast objects. As the contrast is increased, the higher order terms systematically improve the reconstructions until, at sufficiently large contrast, the series diverges.

The remainder of this paper is organized as follows. In Section 2, we develop the scattering theory of diffuse waves in an inhomogeneous medium—this corresponds to the forward problem of optical tomography. We then specialize to the case of a medium with spherical symmetry and obtain estimates on the norms of the forward scattering operators. The inversion of the Born series is taken up in Section 3. In Section 4, we consider the forward problem in the radial case, compute the scattering data for a spherical inhomogeneity and present the results of numerical reconstructions. Some details of the derivation of estimates on the norms of the forward scattering operators are presented in the Appendix.

2. Forward Problem

2.1. General case. We consider the propagation of a diffuse wave in a bounded domain \( \Omega \subset \mathbb{R}^3 \) with a smooth boundary \( \partial \Omega \). The energy density \( u \) of the wave is taken to satisfy
the time-independent diffusion equation with a point source

$$-\nabla^2 u(x) + k^2 (1 + \eta(x)) u(x) = \delta(x - x_1), \quad x \in \Omega$$  \hspace{1cm} (1)

$$u(x) + \ell \nu \cdot \nabla u(x) = 0, \quad x \in \partial \Omega.$$  \hspace{1cm} (2)

Here $x_1$ is the position of the point source on $\partial \Omega$, $\eta(x)$ is the perturbation of the background absorption coefficient, $\nu$ is the outward unit normal to $\partial \Omega$ and the extrapolation length $\ell$ is a positive constant. The function $\eta$ is assumed to be supported in a closed ball $B_a$ of radius $a$.

We can express the solution to (1) as the solution to the integral equation

$$u(x) = u_i(x) - k^2 \int_{\Omega} G(x, y) u(y) \eta(y) dy, \quad x \in \Omega.$$  \hspace{1cm} (3)

where $u_i$ is the energy density of the incident diffuse wave which solves the equation

$$-\nabla^2 u_i(x) + k^2 u_i(x) = \delta(x - x_1), \quad x \in \Omega, \quad x_1 \in \partial \Omega.$$  \hspace{1cm} (4)

Here $G$ is the Green’s function for the operator $-\nabla^2 + k^2$, where $u_i$ and $G$ obey the boundary condition (2). The integral equation (3) has a unique solution $u$. By beginning with the incident wave $u_i$, we can apply fixed point iteration to obtain the well known Born series for $u$

$$u(x) = u_i(x) - k^2 \int_{\Omega} G(x, y) \eta(y) u_i(y) dy$$

$$+ k^4 \int_{\Omega \times \Omega} G(x, y) \eta(y) G(y, y') \eta(y') u_i(y') dy dy' + \cdots,$$  \hspace{1cm} (5)

which we will call the forward scattering series. We also express this series as a formal power series in tensor powers of $\eta$ of the form

$$\phi = K_1 \eta + K_2 \eta \otimes \eta + K_3 \eta \otimes \eta \otimes \eta + \cdots,$$  \hspace{1cm} (6)

where $\phi = u_i - u$ is the scattering data. Each term in the series is multilinear in $\eta$ and the operator $K_j$ is defined by

$$\left( K_j f \right) (x_1, x_2) = (-1)^{j+1} k^{2j} \int_{B_a \times \cdots \times B_a} G(x_1, y_1) G(y_1, y_2) \cdots$$

$$\times G(y_{j-1}, y_j) G(y_j, x_2) f(y_1, \ldots, y_j) dy_1 \cdots dy_j,$$  \hspace{1cm} (7)

for $x_1, x_2 \in \partial \Omega$. In [5] we showed that for $2 \leq p \leq \infty$, the operators

$$K_j : L^p(B_a \times \cdots \times B_a) \to L^p(\partial \Omega \times \partial \Omega)$$  \hspace{1cm} (8)

are bounded and their norms obey the estimate

$$\| K_j \| \leq \nu_p \mu_p^{j-1},$$  \hspace{1cm} (9)

where $\nu_p$ and $\mu_p$ are constants.
2.2. **Two-dimensional radial problem.** We now derive the Born series for the case of a two-dimensional medium which varies only in the radial direction. Here $\Omega$ is assumed to be a disk of radius $R$ centered at the origin and $\eta = \eta(r)$, where $r$ is the radial coordinate. The Green’s function for this problem is given by

$$
G(x, y) = \frac{1}{2\pi} \sum_{n=0}^{\infty} e^{in(\theta_x - \theta_y)} g_n(x, y) ,
$$

where

$$
g_n(x, y) = K_n(kr_>)I_n(kr_<) - \frac{K_n(kR) + k\ell K'_n(kR)}{I_n(kR) + k\ell I'_n(kR)} I_n(k|x|)I_n(k|y|) .
$$

Here have used the notation $\theta_x, \theta_y$ for the angular coordinates of $x$ and $y$, 

$$
r_< = \min(|x|, |y|) ,
$$

$$
r_> = \max(|x|, |y|) ,
$$

and $I_n, K_n$ are the modified Bessel functions of the first and second kind, respectively. Observing that the incident wave $u_i$ is also given by the Green’s function, we can calculate the first term in the forward series

$$
\phi^{(1)}(x_1, x_2) = (K_1\eta)(x_1, x_2)
$$

$$
= k^2 \int_{\Omega} G(x_1, x)G(x_2, x)\eta(x)dx .
$$

This expression corresponds to the usual Born approximation. We note that if $y \in \partial\Omega$, the functions $g_n$ have the simplified form

$$
g_n(x, y) = \tilde{g}_n(x)
$$

$$
= \frac{I_n(k|x|)}{RI_n(kR) + k\ell I'_n(kR)} .
$$

If we introduce polar coordinates on the boundary $x_1 = (R, \theta_1), x_2 = (R, \theta_2)$ we have, by a slight abuse of notation,

$$
\phi^{(1)}(\theta_1, \theta_2) = \frac{k^2}{(2\pi)^2} \int_{\Omega} \sum_{n_1, n_2=0}^{\infty} e^{in_1(\theta_1 - \theta)+in_2(\theta_2 - \theta)} \tilde{g}_{n_1}(x)\tilde{g}_{n_2}(x)\eta(x)dx .
$$

We find that the Fourier coefficients are given by

$$
\phi^{(1)}_{m_1, m_2} = \int_{0}^{2\pi} \int_{0}^{2\pi} e^{im_1\theta_1+im_2\theta_2}\phi^{(1)}(\theta_1, \theta_2)d\theta_1d\theta_2
$$

$$
= k^2 \int_{\Omega} e^{i(m_1+m_2)\theta}\tilde{g}_{m_1}(x)\tilde{g}_{m_2}(x)\eta(x)dx .
$$

It is convenient to introduce a rescaling of the Fourier coefficients. Put

$$
\psi^{(1)}_{m_1, m_2} = \left(\frac{R}{l}\right)^2 (I_{m_1}(kR) + k\ell I'_{m_1}(kR))(I_{m_2}(kR) + k\ell I'_{m_2}(kR))\phi^{(1)}_{m_1, m_2} .
$$

We then have

$$
\psi^{(1)}_{m_1, m_2} = k^2 \int_{\Omega} e^{i(m_1+m_2)\theta}I_{m_1}(k|x|)I_{m_2}(k|x|)\eta(x)dx .
$$
Since we assume that $\eta$ varies only radially, this further simplifies to
\[ \psi^{(1)}_{m_1,m_2} = 2\pi \delta_{m_1,-m_2} k^2 \int_0^R I_{m_1}(kr) I_{m_2}(kr) \eta(r) r dr . \] (20)

Finally, putting
\[ \psi^{(1)}_m = \frac{1}{2\pi} \psi^{(1)}_{m,-m} , \] (21)
we obtain an expression for the Born approximation in terms of the rescaled Fourier coefficients
\[ \psi^{(1)}_m = k^2 \int_0^R (I_m(kr))^2 \eta(r) r dr . \] (22)

The second term in the forward series is given by
\[ \phi^{(2)}(x_1,x_2) = (K_2 \eta \otimes \eta)(x_1,x_2) = -k^4 \int_{\Omega \times \Omega} G(x_1,y_1) G(y_1,y_2) G(y_2,x_2) \eta(y_1) \eta(y_2) dy_1 dy_2 . \] (23)

By using the formula (10) we find that the Fourier coefficients are given by
\[ \phi^{(2)}_{m_1,m_2} = \int_0^{2\pi} \int_0^{2\pi} e^{im_1 \theta_1 + im_2 \theta_2} \phi^{(2)}(\theta_1,\theta_2) d\theta_1 d\theta_2 \]
\[ = -2\pi k^4 \delta_{m_1,-m_2} \int_0^R \int_0^R \tilde{g}_{m_1}(r_1) g_{m_1}(r_1,r_2) \tilde{g}_{m_2}(r_2) \eta(r_1) \eta(r_2) r_1 r_2 dr_1 dr_2 . \] (24)

Since $\eta$ is radial, we need only singly index the data and using the same rescaling we set
\[ \psi^{(2)}_m = \frac{1}{2\pi} \left( \frac{R}{\ell} \right)^2 (I_m(kR) + k\ell I'_m(kR))^2 \phi^{(2)}_{m,-m} , \] (25)
so that
\[ \psi^{(2)}_m = -k^4 \int_0^R \int_0^R I_m(kr_1) g_{m_1}(r_1,r_2) I_m(kr_2) r_1 \eta(r_1) r_2 \eta(r_2) dr_1 dr_2 . \] (26)

The $n$th term in the forward series,
\[ \phi^{(n)} = K_n \eta \otimes \cdots \otimes \eta , \] (27)
has Fourier coefficients which require only a single index, so we put
\[ \psi^{(n)}_m = \frac{1}{2\pi} \left( \frac{R}{\ell} \right)^2 (I_m(kR) + k\ell I'_m(kR))^2 \phi^{(n)}_{m,-m} , \] (28)
which yields the general formula
\[ \psi^{(n)}_m = (-1)^{n+1} k^{2n} \int_0^R \cdots \int_0^R I_m(kr_1) g_{m_1}(r_1,r_2) \cdots \\
\times g_{m}(r_{n-1},r_n) I_m(kr_n) r_1 \eta(r_1) \cdots r_n \eta(r_n) dr_1 \cdots dr_n . \] (29)
2.3. Three-dimensional radial problem. In three dimensions, the radial problem and its corresponding forward series are similar to the two-dimensional case presented above. Here $\Omega$ is a sphere of radius $R$ centered at the origin and $\eta = \eta(r)$, where $r$ is the radial coordinate. The Green’s function is given by [10]

$$G(x, y) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_l(x, y)Y_{lm}(\hat{x})Y_{lm}^*(\hat{y}) ,$$  \hspace{1cm} (30)

where

$$g_l(x, y) = \frac{2k}{\pi} \left( k_l(kr) \hat{i}_l(kr) - \frac{k_l(kR) + k\ell l'_{lm}(kR)}{i_{lm}(kR) + k\ell l'_{lm}(kR)} \hat{i}_l(k|x|) \hat{i}_l(k|y|) \right) .$$  \hspace{1cm} (31)

Here $Y_{lm}$ are spherical harmonics, $\hat{i}_l, k_l$ are the modified spherical Bessel functions of the first and second kind, $x = (|x|, \hat{x})$, $y = (|y|, \hat{y})$ and $\hat{x}, \hat{y} \in S^2$.

The first term in the forward series is given by

$$\phi^{(1)}(x_1, x_2) = (K_1 \eta)(x_1, x_2) = k^2 \int_{\Omega} G(x_1, x)G(x, x_2)\eta(x)dx.$$  \hspace{1cm} (32)

If $y \in \partial\Omega$, the functions $g_l$ have the simplified form

$$g_l(x, y) = \tilde{g}_l(x) = \frac{\ell}{R^2 \hat{i}_l(kR) + k\ell l'_{lm}(kR)} \hat{i}_l(k|x|) .$$  \hspace{1cm} (33)

Introducing spherical coordinates on the boundary, $x_1 = (R, \hat{x}_1)$, $x_2 = (R, \hat{x}_2)$, we have with a slight abuse of notation

$$\phi^{(1)}(\hat{x}_1, \hat{x}_2) = k^2 \int_{\Omega} \sum_{l_1, l_2, m_1, m_2} Y_{l_1m_1}(\hat{x}_1)Y_{l_2m_2}^*(\hat{x}_2)\tilde{g}_{l_1}(x)\tilde{g}_{l_2}(x)\eta(x)dx.$$  \hspace{1cm} (34)

We thus find the generalized Fourier coefficients

$$\phi^{(1)}_{l_1, l_2, m_1, m_2} = k^2 \int_{S^2 \times S^2} Y_{l_1m_1}^*(\hat{x}_1)Y_{l_2m_2}(\hat{x}_2)\phi^{(1)}(\hat{x}_1, \hat{x}_2)d\hat{x}_1d\hat{x}_2 = k^2 \delta_{l_1l_2}\delta_{m_1m_2} \int_0^R \tilde{g}_{l_1}(r)\tilde{g}_{l_2}(r)r^2\eta(r)dr ,$$  \hspace{1cm} (35)

where we have used the orthogonality of the spherical harmonics and the radial dependence of $\eta$. Note that the above expression is independent of $m_1$ and $m_2$, so that we can define a singly indexed family of coefficients

$$\phi^{(1)}_{m} = \phi^{(1)}_{m, m_1, m, m_2} = k^2 \int_0^R \tilde{g}_{m}(r)^2r^2\eta(r)dr .$$  \hspace{1cm} (36)

We now introduce a rescaling in a manner analogous to the two-dimensional case and put

$$\psi^{(1)}_{m} = \left( \frac{R^2}{\ell} \right)^2 \left( i_m(kR) + k\ell l'_{m}(kR) \right)^2 \phi^{(1)}_{m} .$$  \hspace{1cm} (37)
The rescaled Fourier coefficients can be expressed as

$$\psi_m^{(1)} = k^2 \int_0^R (i_m(kr))^2 r^2 \eta(r) dr .$$  \hspace{1cm} (38)$$

Consider now the $n$th term in the forward series,

$$\phi^{(n)}(x_1, x_2) = (K_n \eta \otimes \cdots \otimes \eta)(x_1, x_2)$$

$$= (-1)^{n+1} k^{2n} \int_{\Omega \times \cdots \times \Omega} G(x_1, y_1) \cdots G(y_n, x_2) \eta(y_1) \cdots \eta(y_n) dy_1 \cdots dy_n .$$  \hspace{1cm} (39)$$

Using the formula (30) we can obtain the generalized Fourier coefficients in the form

$$\phi_{l_1,m_1,l_2,m_2}^{(n)} = \int_{S^2 \times S^2} Y_{l_1,m_1}^*(\hat{x}_1) Y_{l_2,m_2}(\hat{x}_2) \phi^{(n)}(\hat{x}_1, \hat{x}_2) d\hat{x}_1 d\hat{x}_2$$

$$= (-1)^{n+1} k^{2n} \delta_{l_1,l_2} \delta_{m_1,m_2} \int_0^R \cdots \int_0^R \hat{g}_{l_1}(r_1) g_{l_1}(r_1, r_2) \cdots g_{l_1}(r_{n-1}, r_n)$$

$$\times \hat{g}_{l_2}(r_n) r_1^2 \eta(r_1) \cdots r_n^2 \eta(r_n) dr_1 \cdots dr_n .$$  \hspace{1cm} (40)$$

If we singly index the coefficients,

$$\phi_m^{(n)} = \phi_{m,m_1,m_2}^{(n)}$$  \hspace{1cm} (41)$$

and using the same rescaling set

$$\psi_m^{(n)} = \left( \frac{R^2}{\ell} \right)^2 (i_m(kR) + k \ell i_m'(kR))^2 \phi_m^{(n)} ,$$  \hspace{1cm} (42)$$

we have the formula

$$\psi_m^{(n)} = (-1)^{n+1} k^{2n} \int_0^R \cdots \int_0^R i_m(kr_1) g_m(r_1, r_2) \cdots$$

$$\times g_m(r_{n-1}, r_n) i_m'(kr_n) r_1^2 \eta(r_1) \cdots r_n^2 \eta(r_n) dr_1 \cdots dr_n .$$  \hspace{1cm} (43)$$

2.4. Analysis of forward series. For the radial problem considered in this paper, the terms of the forward series are discrete sequences:

$$\psi_m = \psi_m^{(1)} + \psi_m^{(2)} + \cdots ,$$  \hspace{1cm} (44)$$

where

$$\psi_m^{(n)} = \{ \psi_m^{(n)} \} \in l^2 .$$  \hspace{1cm} (45)$$

We therefore define the operators

$$K^{(n)} : L^2([0, a] \times \cdots \times [0, a]) \rightarrow l^2$$

by

$$(K^{(n)} f)_m = \int K^{(n)}_m (r_1, \ldots, r_n) \eta(r_1, \ldots, r_n) dr_1 \cdots dr_n .$$  \hspace{1cm} (47)$$

In two dimensions

$$K_m^{(n)} = (-1)^{n+1} k^{2n} I_m(kr_1) g_m(r_1, r_2) \cdots g_m(r_{n-1}, r_n) I_m(kr_n) r_1 \cdots r_n ,$$  \hspace{1cm} (48)$$

where $g_m$ is given by (11). In three dimensions

$$K_m^{(n)} = (-1)^{n+1} k^{2n} i_m(kr_1) g_m(r_1, r_2) \cdots g_m(r_{n-1}, r_n) i_m'(kr_n) r_1^2 \cdots r_n^2 .$$  \hspace{1cm} (49)$$
where \( g_m \) is given by (31). In this way we can write the \( n \)th term in the series as
\[
\psi^{(n)} = K^{(n)} \eta \otimes \cdots \otimes \eta .
\] (50)
It is shown in the Appendix that \( K^{(n)} \) is a bounded operator whose norm obeys the estimate
\[
\|K^{(n)}\| \leq \nu \mu^{n-1} .
\] (51)
In two dimensions
\[
\nu = k^2 \sqrt{a} \left( \sum_{m=0}^{\infty} \sup_{r \in [0,a]} I_m^4(kr)r^2 \right)^{1/2}
\] (52)
and
\[
\mu = k^2 \left( \sum_{m=0}^{\infty} \sup_{r \in [0,a]} \int_0^a g_m^2(r,r')r'^2 dr' \right)^{1/2},
\] (53)
where \( g_m \) is given by (11). In three dimensions,
\[
\nu = k^2 \sqrt{a} \left( \sum_{m=0}^{\infty} \sup_{r \in [0,a]} i_m^4(kr)r^4 \right)^{1/2}
\] (54)
and
\[
\mu = k^2 \left( \sum_{m=0}^{\infty} \sup_{r \in [0,a]} \int_0^a g_m^2(r,r')r'^4 dr' \right)^{1/2},
\] (55)
where \( g_m \) is given by (31).

**Remark 2.1.** The estimate (51) is the counterpart of (9) which holds for the more general continuous problem without radial symmetry and discrete data.

### 3. Inverse Born Series

The inverse problem is to determine the absorption coefficient \( \eta \) everywhere within \( \Omega \) from measurements of the scattering data \( \phi \). Following [4, 7–9], we express \( \eta \) as a power series in tensor powers of \( \phi \):
\[
\eta = \mathcal{K}_1 \phi + \mathcal{K}_2 \phi \otimes \phi + \mathcal{K}_3 \phi \otimes \phi \otimes \phi + \cdots .
\] (56)
By substituting this series into (6) and equating like tensor powers, we find that the operators \( \mathcal{K}_j \) are given by
\[
\mathcal{K}_1 = K_1^+ ,
\] (57)
\[
\mathcal{K}_2 = -\mathcal{K}_1 \mathcal{K}_2 \mathcal{K}_1 \otimes \mathcal{K}_1 ,
\] (58)
\[
\mathcal{K}_3 = -(\mathcal{K}_2 \mathcal{K}_1 \otimes \mathcal{K}_2 + \mathcal{K}_2 \mathcal{K}_2 \otimes \mathcal{K}_1 + \mathcal{K}_1 \mathcal{K}_3) \mathcal{K}_1 \otimes \mathcal{K}_1 \otimes \mathcal{K}_1 ,
\] (59)
\[
\mathcal{K}_j = -\left( \sum_{m=1}^{j-1} \mathcal{K}_m \sum_{i_1 + \cdots + i_m = j} K_{i_1} \otimes \cdots \otimes K_{i_m} \right) \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_1 .
\] (60)
We will refer to (56) with the operators defined by (60) as the inverse Born series. Here \( K_1^+ \) is the pseudoinverse of the operator \( K_1 \), since a true bounded inverse does not exist. The operator \( K_1 \) has singular values which approach zero rapidly; regularization of \( K_1^+ \) is required to control this ill-posedness. The regularized solution to the linearized inverse problem is thus given by \( \eta^{(1)} = K_1^+ \phi \).
In [5,6], we characterized the convergence of the inverse Born series in $L^p$ for $2 \leq p \leq \infty$. The following theorem is an immediate consequence of that work.

**Theorem 3.1** (Convergence of the inverse Born series). *Let $X$ be a domain in $\mathbb{R}^n$. Suppose*

$$K_j : L^2(X) \otimes \cdots \otimes L^2(X) \to l^2$$

*are bounded operators with the property that there exist constants $\mu$ and $\nu$ such that

$$\|K_j\| \leq \nu \mu^{j-1} \quad (61)$$

*Then the inverse series

$$\eta = K_1 \phi + K_2 \phi \otimes \phi + K_3 \phi \otimes \phi \otimes \phi + \cdots,$$

*with $K_j$ defined by (60) converges in the $L^2$ norm to a limit $\tilde{\eta} \in L^2(X)$ if the following conditions hold:

$$\|K_1\| < \frac{1}{\mu + \nu}$$

*and

$$\|K_1 \phi\|_{L^2} < \frac{1}{\mu + \nu}.$$

*Furthermore, the following estimate for the series limit $\tilde{\eta}$ holds

$$\left\| \tilde{\eta} - \sum_{j=1}^{N} K_j \phi \otimes \cdots \otimes \phi \right\|_{L^2} \leq C \left( (\mu + \nu) \|K_1 \phi\|_{L^2} \right)^{N+1} \left( 1 - (\mu + \nu) \|K_1 \phi\|_{L^2} \right),$$

*where $C = C(\mu, \nu, \|K_1\|)$ does not depend on $N$ nor on $\phi$.

**Remark 3.1.** The above result is generic and could be formulated for more general spaces of coefficients and scattering data as long as the condition (61) holds.

It is natural to ask the question: if the limit $\tilde{\eta}$ does exist, how does it relate to $\eta$, the true inverse? In [5] it was shown that

$$\|\eta - \tilde{\eta}\|_{L^2} \leq C \| (I - K_1 K_1) \eta\|_{L^2},$$

*for some $C$ depending on $\mu, \nu, \|K_1\|$ and $\|\eta\|$, where the dependence on $\|\eta\|$ is only on a fixed maximum value (it is independent of $\|\eta\|$ being small). Here $I$ is the identity operator and $K_1 K_1 \eta$ is the projection of $\eta$ onto the subspace generated by the first $n$ eigenmodes of $K_1$. This result reinforces the idea that the nearest projection of $\eta$ onto the $n$th eigenspace is the best to be hoped for in a reconstruction.

The series that we compute fits into the above framework, but it leads us to describe the operators slightly differently than in the continuous case. Here we view the data in terms of eigenmodes, rescaled by a diagonal operator in Fourier space. The forward operators $K^{(n)}$ are given by (47), the forward series is

$$\psi = K^{(1)} \eta + K^{(2)} \eta \otimes \eta + \cdots \quad (63)$$

*and the inverse series is

$$\tilde{\eta} = K^{(1)} \psi + K^{(2)} \psi \otimes \psi + \cdots \quad (64)$$

*where tilde signifies that the limit of the series is not necessarily the actual coefficient $\eta$. 
4. Numerical Results

We now compute the exact forward scattering data by a series solution to (1). We consider an absorption coefficient of the form

\[ \eta(r) = \begin{cases} \eta_1 & 0 \leq r \leq R_1 \\ 0 & R_1 < r \leq R \end{cases} \]  

(65)

We compute the solution on two subdomains, the inner disk or sphere

\[ \Omega_1 = \{ x : |x| \leq R_1 \} \]  

(66)

and the outer annulus

\[ \Omega_2 = \{ x : R_1 < |x| < R \} . \]  

(67)

Eq. (1) then becomes

\[ -\nabla^2 u_1 + k_1^2 u_1 = 0 \text{ in } \Omega_1 \]  

(68)

\[ -\nabla^2 u_2 + k^2 u_2 = \delta(x - x_0) \text{ in } \Omega_2 \]  

(69)

\[ u_1 = u_2 \text{ on } \partial\Omega_1 \]  

(70)

\[ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \text{ on } \partial\Omega_1 \]  

(71)

\[ u_2 + \ell \frac{\partial u_2}{\partial \nu} = 0 \text{ on } \partial\Omega , \]  

(72)

where \( x_0 \) is the source location on \( \partial\Omega \), \( k_1^2 = k^2(1 + \eta_1) \) is the coefficient in the perturbed inner region and the appropriate interface matching conditions have been applied.

4.1. Exact solution in two dimensions. We consider first the free-space fundamental solution \( G(x, x_0) \) for the background medium, where \( x = (r, \theta) \) and \( x_0 = (R, \theta_0) \). It has the Fourier-Bessel series expansion [10]

\[ G_0(x, x_0) = \frac{1}{2\pi} \sum_{n=0}^{\infty} e^{in(\theta-\theta_0)} I_n(kR)K_n(kR) . \]  

(73)

Using this result, we obtain

\[ u_1(x) = \sum_{n=0}^{\infty} a_n e^{in(\theta-\theta_0)} I_n(k_1 r) \]  

(74)

and

\[ u_2(x) = G_0(x, x_0) + \sum_{n=0}^{\infty} e^{in(\theta-\theta_0)} (b_n K_n(kr) + c_n I_n(kr)) . \]  

(75)

We then apply the interface and boundary conditions to obtain the following system of linear equations for the coefficients \( \{a_n, b_n, c_n\} \)

\[
\begin{bmatrix}
I_n(k_1 R_1) & -K_n(kR_1) & -I_n(kR_1) \\
k_1 I'_n(k_1 R_1) & -k K'_n(kR_1) & -k I'_n(kR_1) \\
0 & K_n(kR) + k\ell K'_n(kR) & I_n(kR) + k\ell I'_n(kR)
\end{bmatrix}
\begin{bmatrix}
a_n \\
b_n \\
c_n
\end{bmatrix}
= \frac{1}{2\pi}
\begin{bmatrix}
I_n(kR_1) K_n(kR) \\
k I'_n(kR_1) K_n(kR) \\
k\ell I_n(kR) K'_n(kR) + I_n(kR) K_n(kR)
\end{bmatrix} .
\]  

(76)
In order to obtain an expression for the data \( \phi \), we require the series for the Green’s function

\[
G(x, x_0) = G_0(x, x_0) + \sum_{n=0}^{\infty} d_n e^{in(\theta - \theta_0)} I_n(kr),
\]

(76)

where, by applying the Robin condition, we find that

\[
d_n = \frac{1}{2\pi} \frac{I_n(kR)K_n'(kR) + k\ell I_n(kR)K_n'(kR)}{I_n(kR) + k\ell I_n'(kR)}.
\]

(77)

We thus obtain the data function for \( x \in \partial \Omega \),

\[
\phi(\theta, \theta_0) = u_2(x) - G(x, x_0) = -\sum_{n=0}^{\infty} e^{in(\theta - \theta_0)} \left( b_n K_n(kR) + (c_n - d_n)I_n(kR) \right).
\]

(78)

Next, we compute the Fourier coefficients

\[
\phi_{m,n} = \int_0^{2\pi} \int_0^{2\pi} e^{im\theta_1} e^{-in\theta_2} \phi(\theta_1, \theta_2) d\theta_1 d\theta_2 = -(2\pi)^2 \delta_{m,-n} \left( b_m K_m(kR) + (c_m - d_m)I_m(kR) \right)
\]

(79)

and define

\[
\phi_m = \phi_{m,-m} = -(2\pi)^2 \left( b_m K_m(kR) + (c_m - d_m)I_m(kR) \right).
\]

(80)

We rescale the exact forward data in the same way as the forward series. We thus define, as in (18) and (21),

\[
\psi_m = \frac{1}{2\pi} \left( \frac{R}{l} \right)^2 \left( I_m(kR) + k\ell I'_m(kR) \right)^2 \phi_m
\]

\[= -2\pi \left( \frac{R}{l} \right)^2 \left( I_m(kR) + k\ell I'_m(kR) \right)^2 \left( b_m K_m(kR) + (c_m - d_m)I_m(kR) \right).
\]

(81)

The above expression gives us the scattering data \( \psi \) which we will use to reconstruct \( \eta \) from the inverse Born series (64).

4.2. **Exact solution in three dimensions.** Once again we use the free-space fundamental solution for the background medium,

\[
G_0(x, x_0) = \frac{1}{4\pi} \frac{e^{-k|x-x_0|}}{|x-x_0|},
\]

(82)

which has the expansion [10],

\[
G_0(x, x_0) = \frac{2k}{\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i_l(kr)k_l(kR)Y_{lm}(\hat{x})Y_{lm}^*(\hat{x}_0),
\]

(83)

where \( x = (r, \hat{x}) \) and \( x_0 = (R, \hat{x}_0) \). Using this result, we have

\[
u_1(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} i_l(k_1r)Y_{lm}(\hat{x})Y_{lm}^*(\hat{x}_0)
\]

(84)
and
\[ u_2(x) = G_0(x, x_0) + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (b_m k_l(kR) + c_m i_l(kR)) Y_{lm}(\hat{x}) Y_{lm}^*(\hat{x}_0) . \] (85)

We then apply the interface and boundary conditions to obtain a system of equations for the coefficients \( \{a_{lm}, b_{lm}, c_{lm}\} \):
\[
\begin{bmatrix}
  i_l(k_1 R_1) & -k_l(k_1 R_1) & -i_l(k R_1) \\
  k_1 i_l'(k_1 R_1) & -k_l'(k_1 R_1) & -k_l'(k_1 R_1) \\
  0 & k_l(k R_1) + k_l i_l'(k R_1) & i_l(k R_1) + k l i_l'(k R_1)
\end{bmatrix}
\begin{bmatrix}
  a_{lm} \\
  b_{lm} \\
  c_{lm}
\end{bmatrix}
= \frac{2k}{\pi}
\begin{bmatrix}
  i_l(k R_1) k_l(k R_1) \\
  k i_l'(k R_1) k_l(k R_1) \\
  k l i_l(k R_1) k_l'(k R_1) + i_l(k R_1) k_l(k R_1)
\end{bmatrix}. \] (86)

The coefficients are independent of \( m \) and \( \hat{x}_0 \), so we relabel them as \( a_l = a_{lm}, \ b_l = b_{lm}, \ c_l = c_{lm} \). In order to obtain an expression for the data \( \phi \), we require the series for the Green’s function:
\[ G(x, x_0) = G_0(x, x_0) + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} d_l i_l(kR) Y_{lm}(\hat{x}) Y_{lm}^*(\hat{x}_0) , \] (87)
where by applying the Robin condition we find that
\[ d_l = -\frac{2k}{\pi} \frac{i_l(k R_1) k_l(k R_1) + k l i_l(k R_1) k_l'(k R_1)}{i_l(k R_1) + k l i_l'(k R_1)} . \] (88)

We thus obtain the data function for \( x \in \partial\Omega \),
\[
\phi(\hat{x}, \hat{x}_0) = u_2(x) - G(x, x_0)
= -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} (b_l k_l(k R) + (c_l - d_l) i_l(k R)) Y_{lm}(\hat{x}) Y_{lm}^*(\hat{x}_0) , \] (89)
and its generalized Fourier coefficients,
\[
\phi_{l_{1m}m_1}^{l_{2m}m_2} = \int_{S^2 \times S^2} Y_{l_{1m_1}}(\hat{x}_1) Y_{l_{2m_2}}^*(\hat{x}_2) \phi(\hat{x}_1, \hat{x}_2) d\hat{x}_1 d\hat{x}_2
= -\delta_{l_1 l_2} \delta_{m_1 m_2} (b_l k_l(k R) + (c_l - d_l) i_l(k R)) . \] (90)

Next we define
\[
\phi = \phi_{m m_1}^{m_2} = - (b_{m_2} k_m(k R) + (c_{m} - d_{m}) i_m(k R)) . \] (91)

Finally, we rescale the exact forward data as in (37) and put
\[
\psi_m = \left( \frac{R^2}{\ell} \right)^2 (i_m(k R) + k l i_m'(k R))^2 \phi_m
= - \left( \frac{R^2}{\ell} \right)^2 (i_m(k R) + k l i_m'(k R))^2 (b_{m_2} k_m(k R) + (c_{m} - d_{m}) i_m(k R)) , \] (92)
which is the data we will use in (64) to reconstruct the coefficient \( \eta \).
4.3. **Radius of convergence.** Despite the fact that Theorem 3.1 requires two separate bounds to hold to ensure convergence, we will refer to the quantity

\[
\mathcal{R} = \frac{1}{\mu + \nu}
\]  

(93)

as the radius of convergence of the inverse Born series. We now estimate this radius for the particular geometry (65) in our numerical experiments. In two dimensions, we can bound \( \nu \) from (52) as a function of \( a = R_1 \) by

\[
\nu \leq k^2 a^{3/2} \left( \sum_{m=0}^{\infty} \sup_{r \in [0,a]} \mathcal{I}_m^4(kr) \right)^{1/2}.
\]  

(94)

It can be seen graphically that the maxima are all achieved at \( r = a \). The tail of the sum is negligible after 10 terms, so to estimate \( \nu \) we compute

\[
\nu \leq k^2 a^{3/2} \left( \sum_{m=0}^{10} \mathcal{I}_m^4(ka) \right)^{1/2}.
\]  

(95)

Similarly in three dimensions,

\[
\nu \leq k^2 a^{5/2} \left( \sum_{m=0}^{10} \mathcal{I}_m^4(ka) \right)^{1/2}.
\]  

(96)

We can also bound \( \mu \) in two dimensions. Due to the radial symmetry, the maximum of the Green’s function is attained at \( r = 0 \). This yields

\[
\mu \leq k^2 \left( \sum_{m=0}^{\infty} \int_0^a g_m^2(0,r') r'^2 dr' \right)^{1/2} \\
= k^2 \left( \int_0^a g_0^2(0,r') r'^2 dr' \right)^{1/2},
\]  

(97)
Table 1. Comparison with theoretical radius of convergence for the two- and three-dimensional cases.

<table>
<thead>
<tr>
<th>$k_1$(cm$^{-1}$)</th>
<th>$|K_1^+\phi|_{2D}$</th>
<th>$\mathcal{R}_{2D}$</th>
<th>$|K_1^+\phi|_{3D}$</th>
<th>$\mathcal{R}_{3D}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>0.2077</td>
<td>0.17</td>
<td>0.2020</td>
<td>0.16</td>
</tr>
<tr>
<td>1.3</td>
<td>0.5244</td>
<td>-</td>
<td>0.5130</td>
<td>-</td>
</tr>
<tr>
<td>1.5</td>
<td>0.8220</td>
<td>-</td>
<td>0.8154</td>
<td>-</td>
</tr>
<tr>
<td>1.7</td>
<td>1.166</td>
<td>-</td>
<td>1.190</td>
<td>-</td>
</tr>
</tbody>
</table>

since $I_m(0) = 0$ for $m \geq 1$. Similarly, in three dimensions

$$\mu \leq k^2 \left( \int_0^a g_0^2(0, r')r'^4dr' \right)^{1/2}. \quad (98)$$

In Figure 1 we plot this estimate of the radius as a function of $R_1$, the radius of the inhomogeneity, in both two and three dimensions.

4.4. Two-dimensional numerical experiments. We have performed numerical reconstructions of $\eta$, where $\eta$ is defined by (65), using the inverse series method. When computing the terms of the inverse series, we use a recursive subroutine which implements the formula (60). As input data, we use the exact solution (81). For the necessary forward operators, we call a subroutine which implements the formula (47). We also compute $K_1 = K_1^+$, the pseudo-inverse of $K_1$, by using MATLAB’s built-in singular value decomposition. Since the singular values decay to zero exponentially quickly, we set all but the largest $M = 13$ singular values to zero. One can of course vary the value of $M$, but we found the choice of $M = 13$ to be reasonable in all cases for the computations that follow.

The parameters in the simulated reconstructions are chosen as follows. We take the domain to be the disk of radius $R = 3$cm, the Robin boundary condition parameter $\ell = 0.3$cm and the background coefficient $k = 1$cm$^{-1}$. When computing both (60) and (81), we keep $m = 1, \ldots, 90$ modes and discretize the integral operators using a spatial grid of 90 nodes in each radial direction. We found that increasing the number of modes and/or the size of the spatial grid did not significantly improve the reconstructions.

Figure 2 shows a series of experiments where the inhomogeneity is a disk of fixed radius $R_1 = 1.5$cm whose contrast is varied through a series of values of $k_1 = 1.1, 1.3, 1.5, 1.7$ cm$^{-1}$. In each of the graphs, we show the computed reconstruction using up to five terms of the inverse series. We also display the projection of $\eta$ onto the subspace generated by the first $M = 13$ singular vectors. It is this projection which is the best we can hope to reconstruct. Note that at lower contrast, the series appears to converge quite rapidly to a reconstruction that is close to the projection. As the contrast is increased, the higher order terms make significant improvements to the linear reconstruction, but rapid convergence is not apparent.

Recall that Theorem 3.1 gives sufficient conditions for the convergence of the inverse Born series, namely that $\|K_1^+\| < \mathcal{R}$ and $\|K_1^+\phi\|_{L2} < \mathcal{R}$, where $\mathcal{R}$ is given by (93). The first inequality is in general difficult to satisfy since $\|K_1^+\|$ is quite large. The second condition, however, appears to be more useful. For the numerical simulations shown in Figure 2, $\mu$ and $\nu$ can be computed from the formulas (53) and (52). We thus have $\mathcal{R} = 0.17$. For each
Figure 2. Two-dimensional reconstructions of inhomogeneities with $R_1 = 1.5\text{cm}$ and $R = 3\text{cm}$. The contrast ranges from: top left $k_1 = 1.1\text{cm}^{-1}$, top right $k_1 = 1.3\text{cm}^{-1}$, bottom left $k_1 = 1.5\text{cm}^{-1}$ and bottom right $k_1 = 1.7\text{cm}^{-1}$ of the reconstructions shown in Figure 2, Table 1 shows the result of computing $\| K_1^+ \phi \|_{L^2}$ for varying values of $k_1$. In Figure 3 we illustrate the behavior of the series for a large inhomogeneity of radius $R_1 = 2\text{cm}$. Here we take $k_1 = 2\text{cm}^{-1}$, which corresponds to a contrast of $4:1$. We see that the inverse Born series does not estimate $\eta$ accurately in this case.

4.5. Three-dimensional numerical experiments. We have performed numerical experiments in three dimensions which are strictly similar to those carried out in two dimensions. All parameter values are the same in both cases, allowing for direct comparison of the results. In Figure 4 we show reconstructions of an inhomogeneity of fixed size with increasing contrast. Just as in the two-dimensional case, the series appears to converge quite rapidly
for low contrast objects to a reconstruction that is close to the projection. As the contrast is increased, the higher order terms make significant improvements to the linear reconstruction with expected slower convergence. Figure 5 shows the reconstruction of a large, high contrast inhomogeneity where the inverse series, evidently, does not converge. Finally, in Table 1, we compute the radius of convergence and compare it to $\|K_1^+ \phi\|_{L^2}$.

**Figure 3.** Two-dimensional reconstruction of a large, high contrast inhomogeneity with $R_1 = 2\text{cm}$ and $k_1 = 2\text{cm}^{-1}$.

**Appendix: Derivation of operator bounds**

Here we derive the bounds (51), (54) and (55) on the norms of the three-dimensional forward operators. The corresponding bounds for the two-dimensional case (52) and (53) follow without difficulty. For fixed $n$, corresponding to the order of the term in the forward series, we need to bound the $l^2$ norm of the sequence $\{K^{(n)} f\}_m$ which is indexed by $m$. We have

$$ (K^{(n)} f)_m = \int K^{(n)}_m(r_1, \ldots, r_n) f(r_1, \ldots, r_n) dr_1 \cdots dr_n, $$

where

$$ K^{(n)}_m = (-1)^{n+1} k^{2n} i_m(kr_1) g_m(r_1, r_2) \cdots g_m(r_{n-1}, r_n) i_m(kr_n) r_1^2 \cdots r_n^2, $$

with $g_m$ given by (31). Consider first $n = 1$. For $f \in L^2([0, a])$,

$$ (K^{(1)} f)_m = k^2 \int_0^a i_m(kr) i_m(kr) f(r) r^2 dr, $$

$$ (K^{(2)} f)_m = k^2 \int_0^a i_m(kr_1) i_m(kr_2) g_m(r_1, r_2) f(r_1, r_2) r_1^2 r_2^2 dr_1 dr_2, $$

$$ (K^{(3)} f)_m = k^2 \int_0^a i_m(kr_1) i_m(kr_2) i_m(kr_3) g_m(r_1, r_2) g_m(r_2, r_3) f(r_1, r_2, r_3) r_1^2 r_2^2 r_3^2 dr_1 dr_2 dr_3, $$

and so on.
Figure 4. Three-dimensional reconstructions of inhomogeneities with $R_1 = 1.5\text{cm}$ and $R = 3\text{cm}$. The contrast ranges from: top left $k_1 = 1.1\text{cm}^{-1}$, top right $k_1 = 1.3\text{cm}^{-1}$, bottom left $k_1 = 1.5\text{cm}^{-1}$ and bottom right $k_1 = 1.7\text{cm}^{-1}$

so that

$$
\|K^{(1)} f\|_{L^2}^2 = k^4 \sum_{m} \left( \int_0^a i_m^2(kr) f(r) r^2 dr \right)^2
\leq k^4 \|f\|_{L^2}^2 \sum_{m} \int_0^a i_m^4(kr) r^4 dr,
$$

(102)
Figure 5. Three-dimensional reconstruction of a large, high contrast inhomogeneity with $R_1 = 2 \text{cm}$ and $k_1 = 2 \text{cm}^{-1}$.

where we have used the Cauchy-Schwarz inequality. This yields

$$
\|K^{(1)}\| \leq k^2 \left( \sum_m \int_0^a i_m^4(kr)r^4 dr \right)^{1/2} \\
\leq k^2 a^{1/2} \left( \sum_m \sup_{r \in [0,a]} i_m^4(kr)r^4 \right)^{1/2},
$$

which is exactly $\nu$ in (54). For the general terms,

$$
\|K^{(n)}f\|_2^2 \leq k^{4n} \|f\|_2^2 \sum_m \int_{[0,a] \times \cdots \times [0,a]} i_m^2(kr_1)g_m^2(r_1, r_2) \cdots g_m^2(r_{n-1}, r_n)r_1^4 \cdots r_n^4i_m^2(kr_n)dr_1 \cdots dr_n,
$$

which follows from Cauchy-Schwarz. Hence, we have for $n \geq 2$ that

$$
\|K^{(n)}\|^2 \leq k^{4n} \left( \sum_m \sup_{r \in [0,a]} i_m^4(kr)r^4 \right) \sum_m \int g_m^2(r_1, r_2) \cdots g_m^2(r_{n-1}, r_n)r_n^4dr_1 \cdots dr_n \\
= k^4 \left( \sum_m \sup_{r \in [0,a]} i_m^4(kr)r^4 \right) \mathcal{I}_{n-1}^2,
$$

where we define

$$
\mathcal{I}_{n-1}^2 = k^{4(n-1)} \sum_m \int g_m^2(r_1, r_2)r_1^4 \cdots g_m^2(r_{n-1}, r_n)r_n^4dr_1 \cdots dr_n.
$$
Thus the norm of the \( n \)th order operator obeys the estimate
\[
\|K^{(n)}\| \leq \frac{\nu}{\sqrt{a}} I_{n-1}.
\] (107)

We also have that, for \( n \geq 3 \),
\[
I_{n-1} \leq k^4 \sum_m \sup_{r_{n-1} \in [0,a]} \int_0^a g_m^2(r_{n-1}, r_n) r_n^4 dr_n
\times \left( k^{4(n-2)} \int g_m^2(r_1, r_2) r_2^4 \cdots g_m^2(r_{n-2}, r_{n-1}) r_{n-1}^4 dr_1 \cdots dr_{n-1} \right)
\leq \left( k^4 \sum_m \sup_{r_{n-1} \in [0,a]} \int_0^a g_m^2(r_{n-1}, r_n) r_n^4 dr_n \right) I_{n-2}^2,
\] (108)

which gives the recursive estimate
\[
I_{n-1} \leq \mu I_{n-2},
\] (109)

where \( \mu \) is given by (55). In addition, we have
\[
I_1 = k^4 \sum_m \int_0^a g_m^2(r_1, r_2) r_2^4 dr_1 dr_2
\leq k^4 a \sum_m \sup_{r \in [0,a]} \int_0^a g_m^2(r, r') r'^4 dr',
\] (110)

which implies
\[
I_1 \leq \sqrt{a} \mu.
\] (111)

Finally, we combine (107), (109) and (111) to obtain
\[
\|K^{(n)}\| \leq \nu \mu^{n-1}.
\] (112)

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