RESONANCES OF SMALL VOLUME HIGH CONTRAST LINEAR AND NONLINEAR SCATTERERS: ASYMPTOTIC METHODS

TAOUFIK MEKLACHI, SHARI MOSKOW, AND JOHN C. SCHOTLAND

Abstract. We consider the asymptotic behavior of the resonances of high contrast small volume scatterers for linear and nonlinear scalar waves. The resonance problem is formulated as a nonlinear eigenvalue problem. We derive asymptotic formulae for linear scatterers and nonlinear scatterers of Kerr type. Our results are illustrated with numerical examples.

1. Introduction

Scattering resonances are the rates of oscillation and decay of waves in scattering experiments. They are of central importance in nearly all areas of physics. In some sense, resonances play the role of the discrete spectrum for problems formulated in unbounded domains. More precisely, consider the Helmholtz equation for a scalar field u in $\mathbb{R}^n$:

$$\Delta u + k^2 V(x) u = 0 \ , \tag{1}$$

where k is the wavenumber and the scattering potential V is compactly supported. A resonance is a complex number $\lambda := k^2$ together with a nontrivial solution u of (1) that obeys a suitable outgoing boundary condition. Evidently, (1) can be viewed as a nonlinear eigenproblem. The associated resolvent is well defined when λ is a positive real number and the boundary condition is the Sommerfeld radiation condition. When λ is complex valued, the resolvent is viewed as a meromorphic function that is analytically continued from the positive real axis.

In this paper, we report on the development of methods for calculating the scattering resonances of small volume high contrast scatterers, for both linear and nonlinear waves. This problem arises in applications to optical physics, especially in nonlinear optics, where scattering by metallic nanoparticles is a topic of fundamental interest and considerable applied importance. For instance, in [4,5], scattering from small particles with both linear and nonlinear optical responses was studied using an approach that revealed the existence of scattering resonances. However, explicit characterization or calculations of the resonances were not obtained. In this work, we develop an asymptotic method to calculate scattering resonances within the framework of an integral equation formulation of scattering theory. In particular, we make use of an earlier result on nonlinear eigenvalue problems that allows for an explicit calculation of resonances for both linear and nonlinear waves [7]. Our results are illustrated and validated with numerical calculations.

The remainder of the paper is organized as follows. In Section 2 we consider the problem of scattering resonances for linear scalar waves. We formulate the required Lippmann-Schwinger integral equation and perform an asymptotic analysis of the associated nonlinear eigenvalue problem. In Section 3, we extend these results to the case of scattering by a nonlinear scatterer that exhibits the Kerr effect.

2. Linear scatterers

2.1. Description of the problem.

In this section we study the resonances of small volume high contrast linear scatterers. For simplicity, we work in three dimensions and consider an asymptotic

Date: November 14, 2016.

regime where the dielectric contrast is inversely proportional to the cross sectional area of the scatterer. More precisely, we consider a volume B containing the origin and denote its scaled version by $hB = \{hx : x \in B\}$, where $h > 0$ is a small parameter. The field u satisfies the Helmholtz equation

$$\Delta u(x) + k^2(1 + \eta(x))u(x) = 0 \ , \tag{2}$$

where k is the wavenumber and η is the dielectric susceptibility. We assume that the scaling of η with respect to h is of the form

$$\eta(x) = \frac{1}{h^2}\chi_{hB}(x)\eta_0(x) \ ,$$

for fixed $\eta_0(x)$, which corresponds to a high contrast small volume scatterer in the limit $h \to 0$.

Remark 2.1. *Note that the above scaling does not describe a point scatterer, which is modeled by an approximate δ function, so that η scales as $1/h^3$ in three dimensions.*

The solution to (2) is given by the Lippmann-Schwinger equation

$$u(x) = u_i(x) + \frac{k^2}{h^2}\int_{hB} G(x, y)\eta_0(y)u(y)dy \ , \tag{3}$$

where u_i is the incident field, which satisfies (2) with $\eta = 0$, and the Green's function G is given by

$$G(x, y) = \frac{1}{4\pi}\frac{\exp(ik|x - y|)}{|x - y|} \ .$$

Let $\lambda := k^2$. A *resonance* is defined to be a value of $\lambda \in \mathbb{C} \setminus \mathbb{R}^-$, for which (3) has a nontrivial solution with $u_i = 0$. That is, a resonance λ exists when there is a nontrivial u satisfying

$$u(x) = \lambda \int_{hB} \frac{\eta_0(y)}{h^2} G(x, y)u(y)dy \ .$$

This problem can be viewed as a nonlinear eigenvalue problem since G depends on λ, and thus the dependence of u on the spectral parameter λ is nonlinear. Note that λ cannot be real due to the uniqueness of the solution to the Lippmann-Schwinger equation. However, we will see that complex-valued resonances do exist.

We now consider the problem of finding resonances of small high contrast scatterers in the limit $h \to 0$. To proceed, we introduce the change of variables $x = h\tilde{x}$ and $y = h\tilde{y}$. The Lippmann-Schwinger equation (3) thus becomes

$$\tilde{u}(\tilde{x}) = u_i(h\tilde{x}) + \frac{1}{4\pi}k^2 \int_B \frac{\exp(ikh|\tilde{x} - \tilde{y}|)}{|\tilde{x} - \tilde{y}|}\eta_0(h\tilde{y})\tilde{u}(\tilde{y})d\tilde{y} \ ,$$

where $\tilde{u}(\tilde{x}) = u(h\tilde{x})$. Similarly the resonance problem is to find $(\lambda, \tilde{u})$ such that

$$\tilde{u}(\tilde{x}) = \frac{1}{4\pi}\lambda \int_B \frac{\exp(i\sqrt{\lambda}h|\tilde{x} - \tilde{y}|)}{|\tilde{x} - \tilde{y}|}\eta_0(h\tilde{y})\tilde{u}(\tilde{y})d\tilde{y} \tag{4}$$

for nontrivial $\tilde{u}$. We define the operator $T_h(\lambda)$ on $L^2(B)$ by

$$T_h(\lambda)(u)(\tilde{x}) = \frac{1}{4\pi}\int_B \frac{\exp(i\sqrt{\lambda}h|\tilde{x} - \tilde{y}|)}{|\tilde{x} - \tilde{y}|}\eta_0(h\tilde{y})u(\tilde{y})d\tilde{y}, \tag{5}$$

Then, explicitly indicating the dependence of the above eigenpair on h, we look for nontrivial (u_h, λ_h) that satisfy

$$\lambda_h T_h(\lambda_h)u_h = u_h. \tag{6}$$

If $h \ll 1$, we consider the limiting eigenproblem

$$\lambda_0 T_0(\lambda_0)u_0 = u_0 \tag{7}$$

where

$$T_0(\lambda)(u)(\tilde{x}) = \frac{1}{4\pi}\int_B \eta_0(0)\frac{1}{|\tilde{x}-\tilde{y}|}u(\tilde{y})d\tilde{y}. \tag{8}$$

2.2. **Nonlinear eigenvalues.** Let λ_h be the the perturbative eigenvalue associated to the nonlinear eigenvalue problem (6) and let λ_0 be the eigenvalue associated to the limiting eigenvalue problem (7). In this paper, we present we apply a result in [7] to obtain an approximation to the resonances of small volume high contrast objects. That is, we will explicitly evaluate the correction term λ_1 in the approximation

$$\lambda_h \approx \lambda_0 + \lambda_1 h.$$

We require the following result, which appears as Corollary 4.1 in [7].

Theorem 2.1. *Let $\{T_h(\lambda) : X \to X\}$ be a set of compact linear operator valued functions of λ which are analytic in a region U of the complex plane, such that $T_h(\lambda) \to T_0(\lambda)$ in norm as $h \to 0$ uniformly for $\lambda \in U$. Let $\lambda_0 \neq 0$, $\lambda_0 \in U$ be a simple nonlinear eigenvalue of T_0, define $DT_0(\lambda_0)$ to be the derivative of T_0 with respect to λ evaluated at λ_0, and let ϕ be the normalized eigenfunction and ϕ^* its normalized dual. Then for any h small enough, there exists λ_h a simple nonlinear eigenvalue of T_h, such that if*

$$1 + \lambda_0^2\langle DT_0(\lambda_0)\phi, \phi^*\rangle \neq 0$$

we have the following formula

$$\begin{aligned}\lambda_h = {} & \lambda_0 + \frac{\lambda_0^2\langle\left(T_0(\lambda_0) - T_h(\lambda_0)\right)\phi, \phi^*\rangle}{1 + \langle DT_0(\lambda_0)\phi, \phi^*\rangle} \\ & + O(\sup_{\lambda\in U}\|\left(T_0(\lambda_0) - T_h(\lambda_0)\right)\big|_{R(E)}\|\|\left(T_0(\lambda_0) - T_h(\lambda_0)\right)\big|_{R(E)^*}\|)\end{aligned} \tag{9}$$

where $R(E)$ is the one dimensional eigenspace spanned by ϕ and $R(E)^*$ is its dual.

The proof of this result involves the theory of operator pencils [3,6] and uses the linear eigenvalue corrections theorem of [8].

2.3. **Asymptotic expansion.** We will now apply Theorem 2.1 to solve the nonlinear eigenvalue problem. The limiting equation (7) is a *linear* eigenvalue problem and T_0 is a self adjoint compact operator with positive eigenvalues. Computing λ_0 yields a zeroth order approximation to λ_h. To compute a first order correction, we first need to verify that the hypotheses of Theorem 2.1 hold, which we do in the next Lemma.

Lemma 2.1. *Let U be a domain that is bounded away from the negative real axis in $\mathbb{C}$. Let $T_h(\lambda)$ and T_0 be as defined in (5) and (8) respectively. Suppose further that $\eta_0 \in \mathcal{C}(B)$. For any fixed λ in U, $T_h(\lambda)$ converges T_0 uniformly in the operator norm as $h \to 0$, and we have*

$$\|T_h(\lambda) - T_0\| \leqslant Ch \tag{10}$$

where C depends on the region U, but is otherwise independent of λ and h.

Proof. Let u be a normalized function in $L^2(B)$. We have:

$$(T_h(\lambda) - T_0)(u)(\tilde{x}) = \frac{1}{4\pi}\int_B \left(\eta_0(h\tilde{y})\exp(i\sqrt{\lambda}h|\tilde{x}-\tilde{y}|) - \eta_0(0)\right)\frac{u(\tilde{y})}{|\tilde{x}-\tilde{y}|}d\tilde{y}$$

A Taylor expansion of the function $h :\mapsto \eta_0(h\tilde{y})\exp(i\sqrt{\lambda}h|\tilde{x}-\tilde{y}|)$ gives

$$
\begin{aligned}
(11)\quad (T_h(\lambda)-T_0)(u)(\tilde{x}) &= \frac{h}{4\pi}\int_B \left(\tilde{y}\cdot\nabla\eta_0(0) + i\sqrt{\lambda}|\tilde{x}-\tilde{y}|\eta_0(0)\right)\frac{u(\tilde{y})}{|\tilde{x}-\tilde{y}|}d\tilde{y} + o(h)\\
&= \frac{h}{4\pi}\left(\int_B \frac{\tilde{y}.\nabla\eta_0(0)}{|\tilde{x}-\tilde{y}|}u(\tilde{y})d\tilde{y} + i\sqrt{\lambda}\eta_0(0)\int_B u(\tilde{y})d\tilde{y}\right) + o(h).
\end{aligned}
$$

Applying Cauchy-Schwartz to both integrals we obtain

$$
\left|\int_B \frac{\tilde{y}\cdot\nabla\eta_0(0)}{|\tilde{x}-\tilde{y}|}u(\tilde{y})d\tilde{y}\right|^2 \leqslant \int_B \left|\frac{\tilde{y}\cdot\nabla\eta_0(0)}{|\tilde{x}-\tilde{y}|}\right|^2 d\tilde{y}\int_B |u(\tilde{y})|^2 d\tilde{y}
$$

and

$$
\left|\int_B u(\tilde{y})d\tilde{y}\right|^2 \leqslant |B|\int_B |u(\tilde{y})|^2 d\tilde{y}
$$

so that

$$
\|(T_h(\lambda)-T_0)(u)\|_2 \leqslant Ch\|u\|_2 \ ,
$$

for some positive constant C which depends on the region U, but is otherwise independent of λ and h. □

We can now state our main result for linear media.

Theorem 2.2. *Assume that the hypotheses in Lemma 2.1 hold. Let $\lambda_0 \neq 0$ in U be a simple eigenvalue of T_0, and let u_0 be the normalized eigenfunction. Then for any h small enough, there exists a simple nonlinear eigenvalue λ_h of T_h satisfying the formula:*

$$
(12)\qquad \lambda_h = \lambda_0 + {\lambda_0}^2\left\langle (T_0 - T_h(\lambda_0))u_0, u_0\right\rangle + O(h^2).
$$

Furthermore

$$
(13)\quad \langle (T_0 - T_h(\lambda_0))u_0, u_0\rangle \;=\; -\left(\frac{1}{\lambda_0\eta(0)}\left\langle (x.\nabla\eta(0))u_0(x), u_0(x)\right\rangle \;+\; i\sqrt{\lambda_0}\eta_0(0)U_0^2\right)h \;+\; o(h)
$$

where

$$
U_0 = \int_B u_0(x)dx.
$$

Proof. Lemma 2.1 shows that the uniform convergence in h of the operators $T_h(\lambda)$ to T_0 holds for arbitrary λ in the spectral region U, allowing us to apply Theorem 2.1. The fact that the operator norm convergence is $O(h)$ allows us to discard the remainder term as $O(h^2)$. Note that since T_0 is independent of λ, $D_\lambda T_0 = 0$, and hence (12) holds. Finally, we use the expansion (11) to obtain (13). □

The following corollary follows immediately.

Corollary 2.1. *Assume the hypotheses of Theorem 2.2 hold and suppose also that the coefficient η_0 is constant. We then have*

$$
(14)\qquad \lambda_h = \lambda_0 - i{\lambda_0}^{\frac{5}{2}}\eta_0 U_0^2 h + o(h).
$$

where

$$
U_0 = \int_B u_0(x)dx.
$$

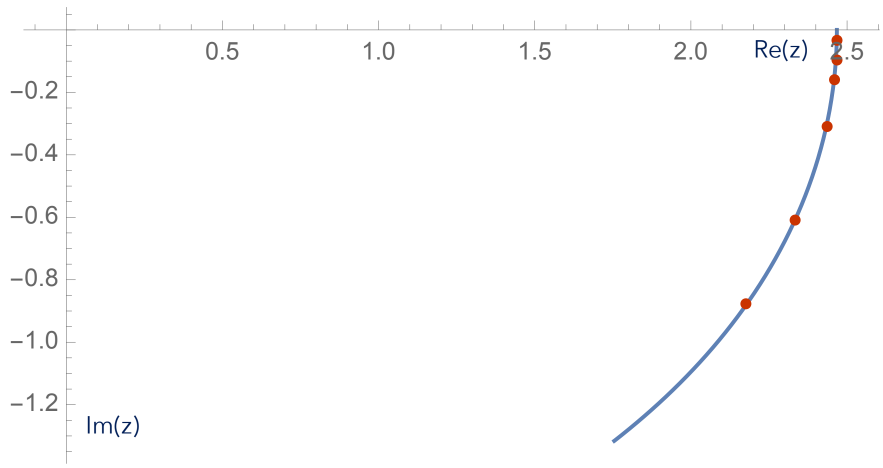

FIGURE 1. Spherical scatterer: Exact resonances in the complex plane as a function of h in [0,0.5]. The red dots show discrete values of λ_h for h in $\{.01, .03, .05, .1, .2, .3\}$.

2.4. **Spherical linear scatterer.** Consider a spherical scatterer of radius h with $\eta_0 = 1$. The governing equations with appropriate interface compatibility conditions satisfied by the resonance modes are given by:

$$\Delta u_1 + k_h^2(1+\eta_h)u_1 = 0 \text{ in } B_h \tag{15}$$

$$\Delta u_2 + k_h^2 u_2 = 0 \text{ in } \mathbb{R}^3\backslash\bar{B}_h \tag{16}$$

$$u_1 = u_2 \text{ on } \partial B_h \tag{17}$$

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \text{ on } \partial B_h. \tag{18}$$

For simplicity, we will restrict our example to radially symmetric modes, as this will allow us to exactly calculate resonances in terms of h. In addition the wave must be outgoing at infinity and bounded inside the sphere, and this yields solutions of the form:

$$u_1 = Aj_0(k_h\sqrt{1+\eta_h}h) \quad \text{in} \quad B_h \tag{19}$$

$$u_2 = Bh_0(k_h h) \quad \text{in} \quad \backslash\, B_h \tag{20}$$

where j_0 and h_0 are respectively the spherical Bessel of the first kind and the spherical Hankel function of the first kind. A and B are constants. Condition (17) gives:

$$Aj_0(k_h\sqrt{1+\eta_h}h) = Bh_0(k_h h) \tag{21}$$

This, along with condition (18) leads to the equation:

$$\sqrt{1+\eta_h}h_0(k_h h){j_0}'(k_h\sqrt{1+\eta_h}h) = {h_0}'(k_h h)j_0(k_h\sqrt{1+\eta_h}h).$$

We obtain an explicit unique solution k_h in terms of h using the symbolic Matlab package:

$$k_h = -\frac{i\log\left(-\frac{\sqrt{\frac{h^2+1}{h^2}}+1}{\sqrt{\frac{h^2+1}{h^2}}-1}\right)}{2h\sqrt{\frac{h^2+1}{h^2}}}\ ,$$

which simplifies to

$$k_h = \frac{\pi}{2\sqrt{h^2+1}} - \frac{i\sin^{-1}(h)}{\sqrt{h^2+1}}\ .$$

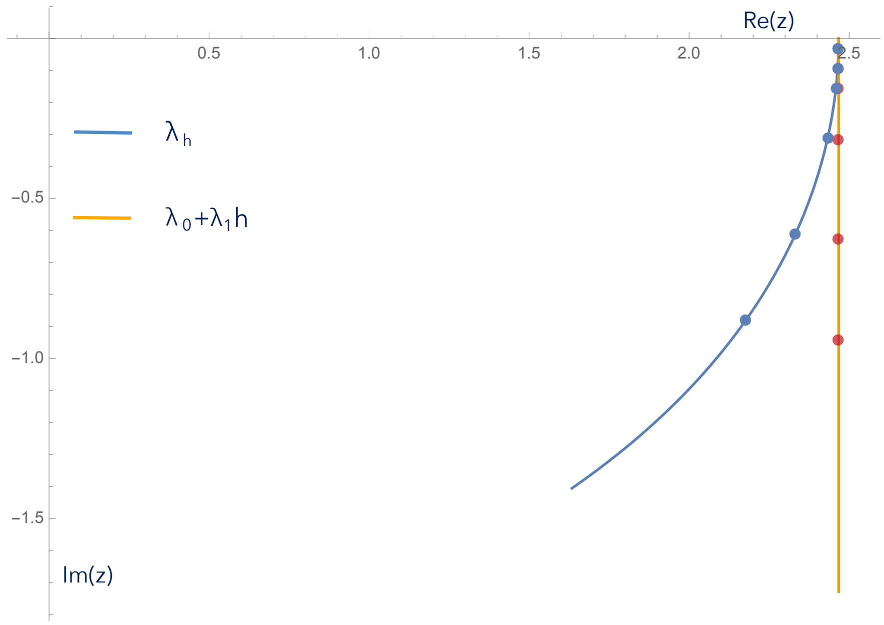

FIGURE 2. Exact λ_h for a spherical scatterer plotted together with its approximation. The blue dots show the discrete exact resonances and red their corresponding corrected approximate values. For the dots the parameter h is taken in the set $\{.01, .03, .05, .1, .2, .3\}$.

The exact resonances are given by

$$\lambda_h = \frac{\pi^2}{4\,(h^2+1)} - \frac{\sin^{-1}(h)^2}{h^2+1} - \frac{i\pi\sin^{-1}(h)}{h^2+1}.$$

See Figure 1 for a parametric plot of the resonances in the complex plane for h in the interval $(0, 0.5)$.

Now we compute the quantity

$$\lambda_0 + \lambda_1 h\ ,$$

which is given explicitly in (14) and compare it to the exact λ_h. First, we start by solving for the pair (λ_0, u_0) satisfying equation (8), or

$$\frac{\lambda_0}{4\pi}\int_B \frac{u_0(y)}{|x-y|}dy = u_0(x).$$

Using a spherical harmonics expansion for radially symmetric fields, we get

$$\begin{aligned} u_0 &= Aj_0(k_0 r) \text{ for } r \le 1 \\ u_0 &= \frac{B}{r} \text{ for } r > 1. \end{aligned}$$

Similarly to (17) and (18), we require u_0 to satisfy the appropriate compatibility conditions on the boundary ∂B, which gives the following equation for λ_0:

$$\cos\sqrt{\lambda_0} = 0$$

which yields the limiting resonance values $(2n+1)\pi/2$ for natural nonnegative numbers n. The resonance value we are interested in is the first positive value:

$$k_0 = \sqrt{\lambda_0} = \frac{\pi}{2}$$

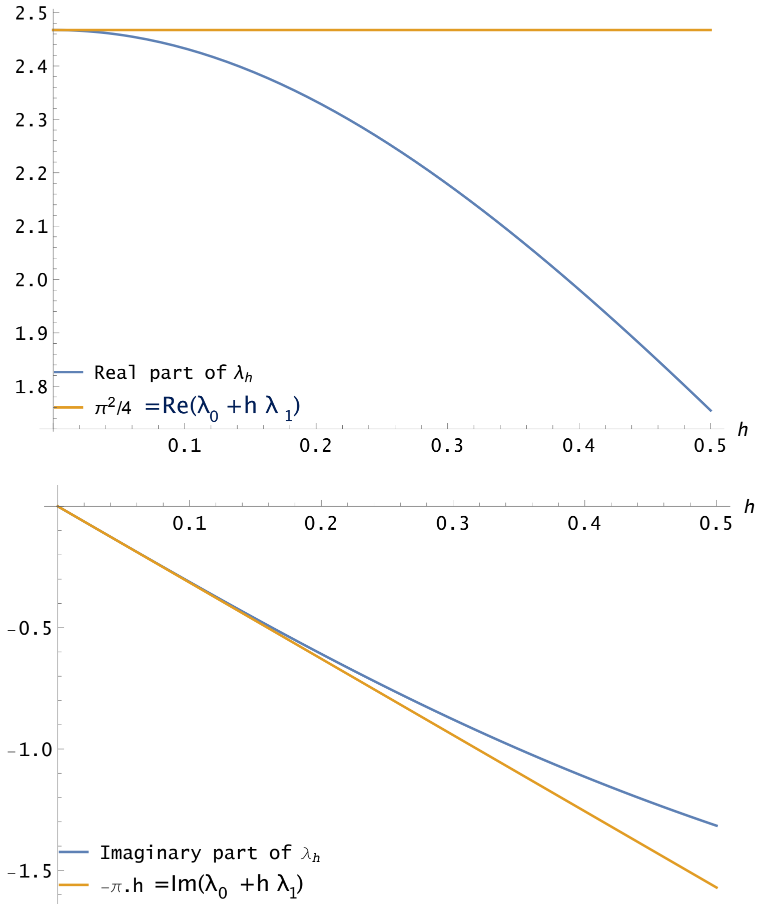

FIGURE 3. Spherical scatterer. Top: Plot of the real part of λ_h vs. its approximation. Bottom: Plot of the imaginary part of λ_h vs. its approximation. Note the better than expected convergence for the imaginary part.

Note that $k_0 = \pi/2$ is the limit of

$$k_h = \frac{\pi}{2\sqrt{h^2+1}} - \frac{i\sin^{-1}(h)}{\sqrt{h^2+1}}$$

as $h \to 0$, and the nonlinear eigenvalues of (5) satisfy $\lambda_h = {k_h}^2$. The normalized eigenfunction u_0 in the $L^2(B)$ norm is given by:

$$u_0(\mathbf{r}) = \frac{1}{\sqrt{2\pi}}\frac{\sin r}{r}$$

Here the vector $\mathbf{r} = r\hat{\theta} \quad (0 \leq r \leq 1)$ and $\hat{\theta}$ is the unit radial vector. Plugging our ingredients in

$$\lambda_h = \lambda_0 - i{\lambda_0}^{\frac{5}{2}}\eta_0 U_0^2 h + o(h).$$

where

$$U_0 = \int_B u_0(x)dx.$$

we get:

$$\lambda_h = \left(\frac{\pi}{2}\right)^2 - i\pi h + o(h)$$

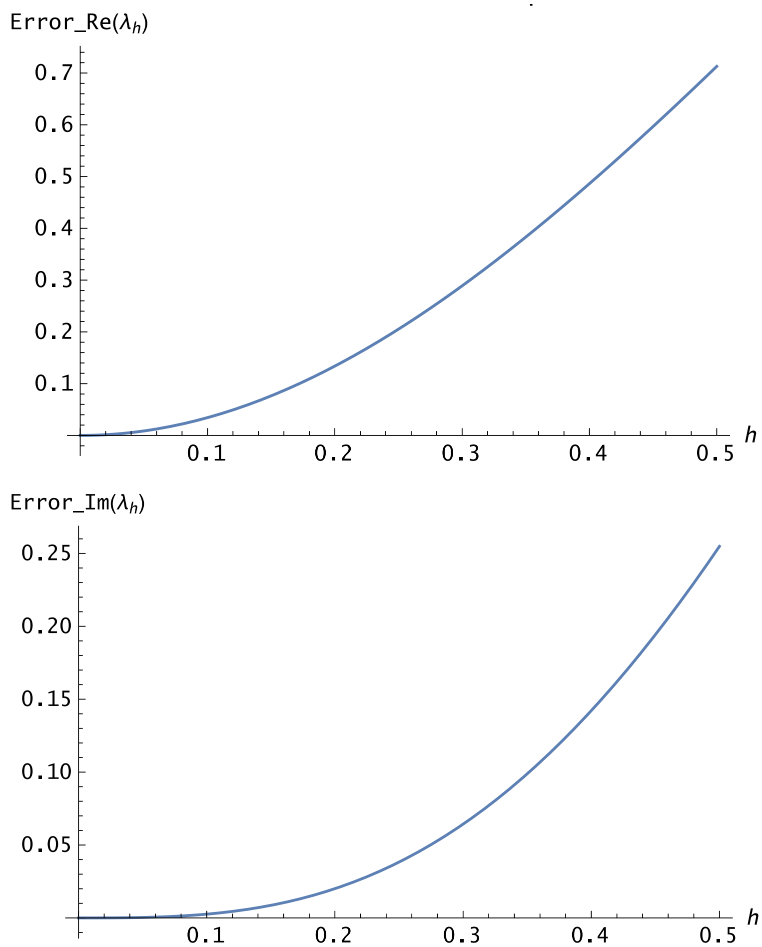

FIGURE 4. Spherical scatterer. Top: Error in the real part of approximation as a function of h. Bottom: Error in the imaginary part as a function of h.

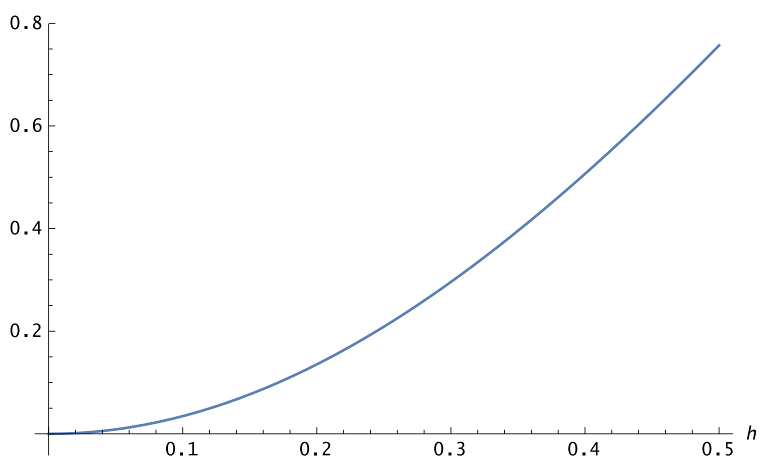

FIGURE 5. Spherical scatterer: Magnitude of error $|\lambda_h - (\lambda_0 + h\lambda_1)|$ of order $\mathcal{O}(h^2)$.

which is the first order approximation to the exact λ_h, as one can clearly see from (22). In fact,

$$\lambda_h = k_h^2 = \frac{\pi^2}{4\,(h^2+1)} - \frac{\sin^{-1}(h)^2}{h^2+1} - \frac{i\pi \sin^{-1}(h)}{h^2+1}$$

with Taylor expansion

$$\lambda_h = \frac{\pi^2}{4} - i\pi h + \left(-1 - \frac{\pi^2}{4}\right) h^2 + \frac{5}{6} i\pi h^3 + \left(\frac{2}{3} + \frac{\pi^2}{4}\right) h^4 + \dots . \tag{22}$$

In Figure 2, we illustrate the exact resonances compared to their asymptotic approximations on the complex plane. Note that our asymptotic formula has one order higher accuracy in the imaginary part than in the real part due to the imaginary part corresponding to odd powers of h. Figure 3 shows plots of the real and imaginary parts of the resonances compared to their approximations vs. h, and in Figures 4 and 5 we show plots of the error.

2.5. **Cubic linear scatterer.** In this section we demonstrate the versatility of the formula (14) by using it to approximately calculate the resonances of a cubic scatterer. Here hB is centered at the origin and has side length h, that is B is a unit cube. As before, we start by solving the limiting eigenvalue problem:

$$T_0(\lambda_0)u_0 = \mu_0 u_0 \ ,$$

where

$$\mu_0 = \frac{1}{\lambda_0} \ .$$

To solve the above limiting equation we use a simple discretization of the integral eigenvalue problem, and since the limiting kernel is independent of λ, we need only solve a standard matrix eigenvalue problem. One could improve upon this by using, for example, the fast multipole method [1, 2], but here we use only a midpoint discretization for simplicity. Let K be the kernel function defined as

$$K(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|} \ .$$

We partition each side of the cube into n segments and thus the cube contains $N = n^3$ cells. Let $\{x_i\}_{i=1}^{i=N}$ be the centers of each cell. The equation $T_0\phi = \mu_0\phi$ can be approximated by

$$\sum_{j=1}^{N} \omega_j K(x_i, x_j)\phi(x_j) = \mu_0 \phi(x_i)$$

where ϕ is the normalized eigenfunction and ω_j are weights associated with each cell that we chose to be the volume of each cell for $j \neq i$:

$$\omega_j = \left(\frac{1}{n}\right)^3 = \frac{1}{N}, \qquad \text{for } j \neq i \ .$$

Note that this approach is similar to the technique that was used in [9] to compute the eigenvectors of the Laplacian via a commuting integral operator. For the cells that contain a singularity in the evaluation of $K(x_i, x_j)$, we use the following approximation of the average:

$$\frac{1}{\mathrm{Vol}(B_i)} \int_{B_i} K(x_i, y)\phi(x_i)dy \approx \frac{1}{\mathrm{Vol}(S_i)} \int_{S_i} K(x_i, y)\phi(x_i)dy$$

where $\mathrm{Vol}(B_i)$ is the volume of the cell B_i centered at x_i and $\mathrm{Vol}(S_i)$ denotes the volume of the circumscribed sphere. The integral on the right hand side is easily evaluated in spherical coordinates and yields the approximation

$$\int_{B_i} K(x_i, y)\phi(x_i)dy \approx \left(\frac{\mathrm{Vol}(B_i)}{\mathrm{Vol}(S_i)} \int_{S_i} K(x_i, y)dy\right) \phi(x_i).$$

Let $\phi = (\phi_1, \phi_2, ..., \phi_N)^t$ where $\phi_i = \phi(x_i)$. The resulting eigenvalue problem is

$$K\phi = \mu\phi$$

where $K = (K_{ij}) \in \mathbb{R}^{N^2}$ with entries:

$$K_{ij} = \frac{1}{N} K(x_i, x_j), \quad i \neq j$$
$$K_{ii} = \frac{\mathrm{Vol}(B_i)}{\mathrm{Vol}(S_i)} \int_{S_i} K(x_i, y) dy$$

Note that K is a symmetric positive definite matrix and has a positive decreasing sequence of eigenvalues converging to 0. The first eigenvalue of K (to five digits of accuracy) is

$$\lambda_0 \approx \frac{1}{\mu_0} = 6.5584 \ ,$$

which we obtain in Matlab along with the associated eigenvector, which is used to approximate

$$U_0 = \int_B u_0(x) dx$$

in formula (14). We then approximate λ_h by

$$\lambda_h \approx 6.5584 - 0.85994i$$

The number of nodes used in this example is $N = 29791$.

3. Nonlinear Scatterers

In this section we consider the resonances of small volume high contrast nonlinear scatterers that exhibit the Kerr effect. The corresponding Helmholtz equation is of the form

$$\Delta u + k^2(1 + \eta(x))u + k^2\beta(x)|u|^2 u = 0 \ , \tag{23}$$

where η and β are functions with support in hB. More explicitly,

$$\eta(x) = \chi_{hB} \frac{\eta_0(x)}{h^2}$$

and

$$\beta(x) = \chi_{hB} \frac{\beta_0(x)}{h^2}$$

for smooth functions η_0 and β_0. As before, the volume hB is centered at the origin and the Sommerfeld radiation condition is satisfied at infinity. The partial differential equation (23) is clearly more complicated than the Helmholtz equation due the extra nonlinear term. Given an incident wave u_i which satisfies the background linear Helmholtz equation, integration by parts shows that the solution to (23) satisfies

$$u(x) = u_i(x) + k^2 \int_{hB} \frac{\eta_0(y)}{h^2} G(x, y) u(y) dy + k^2 \int_{hB} \frac{\beta_0(y)}{h^2} G(x, y) |u(y)|^2 u(y) dy \ , \tag{24}$$

where u_i is the incident field. In a manner similar to the case of the linear scatterer, we define a nonlinear resonance to be a value of $\lambda = k^2$ such that the integral equation (24) has a nontrivial solution when $u_i = 0$.

The change of variables $x = h\tilde{x}$ and $y = h\tilde{y}$ transforms equation (24) with $u_i = 0$ to

(25)

$$\tilde{u}(\tilde{x}) = \frac{1}{4\pi} k^2 \left(\int_B \eta_0(h\tilde{y}) \frac{\exp(i\sqrt{\lambda} h|\tilde{x} - \tilde{y}|)}{|\tilde{x} - \tilde{y}|} \tilde{u}(\tilde{y}) d\tilde{y} + \int_B \beta_0(h\tilde{y}) \frac{\exp(i\sqrt{\lambda} h|\tilde{x} - \tilde{y}|)}{|\tilde{x} - \tilde{y}|} |\tilde{u}(y)|^2 \tilde{u}(\tilde{y}) d\tilde{y} \right) ,$$

where $\tilde{u}(\tilde{x}) = u(x)$. To obtain an approximation to the nonlinear resonances, we follow the same approach as in the previous sections. That is, we view the scaled integral equations in their operator form

$$\lambda_h T_h(\lambda_h) u_h = u_h \ ,$$

where the equations are *nonlinear* in both λ and u. We then approximate the resonances by making use of the limiting spectral problem

$$\lambda_0 T_0(\lambda_0) u_0 = u_0 \ .$$

This equation still manifests a nonlinearity in u_0, but it is linear in λ. Indeed, for small enough h we have

$$\tilde{u}(\tilde{x}) \approx \frac{\eta_0(0)}{4\pi} k^2 \int_B \frac{1}{|\tilde{x}-\tilde{y}|} \tilde{u}(\tilde{y}) d\tilde{y} + \frac{\beta_0(0)}{4\pi} k^2 \int_B \frac{1}{|\tilde{x}-\tilde{y}|} |\tilde{u}(y)|^2 \tilde{u}(\tilde{y}) d\tilde{y} \ .$$

Suppose there exists a nontrivial pair (u_0, λ_0) to the limiting nonlinear integral equation

$$u_0(x) = \frac{\lambda_0}{4\pi} \left(\eta_0(0) \int_B \frac{1}{|\tilde{x}-\tilde{y}|} u_0(\tilde{y}) d\tilde{y} + \beta_0(0) \int_B \frac{1}{|\tilde{x}-\tilde{y}|} |u_0(\tilde{y})|^2 u_0(\tilde{y}) d\tilde{y} \right) \tag{26}$$

Similarly, rewriting (25), suppose for each nonzero h there exists a pair (u_h, λ_h) to the nonlinear integral equation

$$\begin{aligned} u_h(x) = \frac{\lambda_h}{4\pi} \Bigg(& \int_B \eta_0(hy) \frac{\exp(i\sqrt{\lambda} h |x-y|)}{|x-y|} u_h(y) dy \\ & + \int_B \beta_0(hy) \frac{\exp(i\sqrt{\lambda} h |x-y|)}{|x-y|} |u_h(y)|^2 u_h(y) dy \Bigg) . \end{aligned} \tag{27}$$

We will assume apriori that for each h, there exists a pair that solves (27) with $u_h \in L^2(B)$, and that (λ_h, u_h) converges as $h \to 0$ to (λ_0, u_0). We now define the *linear* operators T_h and T_0 by

$$\begin{aligned} T_0(u)(x) &:= \frac{\eta_0(0)}{4\pi} \int_B \frac{u(y)}{|x-y|} dy + \frac{\beta_0(0)}{4\pi} \int_B \frac{1}{|x-y|} |u_0(y)|^2 u(y) dy \\ &= \frac{1}{4\pi} \int_B \frac{\eta_0(0) + \beta_0(0)|u_0(y)|^2}{|x-y|} u(y) dy \end{aligned} \tag{28}$$

and

$$T_h(\lambda)(u)(x) := \frac{1}{4\pi} \int_B \left[\eta_0(hy) + \beta_0(hy)|u_h(y)|^2 \right] \frac{\exp(i\sqrt{\lambda} h |x-y|)}{|x-y|} u(y) dy \ . \tag{29}$$

Note that given these definitions, the known solution pairs satisfy the linear equations

$$u_0 = \lambda_0 T_0 u_0 \tag{30}$$

and

$$u_h = \lambda_h T_h(\lambda_h) u_h. \tag{31}$$

We now show that the conditions of Theorem 2.1 are satisfied for this nonlinear eigenvalue perturbation problem. This will provide an asymptotic formula for the resonances. We first need the following lemma.

Lemma 3.1. *Let U be a bounded domain in $\mathbb{C}$, bounded away from the negative real axis. Suppose that for a sequence of $h \to 0$, $\{\lambda_h\} \subset U, \lambda_0 \in U$, and the pair (λ_h, u_h) is a nontrivial solution to*

$$\lambda_h T_h(\lambda_h) u_h = u_h$$

which converges to the pair (λ_0, u_0), *a nontrivial solution of*

$$\lambda_0 T_0 u_0 = u_0 \ ,$$

where $T_h(\lambda), T_0 : L^2(B) \to L^2(B)$ *are the linear compact operators defined in* (29) *and* (28), *respectively. Suppose furthermore that the convergence of* u_h *to* u_0 *is such that* $\|u_h - u_0\|_\infty \le Ch$, *and that* η_0, $\beta_0 \in C^1$, *and* u_h *and* u_0 *are uniformly bounded on* B. *We then have*

$$\|T_h(\lambda) - T_0\|_2 \le Ch$$

where C *is a positive constant that depends on* U, *but is independent of* h *and* $\lambda \in U$.

Proof. The operator $T_h(\lambda)$ can be rewritten as

$$T_h(\lambda)(u)(x) = \frac{1}{4\pi}\int_B \gamma(hy, y)\frac{\exp(i\sqrt{\lambda}h|x-y|)}{|x-y|}u(y)dy$$

where γ is defined as

$$\gamma(hy, y) = \eta(hy) + |u_h(y)|^2\beta(hy)$$

which on $hB \times B$ is defined as $\gamma(x, y) = \eta(x) + |u_h(y)|^2\beta(x)$. Performing a Taylor expansion with respect to h we obtain

$$T_h(\lambda)u - T_0 u = \beta_0(0)\int_B (|u_h|^2 - |u_0|^2)\frac{u(y)}{|x-y|}dy \tag{32}$$

$$+ h\Big(\int_B \frac{y.\nabla_x\gamma(0,y)}{|x-y|}u(y)dy + i\sqrt{\lambda}\int_B \gamma(0,y)u(y)dy\Big) + o(h) \ , \tag{33}$$

where

$$\gamma(0, y) = \eta_0(0) + \beta_0(0)|u_h(y)|^2$$

and

$$\nabla_x\gamma(0, y) = \nabla\eta(0) + \nabla\beta(0)|u_h(y)|^2.$$

Hence

$$|T_h(\lambda)u - T_0 u| \le |\beta_0(0)|\int_B \mid |u_h|^2 - |u_0|^2 \mid \frac{|u(y)|}{|x-y|}dy \tag{34}$$

$$+ h\Big(\int_B \frac{|y.\nabla_x\gamma(0,y)|}{|x-y|}|u(y)|dy + |\sqrt{\lambda}|\int_B |\gamma(0,y)||u(y)|dy\Big) + o(h) \ .$$

Let M be the upper bound

$$\sup_{y\in B}|u_h(y)|, |u_0(y)| \le M$$

and

$$\sup_{x\in B}\Big(\int_B \frac{1}{|x-y|^2}\Big)^{\frac{1}{2}} \le M.$$

By the assumption of $O(h)$ uniform convergence of u_h to u_0, we have

$$|u_h(y) - u_0(y)| \le Ch \quad \text{for all } y \text{ in } B.$$

for some positive constant C. It follows that in the first integrand above

$$\begin{aligned} \mid |u_h(y)|^2 - |u_0(y)|^2 \mid &= (|u_h(y)| + |u_0(y)|) \mid |u_h(y)| - |u_0(y)| \mid \\ &\le 2M \mid u_h(y) - u_0(y) \mid . \end{aligned}$$

This gives the estimate

$$\begin{aligned} \||u_h|^2 - |u_0|^2\|_\infty &\le 2M\|u_h - u_0\|_\infty \\ &\le 2MCh. \end{aligned}$$

Using Cauchy-Schwarz we obtain

$$\int_B \frac{|\,|u_h(y)|^2 - |u_0(y)|^2\,|}{|x-y|}|u(y)|dy \leq \left(\int_B \frac{|\,|u_h(y)|^2 - |u_0(y)|^2\,|^2}{|x-y|^2}\right)^{\frac{1}{2}} \left(\int_B |u(y)|^2 dy\right)^{\frac{1}{2}}$$

$$\leq \| |u_h|^2 - |u_0|^2 \|_\infty \left(\int_B \frac{1}{|x-y|^2}\right)^{\frac{1}{2}} ||u||_2$$

$$\leq 2M^2 Ch ||u||_2$$

Let M' be a positive number such that

$$\sup_{x\in B}\int_B \frac{|\nabla\eta_0(0)\cdot y|^2}{|x-y|^2} \leq M' \quad \text{and } \sup_{x\in B} \frac{|\nabla\beta_0(0)\cdot y|}{|x-y|^2} \leq M' \ .$$

Then an estimate of the second integral in (34) is

$$\int_B \frac{|y.\nabla_x\gamma(0,y)|}{|x-y|}|u(y)|dy$$

$$\leq \int_B \frac{|y\cdot\nabla\eta(0)|}{|x-y|}|u(y)|dy + \int_B \frac{|y\cdot\nabla\beta(0)|}{|x-y|}|u_h(y)|^2|u(y)|dy$$

$$\leq \left(\int_B \frac{|y.\nabla\eta(0)|^2}{|x-y|^2}dy\right)^{\frac{1}{2}}\|u\|_2 + M^2\left(\int_B \frac{|y.\nabla\beta(0)|^2}{|x-y|^2}dy\right)^{\frac{1}{2}}\|u\|_2$$

$$\leq (M' + M^2 M')\|u\|_2$$

We bound the last integral in a similar manner. Let M'' be defined by

$$\max\{\eta_0(0),\beta_0(0)\}\sup_{x\in B}\int_B \frac{|y|^2}{|x-y|^2} \leq M''$$

We then obtain

$$|\sqrt{\lambda}|\int_B |\gamma(0,y)||u(y)|dy$$

$$\leq |\sqrt{\lambda}|\left(\int_B \frac{|\eta_0(0)\cdot y|}{|x-y|}|u(y)|dy + \int_B \frac{|\beta_0(0)\cdot y|}{|x-y|}|u_h|^2|u(y)|dy\right)$$

$$\leq CM'' + M^2M'')\|u\|_2 \ ,$$

where C is a positive upper bound of $|\sqrt{\lambda}|$ that depends on U. From the above estimates, it follows that

$$\|T_h(\lambda)u - T_0u\|_2 \leqslant Ch\|u\|_2$$

where C depends on M, M', M'' and U, but is otherwise independent of h and λ. □

We can now use Lemma 3.1 to show that the hypotheses of Theorem 2.1 hold, which gives an expression for the resonance. However, to obtain a computable expression for the correction, we assume apriori that there exist expansions for u_h and λ_h in terms of h of the form

(35) $$u_h(y) = u_0(y) + u_1(y)h + o(h)$$

and

(36) $$\lambda_h = \lambda_0 + \lambda_1 h + o(h) \ ,$$

where the $o(h)$ in (35) holds in the sense of the max norm. In view of these assumptions, our final result is in some sense formal. However, as we will see, numerical experiments demonstrate that these assumptions are reasonable in practice.

Theorem 3.1. *Assume the hypotheses in Lemma 3.1 hold, that β_0 and η_0 belong to $C^1(B)$, and that the expansions (35) and (36) also hold. In addition, assume that $\lambda_0 \neq 0$ in U is a simple eigenvalue of T_0, and that the eigenfunction u_0 has unit $L^2(B)$ norm. Then for any h small enough, λ_h is given by the asymptotic formula*

$$\lambda_h = \lambda_0 + \lambda_1 h + o(h) = \lambda_0 + {\lambda_0}^2 \langle (T_0 - T_h(\lambda_0))u_0, u_0 \rangle + o(h) \ . \tag{37}$$

Here the corrector λ_1 is given by

$$\begin{aligned} -4\pi\lambda_1 = {} & 2\beta_0(0)\lambda_0^2 \int_B \int_B \frac{u_0(y) Re(u_1(y))}{|x-y|} u_0(x) dy dx \\ & + \lambda_0^2 \int_B \int_B \frac{\nabla\eta_0(0) \cdot y u_0(y)}{|x-y|} u_0(x) dy dx + i\lambda_0^{\frac{5}{2}} \eta_0(0) U_0^2 \\ & + \lambda_0^2 \int_B \int_B \frac{\nabla\beta_0(0) \cdot y u_0^3(y)}{|x-y|} u_0(x) dy dx + i\lambda_0^{\frac{5}{2}} \beta_0(0) U_0 \int_B u_0^3(y) dy \end{aligned}$$

and the real part of u_1 in (35) *obeys the integral equation*

$$\begin{aligned} 4\pi Re(u_1(x)) = {} & \lambda_0 \int_B \frac{\eta_1 u_0 + \beta_1 u_0^3}{|x-y|} dy + \lambda_0 \int_B \frac{Re(u_1)}{|x-y|} (\eta_0 + 3\beta_0 u_0^2) dy \\ & - \frac{\lambda_0}{4\pi} u_0(x) \Big(2\beta_0(0) \int_B \int_B \frac{u_0(y) Re(u_1(y))}{|x-y|} u_0(x) dy dx \\ & + \int_B \int_B \frac{\nabla\eta_0(0) \cdot y u_0(y)}{|x-y|} u_0(x) dy dx + \int_B \int_B \frac{\nabla\beta_0(0) \cdot y u_0^3(y)}{|x-y|} dy dx \Big). \end{aligned} \tag{38}$$

Proof. From (3.1), we see that the compact operators $T_h(\lambda)$ converge uniformly to the compact operator T_0 for all $\lambda \in U$. Using this result and the fact that T_0 is independent of λ, we can apply Theorem 2.1 to conclude that (37) holds. The remaining part of the proof is devoted to expanding $\langle (T_0 - T_h(\lambda_0))u_0, u_0 \rangle$ to obtain an explicit formula for the correcor λ_1. We thus obtain

$$\begin{aligned} 4\pi \langle (T_h(\lambda_0) - T_0))u_0, u_0 \rangle = {} & \beta_0(0) \int_B \int_B \frac{u_0(y)}{|x-y|} (|u_h(y)|^2 - |u_0(y)|^2) u_0(x) dy dx \\ & + h \Big(\int_B \int_B \frac{\nabla\eta_0(0) \cdot y u_0(y)}{|x-y|} u_0(x) dy dx + i\sqrt{\lambda_0}\eta_0(0) U_0^2 \\ & + \int_B \int_B \frac{\nabla\beta_0(0) \cdot y}{|x-y|} |u_0(y)|^2 u_0(y) u_0(x) dy dx \\ & + i\sqrt{\lambda_0}\beta_0(0) \int_B \int_B |u_0(y)|^2 u_0(y) u_0(x) dy dx \Big) + o(h) \ , \end{aligned}$$

where we have used the fact that $U_0 = \int_B u_0(x)dx$. Let $\bar{u}_h(y)$ be the complex conjugate of $u_h(y)$. Note that T_0 is a symmetric positive definite operator, which means that λ_0 and u_0 are real valued. It follows that

$$\begin{aligned} 4\pi \langle (T_h(\lambda_0) - T_0))u_0, u_0 \rangle = {} & h \Big(2\beta_0(0) \int_B \int_B \frac{u_0(y) Re(u_1(y))}{|x-y|} u_0(x) dy dx \\ & + \int_B \int_B \frac{\nabla\eta_0(0) \cdot y u_0(y)}{|x-y|} u_0(x) dy dx \\ & + i\sqrt{\lambda_0}\eta_0(0) U_0^2 + \int_B \int_B \frac{\nabla\beta_0(0) \cdot y u_0^3(y)}{|x-y|} u_0(x) dy dx \\ & + i\sqrt{\lambda_0}\beta_0(0) U_0 \int_B \int_B u_0^3(y) dy \Big) + o(h) \ . \end{aligned}$$

Since

$$\lambda_1 h = -\lambda_0^2 \langle (T_h(\lambda_0) - T_0))u_0, u_0\rangle + o(h) \ ,$$

we compute

$$\begin{aligned}
&- 4\pi\lambda_1 = 2\beta_0(0)\lambda_0^2 \int_B\int_B \frac{u_0(y)Re(u_1(y))}{|x-y|}u_0(x)dydx\\
&+ \lambda_0^2 \int_B\int_B \frac{\nabla\eta_0(0)\cdot yu_0(y)}{|x-y|}u_0(x)dydx\\
&+ i\lambda_0^{\frac{5}{2}}\eta_0(0)U_0^2 + \lambda_0^2 \int_B\int_B \frac{\nabla\beta_0(0)\cdot yu_0^3(y)}{|x-y|}u_0(x)dydx\\
&+ i\lambda_0^{\frac{5}{2}}\beta_0(0)U_0\int_B u_0^3(y)dy.
\end{aligned}$$

Note that the above expansion has λ_1 expressed in terms of $Re(u_1)$, which is not yet known. To find an equation that must be satisfied by $Re(u_1)$, we expand $\lambda_h T_h u_h = u_h$ and match like powers of h. Note that

$$(\lambda_0 + \lambda_1 h + o(h))T_h(u_0 + u_1 h + o(h)) = u_0 + u_1 h + o(h) \ . \tag{39}$$

Since the above $o(h)$ term is understood in the sense of the uniform norm, $T_h(o(h))$ is $o(h)$. We also need to expand $T_h(u_0 + u_1 h + o(h))$. Defining A_0 and A_1 to be the terms (independent of h) such that

$$T_h(u_h) = T_h(u_0 + u_1 h + o(h)) =: A_0 + A_1 h + o(h). \tag{40}$$

Equation (39) thus becomes

$$(\lambda_0 + \lambda_1 h + o(h))(A_0 + A_1 h + o(h)) = u_0 + u_1 h + o(h),$$

which yields

$$\lambda_0 A_0 + (\lambda_0 A_1 + \lambda_1 A_0)h + o(h) = u_0 + u_1 h + o(h).$$

Hence it follows that u_1 can be expressed as

$$u_1 = \lambda_0 A_1 + \lambda_1 A_0, \tag{41}$$

keeping in mind that A_1 depends on u_1. Now the coefficients A_0 and A_1 can be evaluated by observing that the Taylor expansion of $T_h(v)$ is given by

$$\begin{aligned}
T_h(\lambda)v =& T_0 v + \beta_0(0)\int_B (|u_h|^2 - |u_0|^2)\frac{v(y)}{|x-y|}dy\\
&+ h\Big(\int_B \frac{y.\nabla_x\gamma(0,y)}{|x-y|}v(y)dy + i\sqrt{\lambda}\int_B \gamma(0,y)v(y)dy\Big) + o(h)
\end{aligned}$$

Substituting in the above expansion v and λ, which are given by

$$v = u_0 + u_1 h + o(h)$$

and

$$\lambda = \lambda_0 + \lambda_1 h + o(h)$$

and regrouping the terms in powers of h, we find that A_0 and A_1 are given by

$$A_0 = T_0 u_0(x)$$

and

$$A_1 = \int_B \frac{\eta_1 u_0 + \beta_1 u_0^3}{|x-y|} dy + i\beta_0 \sqrt{\lambda_0} \int_B (u_0 + u_0^3) dy$$
$$+ \eta_0 \int_B \frac{u_1}{|x-y|} dy + 2\beta_0 \int_B \frac{u_0^2 Re(u_1)}{|x-y|} dy + \beta_0 \int_B \frac{u_0^2 u_1}{|x-y|} dy.$$

Using (41), we see that u_1 satisfies

$$4\pi u_1(x) = \lambda_0 \Bigg(\int_B \frac{\eta_1 u_0 + \beta_1 u_0^3}{|x-y|} dy + i\beta_0 \sqrt{\lambda_0} \int_B (u_0 + u_0^3) dy$$
$$+ \eta_0 \int_B \frac{u_1}{|x-y|} dy + 2\beta_0 \int_B \frac{u_0^2 Re(u_1)}{|x-y|} dy + \beta_0 \int_B \frac{u_0^2 u_1}{|x-y|} dy \Bigg) + \lambda_1 T_0 u_0(x) \ ,$$

where all of integrands are functions of the integration variable y and we omit this dependence for simplicity of notation. By taking the real part of the last two coupled equations containing u_1 and λ_1, it follows that

$$4\pi Re(u_1) = \lambda_0 \int_B \frac{\eta_1 u_0 + \beta_1 u_0^3}{|x-y|} dy + \lambda_0 \int_B \frac{Re(u_1)}{|x-y|} (\eta_0 + 3\beta_0 u_0^2) dy) + Re(\lambda_1) \frac{u_0(x)}{\lambda_0} \tag{42}$$

and

$$-4\pi Re(\lambda_1) = 2\beta_0(0)\lambda_0^2 \int_B \int_B \frac{u_0(y) Re(u_1(y))}{|x-y|} u_0(x) dy dx$$
$$+ \lambda_0^2 \int_B \int_B \frac{\nabla\eta_0(0) \cdot y u_0(y)}{|x-y|} u_0(x) dy dx + \lambda_0^2 \int_B \int_B \frac{\nabla\beta_0(0) \cdot y u_0^3(y)}{|x-y|} u_0(x) dy dx.$$

Substituting the above expression for $Re(\lambda_1)$ into (42) results in the following integral equation for $Re(u_1)$

$$4\pi Re(u_1(x)) = \lambda_0 \left(\int_B \frac{\eta_1 u_0 + \beta_1 u_0^3}{|x-y|} dy + \int_B \frac{Re(u_1)}{|x-y|} (\eta_0 + 3\beta_0 u_0^2) dy \right)$$
$$- \frac{\lambda_0}{4\pi} u_0(x) \Bigg(2\beta_0(0) \int_B \int_B \frac{u_0(y) Re(u_1)}{|x-y|} u_0(x) dy dx + \int_B \int_B \frac{\nabla\eta_0(0) \cdot y u_0(y)}{|x-y|} u_0(x) dy dx +$$
$$\int_B \int_B \frac{\nabla\beta_0(0) \cdot y u_0^3(y)}{|x-y|} \Bigg)$$

as desired. □

If the functions η_0 and β_0 are constant, then $\nabla\eta_0(0) = \nabla\beta_0(0) = 0$ and the following corollary follows immediately.

Corollary 3.1. *Assume the hypotheses in Theorem 3.1 hold. Assume furthermore that the functions η_0 and β_0 are constant. Then*

$$\lambda_h = \lambda_0 + \lambda_1 h + o(h) \ , \tag{43}$$

where λ_1 is given by:

$$-4\pi\lambda_1 = 2\beta_0 \lambda_0^2 \int_B \int_B \frac{u_0(y) Re(u_1)}{|x-y|} u_0(x) dy dx + i\lambda_0^{\frac{5}{2}} \eta_0 U_0^2 + i\lambda_0^{\frac{5}{2}} \beta_0 U_0 U_1 \tag{44}$$

where

$$U_0 = \int_B u_0(x) dx \ , \ U_1 = \int_B u_0(x)^3 dx$$

and $Re(u_1)$ solves the integral equation:

$$\begin{aligned} 4\pi Re(u_1(x)) =&\lambda_0 \int_B \frac{Re(u_1)}{|x-y|}(\eta_0 + 3\beta_0 u_0^2)dy \\ &- \frac{\lambda_0}{4\pi} 2\beta_0 u_0(x) \int_B \int_B \frac{u_0(y) Re(u_1)}{|x-y|} u_0(x) dy dx. \end{aligned} \tag{45}$$

3.1. **Nonlinear spherical scatterer.** We consider a spherical scatterer B_h of radius $r_h = h$, with coefficients $\eta_h = \frac{\eta_0}{h^2}\chi_{B_h}$ and $\beta_h = \frac{\beta_0}{h^2}\chi_{B_h}$. Here we take

$$\eta_0 = 1 \quad \text{and} \quad \beta_0 = 0.5 \ .$$

In this subsection we will compute resonances for different values of h obtained by directly solving the nonlinear equation

$$\Delta u_h + k_h^2(1+\eta_h)u_h + \beta_h u_h |u_h|^2 = 0 \text{ in } hB \ ,$$

for radially symmetric and normalized u_h. Tthe resonances will be obtained from the formula (44) in Corollary 3.1.

We compute resonances numerically by discretizing the associated scaled integral equation on B

$$u(x) = k_h^2 \int_B \eta_0(hy) G_h(x,y) u(y) dy + k_h^2 \int_B \beta_0(hy) G_h(x,y) u(y) |u_h(y)|^2 dy \ , \tag{46}$$

where G_h is given by

$$\Delta_x G_h(x,y) + (hk_h)^2 G_h(x,y) = -\delta(x-y)$$

and is given by:

$$G_h(x,y) = \frac{1}{4\pi} \frac{\exp(ihk_h|x-y|)}{|x-y|}.$$

The Green's function is given in spherical harmonics by the expression

$$G_h(x,y) = ihk_h \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_l(x,y) Y_{lm}(\hat{x}) Y_{lm}^*(\hat{y}) \ ,$$

where

$$\begin{aligned} g_l(x,y) &= j_l(hk_h r_<) h_l^{(1)}(hk_h r_>), \\ r_< &= \min(|x|,|y|), \\ r_> &= \max(|x|,|y|), \end{aligned}$$

and $Y_{lm}(\hat{x})$ are spherical harmonics. The functions j_l and $h_l^{(1)}$ are respectively the spherical Bessel functions and the spherical Hankel functions of the first kind. Since u is radially symmetric and independent of the spherical coordinate angles θ and ϕ, we have $l = m = 0$ so that G becomes

$$G(x,y) = ihk_h j_0(hk_h r_<) h_0^{(1)}(hk_h r_>) Y_{00}(\hat{x}) Y_{00}^*(\hat{y}).$$

The nonlinear spectral equation (46) becomes a one-dimensional integral equation of the form

$$\begin{aligned} u(r) = ihk_h^3 \Big(& h_0^{(1)}(hk_h r) \int_0^r j_0(hk_h r') u(r') r'^2 dr' \\ & + j_0(hk_h r) \int_r^1 h_0^{(1)}(hk_h r') u(r') r'^2 dr' + \frac{h_0^{(1)}(hk_h r)}{2} \int_0^r j_0(hk_h r') u(r') |u(r')|^2 r'^2 dr' \\ & + \frac{j_0(hk_h r)}{2} \int_r^1 h_0^{(1)}(hk_h r') u(r') |u(r')|^2 r'^2 dr' \Big). \end{aligned} \tag{47}$$

Discretizing the above equation over the interval $[0,1]$ with respect to the radial variable r and using the trapezoid rule, we obtain a system of nonlinear equations which can be solved with the

h	λ_h
0.3	1.96747-0.716509i
0.2	2.08638-0.501054i
0.15	2.13060-0.382297i
0.1	2.16307-0.258046i
0.7	2.17653-0.181555i
0.05	2.18291-0.129995i
0.03	2.18718-0.0781222i
0.01	2.18932-0.0260617i
0.005	2.18952-0.0130318i

TABLE 1. Resonances for a nonlinear spherical scatterer for small values of h.

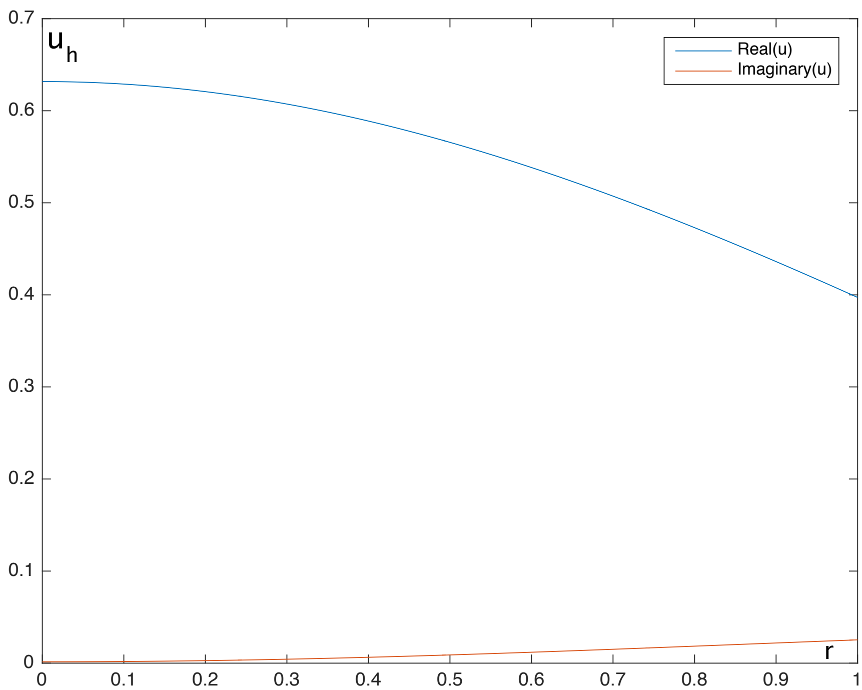

FIGURE 6. Nonlinear spherical scatterer. Real and imaginary parts of the normalized eigenfunction $u_{0.1}$ for $\eta_0 = 1$ and $\beta_0 = 0.5$.

built-in Matlab nonlinear function "fsolve". The computed values for λ_h, u_h are given to six digits in Table 1, and the real and imaginary parts of the eigenfunction u_h for $h = 0.1$ is plotted in Figure 6. In Figure 10 we plot the exact resonances in the complex plane alongside the corrected approximations.

We will compute now the first order approximation $\lambda_0 + \lambda_1 h \approx \lambda_h$ by first solving for (λ_0, u_0) the integral equation (30), then evaluate the correction λ_1 from Corollary 3.1. The nonlinear integral equation $\lambda_0 T_0 u_0 = u_0$ on B is

$$u_0(x) = \lambda_0 \int_B \eta_0 G_0(x,y) u_0(y) dy + \lambda_0 \int_B \beta_0 G_0(x,y) u_0(y) |u_0(y)|^2 dy \tag{48}$$

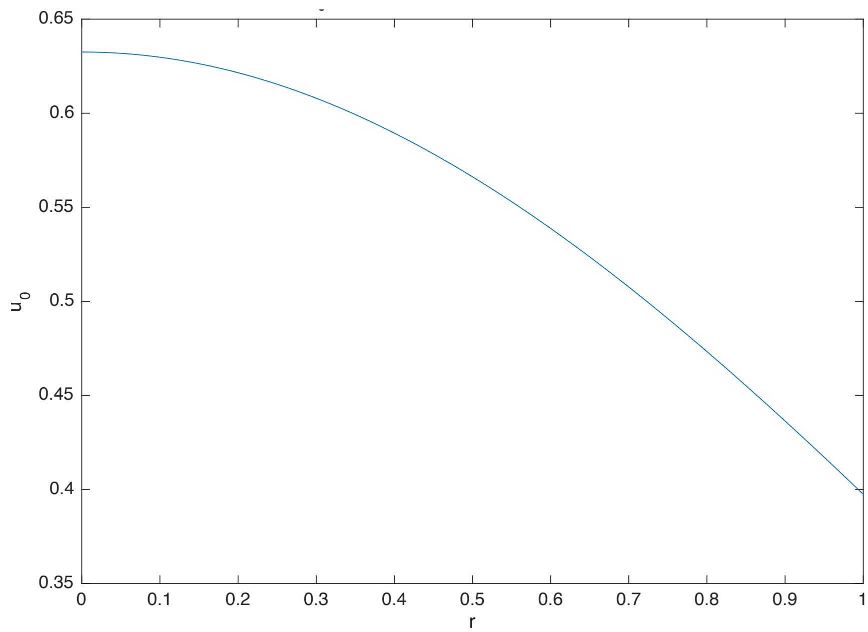

FIGURE 7. Nonlinear spherical scatterer. Normalized eigenfunction u_0 for the limiting scaled equation as $h \to 0$, $\eta_0 = 1$ and $\beta_0 = 0.5$.

where

$$G_0(x, y) = \frac{1}{4\pi|x-y|}.$$

Here G_0 is the Green's function associated to Poisson equation and has the spherical harmonics expansion:

$$G_0(x, y) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\hat{x}) Y_{lm}^*(\hat{y}).$$

For radially symmetric u_0 we have $l = m = 0$ and equation (48) becomes:

$$\begin{aligned} u_0(r) = \lambda_0 \Big(& \frac{1}{r} \int_0^r u_0(r') r'^2 dr' + \int_r^1 u_0(r') r' dr' \\ & + \frac{1}{2r} \int_0^r u_0(r') |u_0(r')|^2 r' dr' + \frac{1}{2} \int_r^1 u_0(r') |u_0(r')|^2 r'^2 dr' \Big) \end{aligned}$$

which is solved numerically, similarly to equation (47). λ_0 is found to be, accurate to six digits

$$\lambda_0 \approx 2.18960$$

and u_0 is shown in Figure 7. Figures 8 and 9 demonstrate numerically the convergence of u_h to u_0.

Next, to evaluate λ_1 we solved first for $Re(u_1)$ in equation (45) which turns out to be zero in this example, thus λ_1 is reduces to

$$\lambda_1 = -\frac{i\lambda_0^{\frac{5}{2}}}{4\pi} \big(\eta_0 U_0^2 + \beta_0 U_0 U_1 \big)$$

which gives the asymptotic approximation for λ_h as

$$\lambda_h \approx 2.18960 - 2.61542hi$$

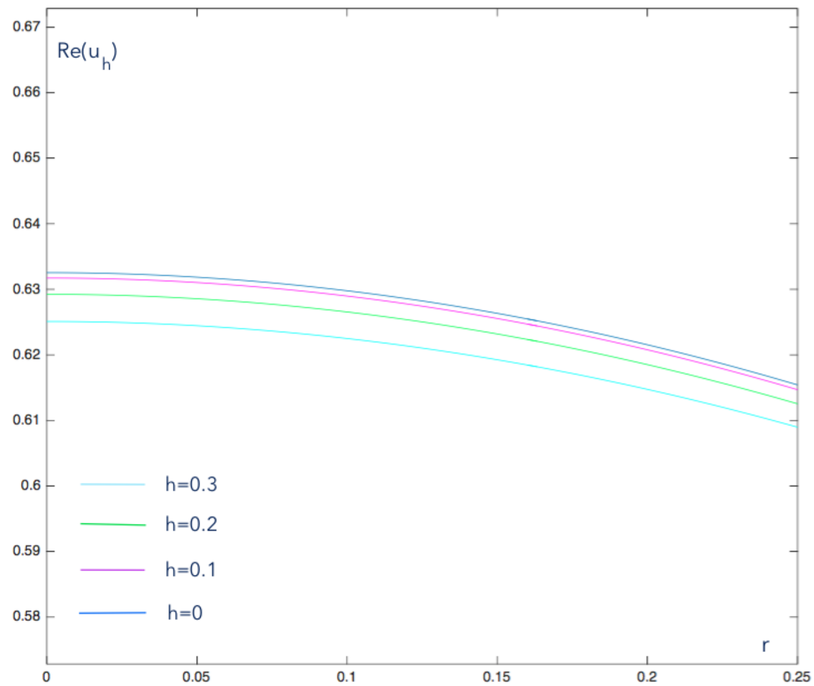

FIGURE 8. Nonlinear spherical scatterer. Convergence of $Re(u_h)$ to $Re(u_0) = u_0$ as h approaches 0.

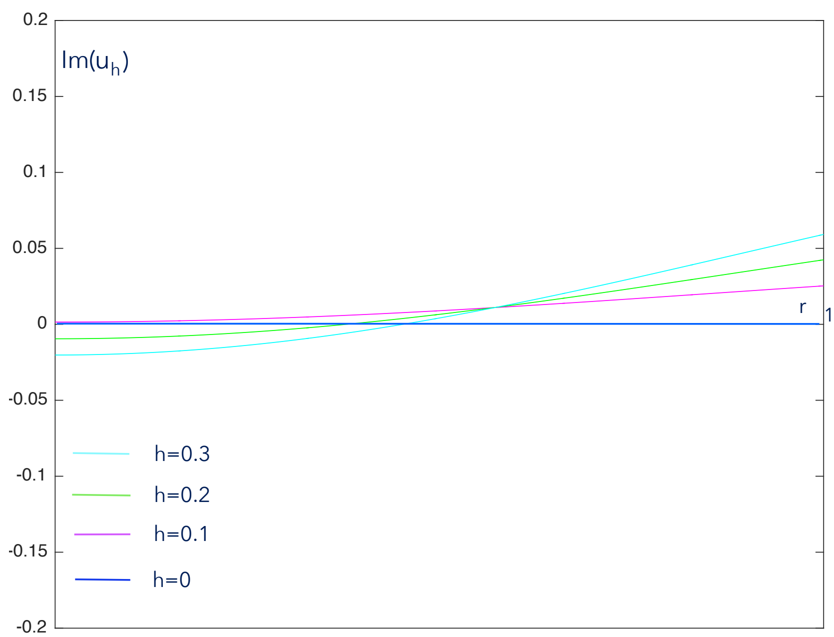

FIGURE 9. Nonlinear spherical scatterer. Convergence of $Im(u_h)$ to $Im(u_0) = 0$ as h approaches 0.

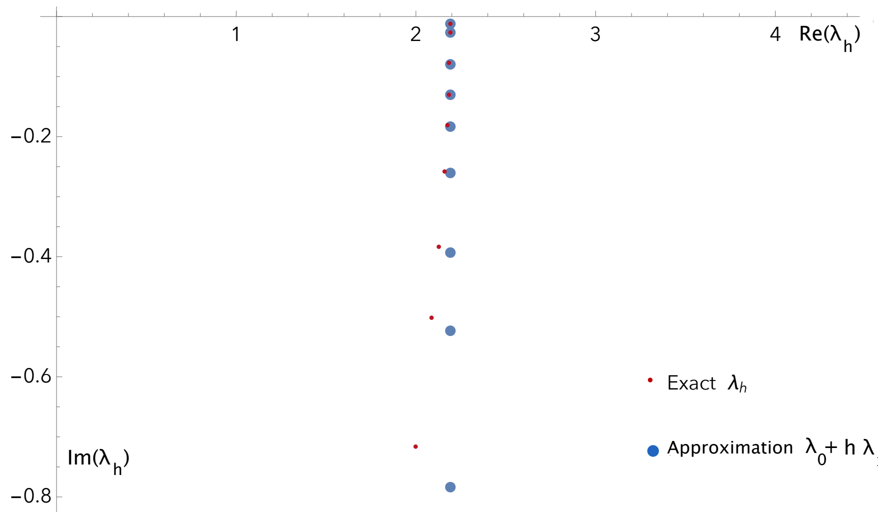

FIGURE 10. Nonlinear spherical scatterer. Exact λ_h plotted against its approximation for values of h in $\{.005, .01, .03, .05, .07, .1, .15, .2, .3\}$, $\eta_0 = 1$ and $\beta_0 = 0.5$. Each coupled dots (blue-red) correspond to a value of h, and as h approaches zero the approximated resonances almost coincide with the exact ones.

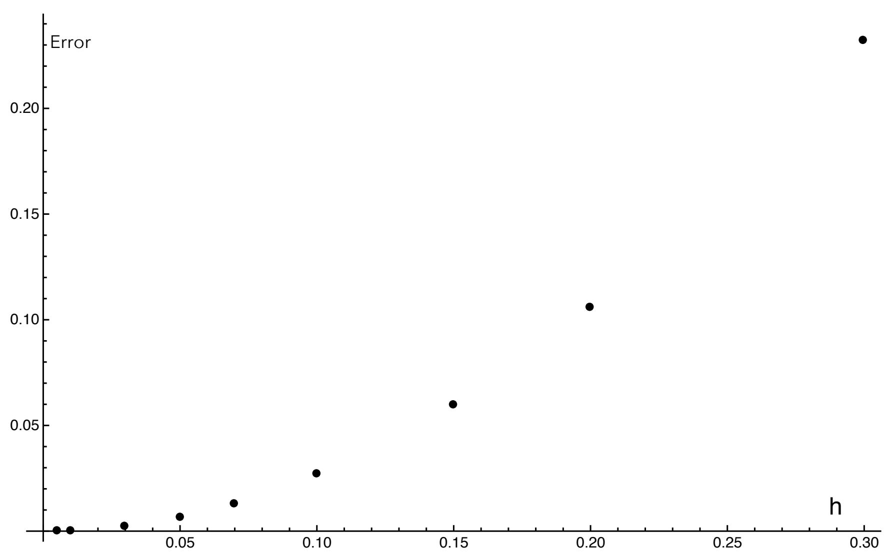

FIGURE 11. Nonlinear spherical scatterer. Magnitude of error between the exact λ_h and its approximation for values of h in $\{.005, .01, .03, .05, .07, .1, .15, .2, .3\}$, $\eta_0 = 1$ and $\beta_0 = 0.5$. The error plot suggests an $\mathcal{O}(h^2)$ order of accuracy.

Figures 10 and 11 show, respectively, the resonances λ_h plotted against their approximations and the magnitude of the error. The eigenpairs (λ_h, u_h), although changed in value from the linear case, behave qualitatively in a similar manner to the linear case. Furthermore, the asymptotic calculation appears to be quite accurate for the nonlinear problem.

References

[1] Eric Darve, *The fast multipole method: Numerical implementation*, Journal of Computational Physics , **160** (2000), 195–240.

[2] Nail A Gumerov and Ramani Duraiswami, *Fast multipole methods for the helmholtz equation in three dimensions*, Elsevier, Netherlands, 2005.

[3] Tosio Kato, *Perturbation theory for linear operators*, Springer Verlag, Berlin, 1995.

[4] Wei Li and John C. Schotland, *Point scatterers in nonlinear optics.*

[5] ———, *Optical theorem for noninear media*, Physical Review A **92** (2015).

[6] A.S. Markus, *Introduction to the spectral theory of polynomial operator pencils*, Shtiintsa, Kishinev, 1986.

[7] Shari Moskow, *Nonlinear eigenvalue approximation for compact operators*, J. Math. Phys. **56(11)** (2015), no. 113512.

[8] John E. Osborn, *Spectral approximation for compact operators*, Math. Comp **29** (1975), 712–725.

[9] Naoki Saito, *Data analysis and representation on a general domain using eigenfunctions of laplacian*, Applied and Computational Harmonic Analysis **25** (2008), 68–97.

Department of Mathematics, Drexel University, Philadelphia, PA, 19104
E-mail address: tm667@drexel.edu

Department of Mathematics, Drexel University, Philadelphia, PA, 19104
E-mail address: moskow@math.drexel.edu

Department of Mathematics and Department of Physics, University of Michigan, Ann Arbor, MI 48109
E-mail address: schotland@umich.edu