1 Introduction

An ordinary differential equation (ODE) is an equation that contains derivative with respect a single variable, such as time $t$. The solution to an ODE is a function or a family of functions that turn the equation into an identity.

Motion of planets is governed by ODE's. ODE's usually come with initial conditions. The number of initial conditions typically matches the order of the equation – which is the "order" of the highest derivative. So: one initial conditions for first order equations, two initial conditions for second order equations and so forth.

1.1 Example

Simple equations

$$y' = f(t),$$

where the right hand side does not contain $y$. The solution can be written out as

$$y(t) = \int_{t_0}^{t} f(t') \, dt + C$$

where $C$ is determined by the initial conditions.

1.2 Example

Classical linear equation

$$y' = Ay$$

There's $y$ on the right hand side.

Divide both sides by $y$:

$$\frac{y'}{y} = A$$

In other words,

$$(\ln y)' = A.$$ 

So

$$\ln y = At + B$$
Finally, raising $e$ to both sides, we get

$$y = Ce^{At},$$

where $C$, is a relabeling of the unknown constant $C = e^B$. $C$ is, of course, determined by the initial conditions.

1.3 Example
Another separation of variables example

$$y' = Ay^4$$

Divide both sides by $y^4$:

$$\frac{y'}{y^4} = A$$

In other words

$$\left(\frac{-1}{3y^3}\right)' = A$$

Or and

$$y = \sqrt[3]{-\frac{1}{At+B}},$$

where $B$ is determined by the initial conditions.

1.4 Example
Yet another separation of variables example

$$y' = Ay^4 e^{-t}.$$  

Divide both sides by $y^4$:

$$\frac{y'}{y^4} = Ae^{-t}.$$  

So

$$\left(\frac{-1}{3y^3}\right)' = Ae^{-t}$$

So

$$\frac{-1}{3y^3} = -Ae^{-t} + B$$

And we conclude that

$$y = \sqrt[3]{-\frac{1}{-Ae^{-t} + B}},$$

where $B$ is determined by the initial conditions.
1.5 Note
Separation of variables is a very limited technique. For example,

\[ y' = y + t \]

cannot be solved by separation of variables.

2 Linear equations with constant coefficients

Consider the following equation

\[ ay'' + by' + cy = 0 \]

We try to solve it by substituting

\[ y = e^{\lambda t} \]

We get

\[ a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0 \]

Cancel \( e^{\lambda t} \) and we are left with

\[ a\lambda^2 + b\lambda + c = 0, \]

and algebraic equation. It, of course, has two solutions

\[ \lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

and the general solution to the equation is

\[ y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, \]

where \( C_1 \) and \( C_2 \) are determined by the initial conditions.

2.1 Initial conditions

Initial conditions for an equation like this are usually given as the initial value of the function \( y \) at time \( t = 0 \) and its derivative \( y' \) at the same time. Suppose that

\[ y(0) = L \]
\[ y'(0) = V \]

Then

\[ C_1 + C_2 = L \]
\[ \lambda_1 C_1 + \lambda_2 C_2 = V, \]
or

\[
\begin{bmatrix}
1 & 1 \\
\lambda_1 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
= \begin{bmatrix}
L \\
V
\end{bmatrix}
\]

Solving

\[
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
\lambda_1 & \lambda_2
\end{bmatrix}^{-1}
\begin{bmatrix}
L \\
V
\end{bmatrix}
= \begin{bmatrix}
\frac{V + \lambda_2 L}{\lambda_2 - \lambda_1} \\
\frac{V + \lambda_1 L}{\lambda_2 - \lambda_1}
\end{bmatrix}
\]

2.2 Example

Solve the spring equation:

\[y'' + 5^2 y = 0\]

subject to

\[
\begin{align*}
y(0) &= L = 10 \\
y'(0) &= V = 0
\end{align*}
\]

The characteristic equation is

\[\lambda^2 + 5^2 = 0\]

or

\[
\begin{align*}
\lambda_1 &= -5i \\
\lambda_2 &= 5i
\end{align*}
\]

and the solution is

\[
y = C_1 e^{-5it} + C_2 e^{5it} = \frac{5i \times 10}{10i} e^{-5it} + \frac{5i \times 10}{10i} e^{5it} = 10 \cos 5t
\]

2.3 Note. Idea generalizes to arbitrary order.

2.4 Note. There are "problems" if eigenvalues match.

2.5 Conversion to a system of equations

Suppose we have a general linear equation with constant coefficients. Third order example

\[a y''' + b y'' + c y' + d y = 0\]

subject to initial conditions

\[
\begin{align*}
y(0) &= X \\
y'(0) &= V \\
y''(0) &= A
\end{align*}
\]
Break the equation up into three:

\[
\begin{align*}
u &= y' \\
v &= u' \\
-bv - cu - dy &= av'
\end{align*}
\]

Summarizing, the system is

\[
\begin{bmatrix}
y' \\
u' \\
v'
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
-\frac{d}{a} & -\frac{c}{a} & -\frac{b}{a}
\end{bmatrix} \begin{bmatrix}
y \\
u \\
v
\end{bmatrix},
\]

subject to the initial boundary condition

\[
\begin{bmatrix}
y \\
u \\
v
\end{bmatrix} \bigg|_{t=0} = \begin{bmatrix}
X \\
V \\
A
\end{bmatrix}
\]

The matrix that figures in the equation is called the companion matrix.

### 3 Linear Systems of Equations

We study

\[
y' = Ay,
\]

subject to the initial condition

\[y(0) = Y\]

Try a solution of the form

\[y = v e^{\lambda t}\]

We get

\[\lambda v e^{\lambda t} = A v e^{\lambda t}\]

Cancel \(e^{\lambda t}\):

\[A v = \lambda v\]

Eigenvalue equation! Assume \(n\) distinct eigenvalues and \(n\) eigenvectors. Then the general solution is given by

\[C_1 v_1 e^{\lambda_{1} t} + \ldots + C_N v_N e^{\lambda_{N} t},\]

where \(C_1, C_2, \ldots, C_N\) are determined by the initial condition. The equation that determines the \(C_n\)’s are

\[Y = C_1 v_1 + \ldots + C_N v_N.\]
3.1 Example

Solve
\[ \frac{dy}{dt} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} y \]
subject to the initial condition
\[ y_0 = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \]

Solve for eigenvalues. Form \( A - \lambda I \)
\[ A - \lambda I = \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} \]

Take the determinant
\[ |A - \lambda I| = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = (2 - \lambda)^2 - 1 \]
\[ = \lambda^2 - 4\lambda + 3 \]

Solve \(|A - \lambda I| = 0\):
\[ \lambda^2 - 4\lambda + 3 = 0 \]

We get
\[ \lambda_1 = 3 \]
\[ \lambda_2 = 1 \]

Find the first eigenvector. Form \( A - 3I \)
\[ A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \]

Solve \((A - 3I) v = 0\)
\[ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Recognize that
\[ v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

Find the second eigenvector. Form \( A - 1I \)
\[ A - 1I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

Solve \((A - 1I) v = 0\)
\[ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
Recognize that
\[ v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]
So the eigenvalues and eigenvectors are
\[ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftrightarrow 3 \]
\[ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \leftrightarrow 1 \]
So the general solution is
\[ y(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t \]
At time \( t = 0 \), we have
\[ y(0) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]
So in order to satisfy the initial condition, we must have
\[ C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \]
Therefore
\[ C_1 = 6 \]
\[ C_2 = 1 \]
and the solution to the problem is
\[ y(t) = 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t. \]

4 Solution by Taylor Series

4.1 Example
Equation
\[ y' = 5y \]
\[ y(0) = 3; \]
We have
\[ y'(0) = 5y(0) = 5 \times 3 \]
\[ y'' = 5y' = 25 \times 3 \]
\[ y''' = 5y'' = 125 \times 3 \]
and so forth.

Thus the solution is

\[
y(t) = y(0) + y'(0) t + \frac{1}{2} y''(0) t^2 + \ldots + \frac{1}{n!} y^{(n)}(0) t^n + \ldots
\]

\[
= 3 \times \left( 1 + 5t + \frac{1}{2} (5t)^2 + \ldots + \frac{1}{n!} (5t)^n + \ldots \right)
\]

\[
= 3e^{5t}
\]

4.2 Example

\[
x^2 y'' + xy' + (x^2 - n^2) y = 0
\]

\[
y(0) = 1
\]

\[
y'(0) = 0
\]

Let

\[
y = \sum c_n x^n
\]

Substitute:

\[
x^2 \sum n (n-1) c_n x^{n-2} + x \sum n c_n x^{n-1} + (x^2 - \alpha^2) \sum c_n x^n = 0
\]

\[
\sum n (n-1) c_n x^n + \sum n c_n x^n + (x^2 - \alpha^2) \sum c_n x^n = 0
\]

\[
\sum n (n-1) c_n x^n + \sum n c_n x^n + \sum c_n x^{n+2} + \sum (-\alpha^2 c_n) x^n = 0
\]

\[
\sum n (n-1) c_n x^n + \sum n c_n x^n + \sum c_n x^{n-2} + \sum (-\alpha^2 c_n) x^n = 0
\]

\[
n (n-1) c_n + nc_n + c_{n-2} - \alpha^2 c_n = 0
\]

So

\[
c_n = \frac{c_{n-2}}{\alpha^2 - n^2}
\]