1 Simpson’s rule

Simpson’s rule is a numerical integration technique that belongs to the family of Newton-Cotes formulas. Observe the following pattern:

1. The Left End rule integrates constant functions exactly. It converges to the true solution as $1/N$. The stencil uses $f_i$.

2. The Trapezoid rule integrates linear functions exactly. It converges to the true solution at the rate of $1/N^2$. The stencil uses $f_i$ and $f_{i+1}$.

This pattern can be extended to any order $n$.

3. The $n$-th order Newton-Cotes formula integrates polynomials up to order $n$ exactly. It converges at least as fast as $1/N^{n+1}$ – sometimes faster as we shall see. (So perhaps I should call it the $n+1$ order formula, but let’s stick to "$n$-th".) The stencil uses $f_i, \ldots, f_{i+n}$.

Second order Newton-Cotes formula is called Simpson’s Rule.

Here’s a good way to derive Simpson’s formula. Formally, we should consider the node points $x_i$, $x_{i+1} = x_i + h$, and $x_{i+2} = x_i + 2h$ and the corresponding values of the function $f_i$, $f_{i+1}$, and $f_{i+2}$. But to simplify the algebra, let us instead consider $x = 0, 1, 2$ and label the corresponding values of the function $f_1$, $f_2$, and $f_3$. The stencil looks like this

$$c_1 f_1 + c_2 f_2 + c_3 f_3$$

The coefficients $c_1$, $c_2$, and $c_3$ are determined from the conditions that the above stencil integrates 1. constant, 2. linear, and 3. quadratic functions exactly:

$$f(x) = 1 : \quad 1c_1 + 1c_2 + 1c_3 = \int_0^2 1\,dx = 2$$
$$f(x) = x : \quad 0c_1 + 1c_2 + 2c_3 = \int_0^2 x\,dx = 2$$
$$f(x) = x^2 : \quad 0c_1 + 1^2c_2 + 2^2c_3 = \int_0^2 x^2\,dx = \frac{8}{3}$$

We get the following system

$$\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 2 \\
0 & 1 & 4
\end{bmatrix} \begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} = \begin{bmatrix}
2 \\
2 \\
\frac{8}{3}
\end{bmatrix}$$
The solution of this system is
\[
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_3 \\
\end{bmatrix} = \begin{bmatrix}
  1 & 1 & 1 \\
  0 & 1 & 2 \\
  0 & 1 & 4 \\
\end{bmatrix}^{-1} \begin{bmatrix}
  2 \\
  2 \\
  8/3 \\
\end{bmatrix} = \begin{bmatrix}
  1/3 \\
  1/3 \\
  1/3 \\
\end{bmatrix}
\]

Therefore, the Simpson stencil is (and don’t forget to multiply by \( h! \))
\[
\left( \frac{1}{3} f_i + \frac{4}{3} f_{i+1} + \frac{1}{3} f_{i+2} \right) h
\]

2 Deriving Simpson’s Rule By Interpolation

Here, the idea is to replace the true function with an approximate one (a parabola) and integrate the approximate one exactly. This approach, too, generalizes easily to any order, but it is more cumbersome algebraically.

Let us interpolate (i.e. pass a curve through) the values of \( f_i, f_{i+1}, \) and \( f_{i+2}. \)

Let the interpolant be a parabola
\[
y(x) = ax^2 + bx + c.
\]

We need to determine \( a, b, \) and \( c \) – three unknowns. But \( y(x) \) needs to pass through three points \((x_i, f_i), (x_{i+1}, f_{i+1}),\) and \((x_{i+2}, f_{i+2}),\) so we have three equations. To make the algebra simple, let \( x_i = 0, x_{i+1} = 1, \) and \( x_{i+2} = 2 \) so the eventual answer will need to be multiplied by \( h. \)

The three equations read:
\[
\begin{align*}
  a0^2 + b0 + c &= f_i \\
  a1^2 + b1 + c &= f_{i+1} \\
  a2^2 + b2 + c &= f_{i+2}
\end{align*}
\]

or, in matrix form:
\[
\begin{bmatrix}
  0 & 0 & 1 \\
  1 & 1 & 1 \\
  4 & 2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  a \\
  b \\
  c \\
\end{bmatrix} =
\begin{bmatrix}
  f_i \\
  f_{i+1} \\
  f_{i+2} \\
\end{bmatrix}
\]

The solution is
\[
\begin{bmatrix}
  a \\
  b \\
  c \\
\end{bmatrix} = \begin{bmatrix}
  0 & 0 & 1 \\
  1 & 1 & 1 \\
  4 & 2 & 1 \\
\end{bmatrix}^{-1} \begin{bmatrix}
  f_i \\
  f_{i+1} \\
  f_{i+2} \\
\end{bmatrix} = \begin{bmatrix}
  \frac{1}{3} f_i - f_{i+1} + \frac{1}{3} f_{i+2} \\
  -\frac{3}{2} f_i + 2 f_{i+1} - \frac{1}{2} f_{i+2} \\
  f_i \\
\end{bmatrix}
\]

Therefore, \( y(x) \) is given by
\[
y(x) = \left( \frac{1}{2} f_i - f_{i+1} + \frac{1}{2} f_{i+2} \right) x^2 + \left( -\frac{3}{2} f_i + 2 f_{i+1} - \frac{1}{2} f_{i+2} \right) x + f_i
\]
Finally, we need to integrate $y(x)$ from 0 to 2. Since $\int_0^2 1 \, dx = 2$, $\int_0^2 x \, dx = 2$, and $\int_0^2 x^2 \, dx = \frac{8}{3}$, we have

$$\int_0^2 y(x) \, dx = \frac{8}{3} \left( \frac{1}{2} f_i - f_{i+1} + \frac{1}{2} f_{i+2} \right) + 2 \left( -\frac{3}{2} f_i + 2 f_{i+1} - \frac{1}{2} f_{i+2} \right) + 2 f_i$$

$$= \frac{1}{3} f_i + \frac{4}{3} f_{i+1} + \frac{1}{3} f_{i+2}$$

3 Matlab code

The longer-than-necessary Matlab code is:

```matlab
function s = simpsonIntegration(fString, A, B, N)
if mod(N, 2) == 1
    N=N+1 ;
end
f = inline(fString, 'x');
h = (B-A)/N;
s=0 ;
for stencil = 1:N/2
    xLeft = A + 2*(stencil - 1)*h;
xMiddle = xLeft + h;
xRight = xMiddle + h;
fLeft = f(xLeft);
fMiddle = f(xMiddle);
fRight = f(xRight);
s = s + h*(1/3*fLeft + 4/3*fMiddle + 1/3*fRight);
end
```

4 Order of Convergence

The order of convergence can be determined by building a loglog plot. Let us test our numerical scheme by integrating $f(x) = \sin x$ for 0 to 1, the true answer being

$$\int_0^1 \sin x \, dx = 0.4596976941318602826.$$ 

Build the plot:

```matlab
>> N = [ 4 16 64 256 1024 ];
>> for ii = 1:5
    err(ii) = abs(0.459697694131860 - simpsonIntegration('sin(x)', 0, 1, N(ii)));
end
>> loglog(N, err, ‘ro-’, ’LineWidth’, 2); grid on; grid minor;
```
And we get the following result:
We observe fourth order convergence. We were aiming for third order, but got an added bonus.