Quadratic Form Minimization and FEM in 1D

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1 Quadratic form minimization

Given a positive definite matrix $A$ and an arbitrary vector $b$ find the vector $x$ that minimizes the quadratic form $f(x)$:

$$ f(x) = \frac{1}{2} x^T A x - x^T b $$

or

$$ f(x_1, x_2, ..., x_n) = \frac{1}{2} \sum_{i,j=1}^{n} A_{ij} x_i x_j - \sum_{i=1}^{n} x_i b_i $$

1.1 Example

Suppose that

$$ A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} $$

and

$$ b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} ; $$

Then

$$ f(x) = \frac{1}{2} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} $$

$$ = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 - x_1 - 2x_2 - 3x_3 $$

1.2 One dimensional analogy

First consider the one dimensional version:

$$ f(x) = \frac{1}{2} ax^2 - bx $$
It is minimized where $df/dx = 0$, in other words

$$ax - b = 0$$

or

$$ax = b.$$ 

The multidimensional equivalent is

$$Ax = b$$

but how to derive it?

### 1.3 Three dimensional case

Let us carry out the derivation for the three dimensional case.

$$f(x_1, x_2, x_3) = \frac{1}{2} \begin{pmatrix} A_{11}x_1 + A_{12}x_1x_2 + A_{13}x_1x_3 + A_{21}x_2x_1 + A_{22}x_2x_2 + A_{23}x_2x_3 + A_{31}x_3x_1 + A_{32}x_3x_2 + A_{33}x_3x_3 \end{pmatrix} - (b_1x_1 + b_2x_2 + b_3x_3)$$

Now,

$$\frac{\partial f}{\partial x_1} = \frac{1}{2} \begin{pmatrix} 2A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{21}x_1 + 0 + 0+ A_{31}x_3 + 0 + 0 \end{pmatrix} - b_1 = A_{11}x_1 + A_{12}x_2 + A_{13}x_3 - b_1$$

$$\frac{\partial f}{\partial x_2} = \frac{1}{2} \begin{pmatrix} 0 + A_{12}x_1 + 0+ A_{21}x_1 + 2A_{22}x_2 + A_{23}x_3 + 0 + A_{32}x_3 + 0 \end{pmatrix} - b_2 = A_{21}x_1 + A_{22}x_2 + A_{23}x_3 - b_2$$

$$\frac{\partial f}{\partial x_3} = \frac{1}{2} \begin{pmatrix} 0 + 0 + A_{13}x_1 + 0 + 0 + A_{23}x_2 + A_{31}x_1 + A_{32}x_2 + 2A_{33}x_3 \end{pmatrix} - b_3 = A_{31}x_1 + A_{32}x_2 + A_{33}x_3 - b_3$$

Summarizing

$$\frac{\partial f}{\partial x} = Ax - b$$

and the equation $\partial f/\partial x = 0$ reads

$$Ax = b.$$ 

### 1.4 Example

Consider the quadratic form in the example above:

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_1 - 2x_2 - 3x_3$$
The partial derivative are
\[
\frac{\partial f(x_1, x_2, x_3)}{\partial x_1} = 2x_1 - x_2 - 1
\]
\[
\frac{\partial f(x_1, x_2, x_3)}{\partial x_2} = -x_1 + 2x_2 - x_3 - 2
\]
\[
\frac{\partial f(x_1, x_2, x_3)}{\partial x_3} = -x_2 + 2x_3 - 3
\]

Equating each derivative to zero and rewriting this system of equations in matrix form, we get
\[
\begin{bmatrix}
 2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix},
\]
or
\[Ax = b.\]

1.5 General multidimensional case
The general multidimensional case can be analyzed in an analogous way, but the notation becomes a little tricky. I will soon post a short linear algebra essay that I wrote on this topic.

2 Constrained quadratic form minimization
Slightly problem: minimize \( \frac{1}{2} x^T Ax \), subject to a constraint, such as \( x_1 = 5 \) or, more generally, \( c(x_1, \ldots, x_n) = 0 \).

2.1 Approach 1: Direct substitution
Let’s illustrate this with a 3 \( \times \) 3 example:
\[
f(x_1, x_2, x_3) = \frac{1}{2} \left( \begin{array}{c}
A_{11}x_1x_1 + A_{12}x_1x_2 + A_{13}x_1x_3 \\
A_{21}x_2x_1 + A_{22}x_2x_2 + A_{23}x_2x_3 \\
A_{31}x_3x_1 + A_{32}x_3x_2 + A_{33}x_3x_3
\end{array} \right)
\]
Substituting \( x_1 = 5 \), we get
\[
g(x_2, x_3) = \frac{1}{2} \left( \begin{array}{c}
25A_{11} + 5A_{12}x_2 + 5A_{13}x_3 + 5A_{21}x_2 + A_{22}x_2x_2 + A_{23}x_2x_3 + 5A_{31}x_3 + A_{32}x_3x_2 + A_{33}x_3x_3
\end{array} \right)
\]
The terms \( 25A_{11} \) is independent of \( x_2 \) and \( x_3 \) and can be ignored. Rewrite the rest as
\[
g(x_2, x_3) = \frac{1}{2} \left( \begin{array}{c}
A_{22}x_2x_2 + A_{23}x_2x_3 + A_{32}x_3x_2 + A_{33}x_3x_3 + 5A_{21}x_2 + 5A_{31}x_3
\end{array} \right)
\]
We have reduced the problem to one we have already considered since this \( g \) is of the form \( Ax - b \). The minimum occurs at the solution to the following equation:

\[
\begin{bmatrix}
A_{22} & A_{23} \\
A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
-5A_{21} \\
-5A_{31}
\end{bmatrix}
\]

### 2.2 Approach 2: Lagrange multipliers

An alternative, and more general, way of carrying out constrained optimization is the method of Lagrange multipliers.

According to this method, a new variable, called a Lagrange multiplier, is introduced for each constraint. The original objective function \( f(x_1, \ldots, x_n) \) is replaced by a new one \( f_{\text{new}}(x_1, \ldots, x_n, \lambda) \) in which a new term is added for each constraint. The additional term is a product of the Lagrange multiplier and the constraint expressed in the form \( c(x_1, \ldots, x_n) = 0 \). For example, \( x_1 = 5 \) must be rewritten as \( x_1 - 5 = 0 \). For example:

\[
x_1 = 5 \quad f_{\text{new}}(x_1, \ldots, x_n, \lambda) = \frac{1}{2}x^TAx + \lambda(x_1 - 5)
\]

\[
c(x_1, \ldots, x_n) = 0 \quad f_{\text{new}}(x_1, \ldots, x_n, \lambda) = \frac{1}{2}x^TAx + \lambda c(x_1, \ldots, x_n)
\]

The new objective function has as many additional variables as there are constraints, but the equations \( \partial f / \partial x_i = 0 \) are supplemented by the constraints (so the number of equations once again equals the number of unknowns). The new set of equations is, say in the \( c(x_1, \ldots, x_n) = 0 \) case, is \( \frac{\partial f_{\text{new}}}{\partial x_i} = 0 \) and \( c(x_1, \ldots, x_n) = 0 \). The constraint can actually be rewritten as \( \frac{\partial f_{\text{new}}}{\partial \lambda} = 0 \), so the method may be worded as replacing \( f \) by \( f_{\text{new}} \) and the performing unconstrained optimization on \( f_{\text{new}} \).

### 2.3 Examples

**Example 1** Minimize

\[
f(x_1, x_2, x_3) = \frac{1}{2} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3
\]

subject to

\[
x_1 = 5
\]

Let

\[
f(x_1, x_2, x_3, \lambda) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 + \lambda(x_1 - 5)
\]
Then
\[
\frac{\partial f(x_1, x_2, x_3, \lambda)}{\partial x_1} = 2x_1 - x_2 + \lambda = 0
\]
\[
\frac{\partial f(x_1, x_2, x_3, \lambda)}{\partial x_2} = -x_1 + 2x_2 - x_3 = 0
\]
\[
\frac{\partial f(x_1, x_2, x_3, \lambda)}{\partial x_3} = -x_2 + 2x_3 = 0
\]
\[
\frac{\partial f(x_1, x_2, x_3, \lambda)}{\partial \lambda} = x_1 - 5 = 0
\]

Rewriting this system in matrix form, we get
\[
\begin{bmatrix}
2 & -1 & 0 & 1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\lambda
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
5
\end{bmatrix}
\]

In this eventual system we recognize the matrix $A$ in the top left corner plus two 1’s that correspond to the Lagrange multiplier. The system is still symmetric, but no longer positive definite!

For the record, the solution of the system is
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\lambda
\end{bmatrix}
= 
\begin{bmatrix}
2 & -1 & 0 & 1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
0 \\
0 \\
5
\end{bmatrix}
= 
\begin{bmatrix}
5 \\
10 \\
25 \\
10
\end{bmatrix}
\]

\[2.3.1 \text{ Example 2}\]

Minimize the same quadratic form, subject to two constraints $x_1 = 5$, and $x_3 = 10$.

Form
\[
f(x_1, x_2, x_3, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 + \lambda_1 (x_1 - 5) + \lambda_2 (x_3 - 10)
\]

Omitting the rest of the details, we present the eventual linear system
\[
\begin{bmatrix}
2 & -1 & 0 & 1 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\lambda_1 \\
\lambda_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
5 \\
10
\end{bmatrix}
\]

whose solution is
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\lambda_1 \\
\lambda_2
\end{bmatrix}
= 
\begin{bmatrix}
2 & -1 & 0 & 1 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
0 \\
0 \\
5 \\
10
\end{bmatrix}
= 
\begin{bmatrix}
5 \\
\frac{15}{2} \\
\frac{25}{2} \\
5 \\
10
\end{bmatrix}
\]
2.3.2 Example 3

Minimize the same quadratic form subject to the constraint \( x_1 + 2x_2 + 3x_3 = 6 \).

Form

\[
f(x_1, x_2, x_3, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 + \lambda (x_1 + 2x_2 + 3x_3 - 6)
\]

The eventual linear system is

\[
\begin{bmatrix}
2 & -1 & 0 & 1 \\
-1 & 2 & -1 & 2 \\
0 & -1 & 2 & 3 \\
1 & 2 & 3 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\lambda
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0 \\
6
\end{bmatrix}
\]

and its solution is

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\lambda
\end{bmatrix}
=
\begin{bmatrix}
2 & -1 & 0 & 1 \\
-1 & 2 & -1 & 2 \\
0 & -1 & 2 & 3 \\
1 & 2 & 3 & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
0 \\
0 \\
6
\end{bmatrix}
=
\begin{bmatrix}
\frac{5}{4} \\
-\frac{1}{4}
\end{bmatrix}
\]

2.3.3 Example 4

Minimize the same quadratic form subject to the constraint \( x_1^2 + x_2^2 + x_3^2 = 1 \).

Form

\[
f(x_1, x_2, x_3, \lambda) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 + \lambda (x_1^2 + x_2^2 + x_3^2 - 1)
\]

and the resulting system (no longer linear because the constraint is not linear!) is

\[
\begin{align*}
\frac{\partial f}{\partial x_1} (x_1, x_2, x_3, \lambda) &= 2x_1 - 2x_2 + 2\lambda x_1 = 0 \\
\frac{\partial f}{\partial x_2} (x_1, x_2, x_3, \lambda) &= 2x_2 - x_1 - x_3 + 2\lambda x_2 = 0 \\
\frac{\partial f}{\partial x_3} (x_1, x_2, x_3, \lambda) &= 2x_3 - x_2 + 2\lambda x_3 = 0 \\
\frac{\partial f}{\partial \lambda} (x_1, x_2, x_3, \lambda) &= x_1^2 + x_2^2 + x_3^2 - 1 = 0
\end{align*}
\]

Nonlinear systems, such as this one, are typically not easy to solve. But this one is not so bad. Assume for a moment that \( \lambda \) is known and rewrite the first three equations as

\[
\begin{bmatrix}
2 (1 + \lambda) & -1 & 0 \\
-1 & 2 (1 + \lambda) & -1 \\
0 & -1 & 2 (1 + \lambda)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Since \( x_1 = x_2 = x_3 = 0 \) does not satisfy the fourth equation, the above matrix must be singular and have zero determinant:

\[
(1 + \lambda) (2\lambda^2 + 4\lambda + 1) = 0
\]
One of the solutions is $\lambda = -1$ and the corresponding overall solution is

$$
\begin{align*}
  x_2 &= 0 \\
  x_1 &= x_2 = \pm \frac{\sqrt{2}}{2}
\end{align*}
$$

Other solutions can be found by considering solutions of $2\lambda^2 + 4\lambda + 1 = 0$.

For reference, here’s the plot of the quadratic form on the unit sphere:

2.4 Notes about the method of Lagrange multipliers

1. Constrained minimization of $\frac{1}{2} x^T A x - x^T b$ can be derived in similar fashion.
2. The matrix that includes Lagrange multipliers is not positive definite because there are zeros on the diagonal.

3 The finite element method (FEM) in 1D

The finite element method belongs to the category of numerical methods that adhere to following philosophy:

\[ \text{Replace the exact physical problem} \]
\[ \text{with an approximate discrete one} \]
\[ \text{and solve the latter exactly} \]

The exact problem that we need to solve is this. Minimize the integral

$$
E = \frac{1}{2} \int_0^1 \frac{du}{dx} \frac{du}{dx} \, dx
$$

among all differentiable functions $u(x)$, $x \in [0, 1]$, subject to the boundary conditions $u(0) = A$, $u(1) = B$.

The approximate replacement problem is this. Given a partitioning $[x_0 = 0, x_1, \ldots, x_{N-1}, x_N = 1]$ of $[0, 1]$, minimize the integral

$$
E = \frac{1}{2} \int_0^1 \frac{du}{dx} \frac{du}{dx} \, dx \quad \text{(1)}
$$
among all piecewise linear continuous functions $u(x)$ (linear on each $[x_i, x_{i+1}]$)
subject to the boundary conditions $u(0) = A$, $u(1) = B$.

We have replaced the original problem with an infinite number of degrees of
freedom with a discrete problem that only has $N - 1$ degrees of freedom.

3.1 The FEM basis

We have effectively chosen a basis (in the linear algebra sense) with which to
represent the function $u(x)$.

Consider the set of piecewise linear functions $\psi_r$, $r = 0, ..., N$, such that

$$\psi_r(x_i) = \begin{cases} 1, & i = r \\ 0, & i \neq r \end{cases}$$

This is called an "atomic" basis of hat functions. Three representatives of this
basis are seen here:

Any linearly independent set of piecewise linear functions could have been
chosen as the bases. This set, however, provides to significant advantages. If
the piecewise linear function $u(x)$ has values $u_0, u_1, ..., u_N$ at the nodes then it
is represented in this basis simply as

\[ u = \sum_{r=0}^{N} u_r \psi_r \]

The second advantage is that each basis function has a very narrow support (region where the function is nonzero). As a result, the eventual linear system will be highly sparse!

### 3.2 The FEM algorithm

Represent function \( u(x) \) in the above basis:

\[ u(x) = \sum_{r=0}^{N} c_r \psi_r(x). \quad (2) \]

We could have written \( u_r \psi_r \), but \( c_r \psi_r \) is a little more general and will work for other bases. Our goal is to determine \( c_0, c_1, ..., c_N \) by minimizing the objective function. Substitute equation (2) into the objective function

\[
E(c_0, ..., c_N) = \int_{0}^{1} \left( \sum_{r=0}^{N} c_r \frac{d\psi_r}{dx} \right) \left( \sum_{s=0}^{N} c_s \frac{d\psi_s}{dx} \right) dx
\]

\[
= \frac{1}{2} \sum_{r,s=0}^{N} c_r c_s \int_{0}^{1} \frac{d\psi_r}{dx} \frac{d\psi_s}{dx} dx.
\]

The objective function reduced to the quadratic form

\[ E(c_0, ..., c_N) = \frac{1}{2} c^T M c, \quad (3) \]

where

\[ M_{rs} = \int_{0}^{1} \frac{d\psi_r}{dx} \frac{d\psi_s}{dx} dx. \quad (4) \]

It is to be minimized with respect to \( c_0, c_1, ..., c_N \) subject to the following constraints which come from the boundary conditions:

\[ c_1 = A \quad (5) \]

\[ c_{N+1} = B \quad (6) \]

This is a problem that we have solved several times in the first half of this document! What we have not yet demonstrated – and it is the only remaining task – is how to form the matrix \( M \).
3.3 Example: Forming the matrix $M$

We will form the matrix $M$ explicitly for the following partitioning of $[0, 1]$:

$$
\begin{bmatrix}
0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1
\end{bmatrix}
$$

This is a regular grid for $N = 4$. One of the main strengths of the finite element method is that the grid need not be regular. However, we have chosen a regular grid to simplify the computations in this example. The atomic basis for this partitioning looks like this:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example_matrix}
\caption{Atomic basis for the partitioning.}
\end{figure}

In this example, we will go "matrix entry by matrix entry". (When writing code, it is customary to go "segment by segment").

We start with the element $M_{11}$. We have

$$
M_{11} = \int_0^1 \frac{d\psi_1}{dx} \frac{d\psi_1}{dx} dx
$$

The function $\psi_1(x)$, seen on the first subplot of the above figure, is only nonzero on the segment $[0, \frac{1}{4}]$, where it drops from $\psi_1(0) = 1$ to $\psi_1(\frac{1}{4}) = 0$. In other
words, \( \frac{d\psi_1}{dx} = -4 \), and we have

\[
M_{11} = \int_{0}^{\frac{1}{4}} (-4) (-4) \, dx = 4
\]

Let us next consider another diagonal entry \( M_{22} \):

\[
M_{22} = \int_{0}^{1} \frac{d\psi_2}{dx} \frac{d\psi_2}{dx} \, dx
\]

The basis function \( \psi_2(x) \) is nonzero on \( [0, \frac{1}{4}] \) where it first slopes up from \( \psi_2(0) = 0 \) to \( \psi_2(\frac{1}{4}) = 1 \) and then slopes back down to \( \psi_2(\frac{1}{2}) = 0 \). Therefore,

\[
M_{11} = \int_{0}^{\frac{1}{4}} 4 \times 4 \, dx + \int_{\frac{1}{4}}^{\frac{1}{2}} (-4) (-4) \, dx = 8
\]

The same computation applies to all other diagonal entries except \( M_{5,5} \) which follows the pattern of \( M_{11} \).

Let us next look at an off-diagonal element, \( M_{12} = M_{21} \):

\[
M_{12} = \int_{0}^{1} \frac{d\psi_1}{dx} \frac{d\psi_2}{dx} \, dx.
\]

The segment where the functions \( \psi_1(x) \) and \( \psi_2(x) \) are both nonzero is \( [0, \frac{1}{4}] \), where \( d\psi_1/dx = -4 \) and \( d\psi_2/dx = 4 \). Therefore

\[
M_{12} = \int_{0}^{\frac{1}{4}} (-4) 4 \, dx = -4.
\]

The same computation applies to every entry on the sub- and super-diagonal. These entries correspond to the interactions of neighboring element functions.

It is evident that the rest of the entries is zero since element function that are not immediate neighbors do not share a region where both are nonzero.

In summary, the matrix \( M \) is this:

\[
M = \begin{bmatrix}
4 & -4 & 0 & 0 & 0 \\
-4 & 8 & -4 & 0 & 0 \\
0 & -4 & 8 & -4 & 0 \\
0 & 0 & -4 & 8 & -4 \\
0 & 0 & 0 & -4 & 4 \\
\end{bmatrix}
\]

This matrix is symmetric (that can be seen) and positive definite, since

\[
c^T M c = \int_{0}^{1} \left( \sum c_r \frac{d\psi_r}{dx} \right)^2 \, dx
\]

and is therefore always positive as long as at least one of \( c_r \) is not zero.
3.4 Example: Incorporating boundary conditions by Lagrange multipliers

Omitting the details, incorporating the boundary conditions (5), (6) according to the method of Lagrange multipliers leads to the following system

\[
\begin{bmatrix}
4 & -4 & 0 & 0 & 0 & 1 & 0 \\
-4 & 8 & -4 & 0 & 0 & 0 & 0 \\
0 & -4 & 8 & -4 & 0 & 0 & 0 \\
0 & 0 & -4 & 8 & -4 & 0 & 0 \\
0 & 0 & 0 & -4 & 4 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5 \\
\lambda_1 \\
\lambda_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
A \\
B
\end{bmatrix}
\]

which leads to the following solution

\[
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5 \\
\lambda_1 \\
\lambda_2
\end{bmatrix}
= 
\begin{bmatrix}
4 & -4 & 0 & 0 & 0 & 1 & 0 \\
-4 & 8 & -4 & 0 & 0 & 0 & 0 \\
0 & -4 & 8 & -4 & 0 & 0 & 0 \\
0 & 0 & -4 & 8 & -4 & 0 & 0 \\
0 & 0 & 0 & -4 & 4 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
^{-1}
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
A \\
B
\end{bmatrix}
= 
\begin{bmatrix}
A \\
\frac{3}{4}A + \frac{1}{2}B \\
\frac{3}{4}A + \frac{1}{2}B \\
\frac{3}{4}A + \frac{1}{2}B \\
\frac{3}{4}A + \frac{1}{2}B \\
B \\
A - B
\end{bmatrix}
\]

3.5 Important note

The solution in the linear interpolation from A to B. We have actually solved the problem exactly! This is a case of "superconvergence" – obtaining a better answer that could be expected. In our case, this is not surprising. We restricted the candidates for minimization to piecewise linear functions. But the true solution is a piecewise linear function. In other words, the true solution was still in the mix and was therefore retrieved.

3.6 Example: Incorporating boundary conditions by direct substitution

Once again omitting the details we state the eventual linear system:

\[
\begin{bmatrix}
8 & -4 & 0 \\
-4 & 8 & -4 \\
0 & -4 & 8
\end{bmatrix}
\begin{bmatrix}
c_2 \\
c_3 \\
c_4
\end{bmatrix}
= 
\begin{bmatrix}
-4 & 0 & 0 \\
0 & -4 & 0 \\
0 & 0 & -4
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
8 & -4 & 0 \\
-4 & 8 & -4 \\
0 & -4 & 8
\end{bmatrix}
\begin{bmatrix}
c_2 \\
c_3 \\
c_4
\end{bmatrix}
= 
\begin{bmatrix}
4A \\
0 \\
4B
\end{bmatrix}
\]

12
The solution is

\[
\begin{bmatrix}
  c_2 \\
  c_3 \\
  c_4 \\
\end{bmatrix} = \begin{bmatrix}
  8 & -4 & 0 \\
  -4 & 8 & -4 \\
  0 & -4 & 8 \\
\end{bmatrix}^{-1} \begin{bmatrix}
  4A \\
  0 \\
  4B \\
\end{bmatrix} = \begin{bmatrix}
  \frac{3}{4}A + \frac{1}{4}B \\
  \frac{1}{4}A + \frac{1}{4}B \\
  \frac{3}{4}A + \frac{1}{4}B \\
\end{bmatrix}
\]