Towards thermodynamics of elastic electric conductors
M. Grinfeld a, P. Grinfeld b
a The Educational Testing Service, Princeton, New Jersey, USA
b Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA

Online Publication Date: 01 May 2001
To cite this Article: Grinfeld, M. and Grinfeld, P. (2001) 'Towards thermodynamics of elastic electric conductors', Philosophical Magazine A, 81:5, 1341 - 1354
To link to this article: DOI: 10.1080/01418610108214445
URL: http://dx.doi.org/10.1080/01418610108214445

Philosophical Magazine A
Publication details, including instructions for authors and subscription information: http://www.informaworld.com/smpp/title~content=t713396797

PLEASE SCROLL DOWN FOR ARTICLE
Towards thermodynamics of elastic electric conductors

MICHAEL GRINFELD†

The Educational Testing Service, Rosedale Road, Princeton, New Jersey 08541, USA

and PAVEL GRINFELD

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, 2-236, Cambridge, Massachusetts 02139-4307, USA

[Received 29 August 2000 and accepted in revised form 29 October 2000]

Abstract

We discuss some problems of equilibrium shapes and possible morphological patterns of crystalline deformable conductors. To this end we propose a master system describing quasistatic evolution of shapes of such conductors owing to rearrangement of their material particles. The driving force of the evolution is the diminishing of the total energy of the system consisting of the elastic, electrostatic and surface energies. The proposed master system is based on the assumption that the characteristic time scale of establishing equilibrium with respect to rearrangement of materials particles is much greater than the time scales of establishing equilibrium with respect to mechanical and electrostatic degrees of freedom. The proposed system is based on principles and notions of the exact nonlinear continuum mechanics in the Eulerian description. We use the exact master system to derive an explicit formula of increment or decrement in the growth of small disturbances of an isotropic, uniformly stressed elastic half-plane in a uniform electrostatic field. In the case of small deviations from the unstressed state we express the coefficients of the dispersion relation in terms of the Lamé and Poisson modules. After establishing the key dispersion equation on the solid basis of the nonlinear theory, we show some modifications of the master system that will allow us to establish the same dispersion equations without any appeal to the notions and concepts of the exact theory. Finally we apply our system to the case of incompressible substances and demonstrate that the suggested master system leads to the same (in)stability criterion as that established long ago for a liquid conductor in an electrostatic field.

§ 1. Introduction

In recent decades, the theory and applications of the Gibbs (1876, 1878) equilibrium thermodynamics of heterogeneous systems has been showing substantial progress in several directions. In particular, the theory of stability of heterogeneous systems with solid phases (Nozières 1991, Grinfeld, 1993, 1995) has proven to be useful in many traditional areas as well as in a number of new areas. For some of these new areas the stress-driven rearrangement destabilization should be considered along with the action of electrostatic and/or magnetostatic fields (for example

† Email: mgrinfeld@ets.org
We think that studying these phenomena is important for a number of industries including protecting ships and pipelines from corrosion (the so-called anodic protection of ships and oil pipelines). We believe that the appropriate extensions of the Gibbs thermodynamics which include both the nonlinear elasticity and electrostatic or magnetostatic fields might be helpful for further progress in these areas.

The problems discussed in this paper are not new. On the contrary, they are so old that the present authors are not able to present here a full historical review of relevant thermodynamics which benefited from contributions of such great thermodynamicists as Kelvin, Helmholtz and Gibbs. In this historical context, it is worth noting that the theories of elasticity and of electromagnetism were greatly influenced by each other in the early stages of their development (Maugin 1988, Whittaker 1989). Further, even a perfunctory investigation will reveal that thermodynamics is the central topic in electrochemistry. To a reader interested in learning some relevant applications and theoretical difficulties in the thermodynamic foundations we recommend the classic textbooks by Stratton (1941), Landau and Lifshitz (1965), Becker (1982) and Tamm (1989).

Despite all the remarkable achievements of thermodynamics of heterogeneous systems with dipolar interaction, some major gaps in this branch of thermodynamics remain. The most obvious of these gaps is the absence of relevant theory concerning the second energy variation (i.e. of the thermodynamic stability) of solid heterogeneous system with electrostatic or magnetostatic interactions. The classic books written on the subject establish various formulae of ponderomotive forces the derivation of which is basically equivalent to the analysis of the first variation problem of the relevant energy functionals. (It will not escape the reader's attention that some of the key energy formulae for the ponderomotive forces are incompatible. For clarification we recommend especially the books by Brown (1966) and Rosensweig (1985) and the references therein). There is no doubt that solid heterogeneous systems with dipolar interaction should demonstrate various morphological instabilities. Such instabilities are well known for a number of liquid systems (for example Rayleigh (1900), Tonks (1935), Frenkel' (1936) and Rosensweig (1985)). In these publications the fluid property of the substance is used in intermediate calculations but does not appear in the final stability criteria which are purely energetic in nature. Clearly it is more natural to explore such phenomena using a purely thermodynamic method of the second energy variation which is equally applicable to liquids and solids. Because of ponderomotive forces, in addition to the above-mentioned electricity-driven destabilization, crystalline substances should undergo the now widely known stress-driven rearrangement instabilities at various interfaces. These two wide classes of instabilities can 'work' together and the interaction of these two classes of morphological instabilities has never been explored. It is of importance from academic and practical viewpoints.

In this relatively short paper we touch upon a few of the most elementary aspects of thermodynamics of crystalline conductors with rearrangement in the presence of stresses and quasistatic dipolar fields. We discuss some general thermodynamics aspects considering the simplest problem of equilibrium shape of an elastic crystalline conductor placed in an electrostatic field. This simplest problem (which is, in fact, quite complicated) requires several concepts and possesses several features which are identical with or similar to those of general heterogeneous systems with crystalline components.
The current situation in thermodynamics of heterogeneous systems in electrostatic or magnetostatic fields is reminiscent of the situation some time ago in the thermodynamic theory of heterogeneous systems with solid phases, especially in what concerns the absence of the second variation theory. The last 20 years of remarkable progress in the theory of heterogeneous systems with solid phases demonstrated that all interested communities easily accepted the main conclusions of the stability theory based on the analysis of the second energy variation but not the theory of the second energy variation itself. Instead there is a tendency of using specific simple examples and/or different quasistatic approaches which yield analysis of the equilibrium, stability and, to some extent, post-stable evolution of configuration. Although we share with other researchers several points of criticism of the quasistatic approach (which is conceptually less reliable than the approach based on the theory of the second energy variation), we agree that the quasistatic approach is technically simpler and, when used properly, it leads to the conclusions about equilibrium and stability which are equivalent to those established by the analysis of the first and second energy variations. To facilitate the progress in the thermodynamics of heterogeneous systems with electrostatic fields we present here the results of the analysis of the first and second energy variations in terms of the quasistatic theory of evolution. Although the quasistatic approximation has obvious limitations, there is no doubt that the system can be legitimately and successfully used for a numerical calculation of stable equilibrium configurations.

In this paper, contrary to the traditional approach of linear elasticity, our consideration is based on exact nonlinear elasticity which is more suitable for thermodynamic foundations for various reasons. The present authors believe that it is important to use nonlinear elasticity even when elastic stresses and deformations are small; the effects of the stress-driven rearrangement instabilities are formally of the second order of magnitude in elastic deformation. However, this is exactly the orders of magnitude which are neglected in the linear elasticity. During the last 40 years a rather large amount of publications has been devoted to the nonlinear analysis of stresses in electromagnetic systems (for example the monograph by Brown (1966) and the references therein). The nonlinear elasticity approach suggested in these publications is based on the Lagrangian description of elastic continuum which is quite convenient for the analysis of relevant bulk effects in elasticity but is much less convenient for the analysis of the interfaces at which geometric and material parameters and fields experience discontinuities of various orders. The Eulerian description appears to be much more convenient for the analysis of the interfaces in the presence of electrostatic or magnetostatic fields and, for some quite simple and useful models of the interfaces, it appears to be the only practical description. In this paper, the analysis is based on the Eulerian description of continuum.

§ 2. THE MODEL FOR AN ELASTIC ELECTRIC CONDUCTOR

According to the Eulerian description of continuum, the spatial coordinates $z'$ (each index $i$ has the values $1, 2, 3$ and, as always, summation over repeated covariant and contravariant indices is implied), together with time (or the time-like variation parameter) $t$, play the role of independent variables. We use the notation $z^\alpha (z), z_\beta (z)$ for the covariant and contravariant components of the metric tensor of the spatial coordinate system $z'$. This tensor is responsible for raising and lowering the indices as well as the spatial covariant differentiation denoted as $\nabla_i$. 
Let us consider a certain amount of a crystalline elastic substance occupying the domain $\Omega$ bounded by the surface $S$ (Figure 1). We assume that the part $S_1$ of the external boundary $S$ is free of any external forces. The positions of material particles that comprise the remaining part $S_2$ are prescribed. In other words, we assume that certain displacements $\vec{u}$ are specified on $S_2$. Also, it is assumed that rearrangement of materials particles can occur on the part $S_1$. There exist a number of rearrangement mechanisms: surface diffusion, melting-crystallization, evaporation-sublimation, transport of ions in surrounding electrolyte, etc. Particular mechanisms are of secondary importance for morphological destabilizations caused by accumulated energy release. Below we suggest a certain particularly simple model which yields an easy numerical implementation and reflects the most robust features of various rearrangement mechanisms. Of course, for each specific system the mechanism and rate of the rearrangement are essential and they are able to prevent the occurrence of the instability even if the flat interface is thermodynamically unstable. In fact, in the Eulerian description (contrary to the Lagrangian description) both the domain $\Omega$ and the surface component $S_1$ change their spatial positions if the displacements of a deformable substance are not equal to zero. It should be clear to the reader that the domain $\Omega$ and the surface component $S_1$ change in both descriptions if rearrangement occurs on $S_1$.

The total energy $E_{\text{total}}$ consists of the following energy components: electrostatic energy $\Phi_{\text{el}}$, gravitational energy $\Phi_{\text{mass}}$, surface energy $\Phi_{\text{surf}}$ and elastic energy $\Phi_{\text{elast}}$:

$$E_{\text{total}} = \Phi_{\text{el}} + \Phi_{\text{mass}} + \Phi_{\text{surf}} + \Phi_{\text{elast}}$$

$$= \frac{1}{8\pi} \int_{\text{Spacc}} d\Omega \nabla_j \varphi \nabla^j \varphi + \int_{\Omega} d\Omega \rho G(z) + \int_{S} dS \sigma_L + \int_{\Omega} d\Omega \rho e (\nabla_j U_i), \quad (1)$$

where we use $U'(z)$ for the displacements of the material points, $\rho(z)$ for the mass densities of the substance, and $\varphi(z)$ and $E_i(z) \equiv -\nabla_i \varphi$ for the electrostatic potential and the electric field respectively. Further, $\nabla_j U_i$ denote the spatial displacement gradients, $e(\nabla_j U_i)$ the elastic energy density per unit mass, $G$ the potential of mass forces and $\sigma$ the surface energy density per unit area. For a crystalline substance the surface energy $\sigma$ depends strongly on the orientation of the interface with respect to the crystallographic axes of the substance (which is usually described by a function $\sigma(N_i)$ of the unit normal $N_i$ to the surface). For simplicity we ignore this dependence and assume $\sigma_L$ to be a certain positive constant. We assume that the system is kept under a fixed absolute temperature $T$. Therefore, the elastic energy $e$ is actually the
free energy density of the crystal. We assume that the external electrostatic field is generated by immobile electric charges \(q(z)\). Thus, the total potential \(\varphi\) is generated by the external charges \(q\) and the surface charges distributed with the density \(\tau\) on the surface of the crystalline conductor.

We need to make a few further assumptions concerning the evolution of the system.

(i) At all times the body is in the state of electrostatic equilibrium.
(ii) At all times the body is in the state of mechanical equilibrium both within the bulk and on \(S_1\).
(iii) In the process of the shape evolution, because of the rearrangement of material particles in vicinity of \(S_1\) the total mass of the crystalline body remains fixed.
(iv) In the process of the shape evolution, because of the rearrangement of material particles in vicinity of \(S_1\) the total energy of the crystalline body monotonically decreases.

§3. The master system of quasistatic evolution

The postulated master system of quasistatic evolution contains three components:

(i) the system of electrostatics;
(ii) the system of mechanical equilibrium;
(iii) the system of kinetics of rearrangement.

3.1. Equations of electrostatics

This subsystem is standard and it is valid for the static field in any conductor of electricity.

(a) Within the conductor's body,

\[ \varphi = \text{constant}. \]

(b) Outside the conductor,

\[ \nabla^i\nabla_i\varphi = -4\pi q. \]

(c) Across the conductor's interface

\[ [\varphi]^+_- = 0, \]

\[ [\nabla_i\varphi]^+_- N^i = 4\pi \tau \]

where + corresponds to the domain occupied by the conductor, \(N^i\) is the external unit normal to the conductor's boundary and \(\tau\) is the surface density of the electric charge.

3.2. Equations of mechanical equilibrium

This system is standard as well. We make one modification, however. We formulate it in the exact nonlinear form. Fortunately, the exact nonlinear form of this system and its linear approximation look the same. The nonlinearity affects only the expression for the tensor of finite deformations and the thermodynamics formula expressing the stress tensor in terms of the elastic energy density. It is necessary to
distinguish between the location of the interfaces in the undeformed and deformed configurations.

We assume the following equations of 'mechanical' equilibrium within the bulk \( \Omega \) of the crystal, at the fixed interface \( S_2 \), and at the free interface \( S_1 \).

(a) Within the bulk,
\[
\nabla_j \sigma^{ij} = \rho \nabla^i G.
\]

(b) At the fixed interface \( S_2 \),
\[
U^i = \tilde{u}^i,
\]
where \( \tilde{u}^i \) is a given boundary displacement.

(c) At the free interface \( S_1 \),
\[
\sigma^{ij} N_j = (2\pi r^2 + \sigma L B'^{ij} ) N^i ,
\]
where \( B'^{ij} \) is twice the mean curvature of the interface (in the deformed configuration) and \( \sigma^{ij} \) is the Cauchy (or actual) stress tensor defined by the Murnaghan (1951) formula:
\[
\sigma^{ij} = \rho \frac{\partial e}{\partial \nabla^i U^i_m} (\delta^i_m - \nabla^i U^i_m) = \rho \frac{\partial e}{\partial \nabla^i U^i_m} A^i_m ,
\]
where \( A^i_m = \delta^i_m - \nabla^i U^i_m \).

### 3.3. Mechanism and kinetics of rearrangement

We write the surface (local) chemical potential \( \mu^{(N)}(\text{sur}) \) of the boundary particles in the form
\[
\mu^{(N)} = \chi^{ij} N_i N_j,
\]
where \( \chi^{ij} \) is the 'generalized' Bowen (1967) chemical potential tensor given by
\[
\chi^{ij} = (e + G) z^{ij} - \frac{1}{\rho} \sigma^{ij} .
\]

We postulate that the surface mass flux \( J(\xi, t) \) at each point \( (\xi, t) \) of the interface \( S_1 \) has the following form:
\[
J(\xi, t) = \rho (C - f^i N_i) = -K [\mu^{(N)}(\xi, t) - \langle \mu^{(N)} \rangle] ,
\]
where \( C(\xi, t) \) is the velocity of the interface \( S_1 \), \( f^i \) is the spatial velocity of materials' particles in the process of quasistatic evolution and \( \langle A \rangle \) represents the average of a surface function \( A(\xi, t) \) over the interface \( S_1 \):
\[
\langle A \rangle = \frac{1}{|S_1|} \int_S dS A.
\]

The present authors hope that the suggested kinetic equation (12) is quite intuitive. It is of primary importance for us that the kinetic equation (12) guarantees a monotonic decrease in the total energy of the system and conservation of its total mass implied by the relationship
\[
\int_{S_1} dS J = 0.
\]
3.4. An elastic conducting layer in a uniform electric field

Equations (2)-(9) can be used to determine the equilibrium electrostatic and stress–deformation fields in crystalline conductors. Let us consider the simplest problem of a conducting layer of a constant thickness $2h$ (in the undeformed configuration) placed perpendicularly to a uniform electric field of intensity $R$. This model problem can be used to describe electromechanical fields in finite elastic layers with the in-plane sizes much greater than the thickness of the layer in the layer’s points located far enough from the layer’s edge. The electrostatics equations (2)-(5) show that uniform distribution of surface electric charges of the density $\tau = \pm R/4\pi$ will appear on the opposite sides of the conducting layer. The mechanical equilibrium equations (6)-(9) show that the layer will be uniformly stressed with the principal stress component $P_\bot = 2\pi \tau^2 = R^2/8\pi$. The in-plane principal stresses depend on the boundary conditions at the layer’s edge. In particular, if the edge is stress free, the in-plane principal stresses are equal to zero. After finding the principal stresses, the deformations can be found with the help of the equations (9) (of course, it can be done if the elastic energy density is specified as a function of deformation gradients or as a certain equivalent function, say, as the function of the main elongations for isotropic elastic substances). If the layer is clamped at its edge, then the in-plane elongations are equal to zero, and the ‘stress-deformation’ relationship is used to determine the in-plane principal stresses. Another important situation is when a conducting layer is attached to a certain flat substrate. Similar configurations appear when a metallic layer is deposited by a certain mechanism of mass transport. Corresponding stresses are determined by the misfit in the lattice parameters of the substrate and depositing substance.

§ 4. Dispersion relation for a deformable substance

We limit ourselves to the study of an isotropic elastic solid two-dimensional conductor (all fields depend on the variables $x^1 = x$ and $x^2 = z$ only and the $x^3$ component of every field vanishes identically). We assume that a uniform electric field is directed along the $z$ axis (figure 2). In an isotropic solid the so-called principal stresses $P_M (M, N = 1, 2, 3)$ are certain functions of the principal elongations $A_N$ only: $P_M = P_M (A_N)$. The following quantities $b_{MN}$ and $c_{MN}$ play the role of the so-called ‘instant’ elastic moduli:

\[
\begin{align*}
  b_{21} &= A_1^2 \frac{P_1^2 - P_2^2}{A_1^2 - A_2^2}, \\
  b_{12} &= A_1^2 \frac{P_1^2 - P_2^2}{A_1^2 - A_2^2}, \\
  c_{21} &= A_1 \frac{\partial P_1^2}{\partial A_1}, \\
  c_{12} &= A_2 \frac{\partial P_1^2}{\partial A_2}, \\
  c_{11} &= A_1 \frac{\partial P_1^2}{\partial A_1}, \\
  c_{22} &= A_2 \frac{\partial P_2^2}{\partial A_2},
\end{align*}
\]

(15)

In particular, if the system is close to being undeformed (i.e. when $A_N = 1$) the instant moduli can be expressed in terms of the Lamé elastic moduli:

\[
\begin{align*}
  b_{12} &= b_{21} = \mu, \\
  c_{21} &= c_{12} = \lambda, \\
  c_{11} &= c_{22} = \lambda + 2\mu.
\end{align*}
\]

(16)

In the case of unequal principal deformations, when at least one of the principal elongations is not equal to the other two, the material becomes anisotropic (this is a
stress-generated anisotropy) and its ‘instant’ moduli cannot be expressed in terms of the Lamé elastic moduli.

By considering rearrangements of the length scale much smaller than the characteristic sizes of the crystalline conductor we can limit ourselves to the special case when the conductor occupies the lower half-plane, \( z < 0 \) (or, correspondingly, of the lower half-space in the three-dimensional case). Let us consider a uniformly deformed equilibrium configuration of this half-plane and explore increment and decrements of small rearrangement disturbances of the upper border of the solid. We use a superscript open circle to denote various parameters of the equilibrium configuration and a tilde for the small disturbances.

The disturbed electrostatics equations (2)-(5) are as follows.

(a) Within the conductor,
\[
\tilde{\varphi} = \text{constant.} \tag{17}
\]

(b) Outside the conductor,
\[
\nabla \tilde{\varphi} = 0. \tag{18}
\]

(c) Across the conductor's interfaces,
\[
\left[ \frac{\partial \tilde{\varphi}}{\partial t} \right]^+ R \tilde{C} = 0, \tag{19}
\]
\[
\nabla \left[ \frac{\partial \tilde{\varphi}}{\partial t} \right]^+ N^I = 4\pi \frac{\delta \tau}{\delta t}, \tag{20}
\]
where \( \delta/\delta t \) differentiation is understood in the sense of the work by Grinfeld (1991).
With the help of the notation in equation (15) the two non-trivial linearized bulk equilibrium equations (6) are (in the absence of gravity)

\[ \frac{\partial}{\partial x} \left( c_{11}^0 \frac{\partial f^1}{\partial x} + c_{12}^0 \frac{\partial f^2}{\partial z} \right) + \frac{\partial}{\partial z} \left( h_{12}^0 \frac{\partial f^2}{\partial x} + h_{21}^0 \frac{\partial f^1}{\partial z} \right) = 0, \]

\[ \frac{\partial}{\partial x} \left( b_{12}^0 \frac{\partial f^2}{\partial x} + b_{21}^0 \frac{\partial f^1}{\partial z} \right) + \frac{\partial}{\partial z} \left( c_{21}^0 \frac{\partial f^1}{\partial x} + c_{22}^0 \frac{\partial f^2}{\partial z} \right) = 0. \]  

(21)

The linearized boundary mechanical equilibrium equations (8) are

\[ b_{12}^0 \frac{\partial f^1}{\partial z} + b_{21}^0 \frac{\partial f^2}{\partial x} + (2\pi r_0^0 - \sigma^{01}) \frac{\partial \tilde{C}}{\partial x} = 0, \]

\[ c_{21}^0 \frac{\partial f^1}{\partial x} + c_{22}^0 \frac{\partial f^2}{\partial z} - 4\pi r_0 \frac{\delta \tilde{r}}{\delta t} - \sigma \frac{\partial^2 \tilde{C}}{\partial x^2} = 0. \]  

(22)

(23)

Similarly, equation (12) is

\[ \frac{\delta \tilde{C}}{\delta t} = \frac{\partial \tilde{f}_1}{\partial t} N_i^0 - \frac{\rho^0}{\rho^2} \left( \sigma^{0i} \nabla_j \tilde{f}_1 - 2\pi r_0 \nabla_i \tilde{f}_1 - 4\pi r_0 \frac{\delta \tilde{r}}{\delta t} - \sigma \nabla_{\alpha} \nabla^\alpha \tilde{C} - \rho^0 \frac{d\tilde{\Theta}}{dt} \right). \]

(24)

where

\[ \rho^0 \frac{d\tilde{\Theta}}{dt} = \left\langle \sigma^{0i} \nabla_j \tilde{f}_1 - 2\pi r_0 \nabla_i \tilde{f}_1 - 4\pi r_0 \frac{\delta \tilde{r}}{\delta t} - \sigma \nabla_{\alpha} \nabla^\alpha \tilde{C} \right\rangle_S. \]

(25)

The bulk equilibrium system (21) allows an exponentionally decaying solution (in the negative \( z \) direction):

\[ f^1(x, z) = \exp (\eta t + ikx) [H_1 \exp (k|r_1|z) + H_2 \exp (k|r_2|z)], \]

\[ f^2(z) = \exp \eta t + ikx i[\Theta_1 H_1 \exp (k|r_1|z) + \Theta_2 H_2 \exp (k|r_2|z)], \]

(26)

where \( H_1 \) and \( H_2 \) are constants, \( |r_1| \) and \( |r_2| \) are the two positive roots of the biquadratic equation

\[ r^4 - r^2 \frac{c_{11}^0 c_{22}^0 + b_{12}^0 b_{21}^0 - (c_{12}^0 + b_{12}^0)(c_{21}^0 + b_{21}^0)}{c_{22}^0 b_{21}^0} + \frac{c_{11}^0 b_{12}^0}{c_{22}^0 b_{21}^0} = 0 \]  

(27)

and \( \Theta_1 \) and \( \Theta_2 \) are defined as

\[ \Theta_{1,2} = - \frac{c_{11}^0 - b_{21}^0 r_{1,2}}{(c_{12}^0 + b_{12}^0)r_{1,2}}. \]  

(28)

The solution (26) should be extended to include other unknown fields:

\[ \bar{\varphi}(x, z, t) = \exp (\eta t + ikx) \left\{ \begin{array}{ll} A_+ \exp (-kz) & \text{at } z \geq h, \\
0 & \text{at } z \leq 0, \end{array} \right. \]

\[ \tilde{\tau}(x, t) = \exp (\eta t + ikx) T, \quad \tilde{C}(x, t) = \exp (\eta t + ikx) \tilde{T} \]

(29)
Substituting equations (26) and (29) into the linearized master system, we arrive at the following formula for the increment or decrement $\eta$:

$$\eta = \frac{K}{\rho^{02}} \frac{k\Gamma B_2 - k(\sigma k - 4\pi \tau^{02})}{1 + B_1},$$

where

$$\Gamma = \sigma^{11} - 2\pi \tau^{02} = \sigma^{11} - \sigma^{022}$$

is the parameter of non-hydrostatics which is equal to the doubled maximum shear stress.

The coefficients $B_1$ and $B_2$ are given by

$$B_1 = \frac{1}{D} \left[ \Gamma c_1^0 (\Theta_1 - \Theta_2) - \Gamma c_2^0 \Theta_1 \Theta_2 (r_1 - r_2) + (4\pi \tau^{02} - \sigma k) b_{12}^0 (r_2 \Theta_1 - r_1 \Theta_2) \right],$$

$$B_2 = \frac{1}{D} \left[ -\Gamma c_2^0 (\Theta_1 r_1 - \Theta_2 r_2) + (4\pi \tau^{02} - \sigma k) [b_{21}^0 (\Theta_1 - \Theta_2) - b_{12}^0 (r_1 - r_2)] \right]$$

and $D$ is given by

$$D \equiv (b_{21}^0 r_1 - b_{21}^0 \Theta_1) (c_{21}^0 + c_{22}^0 \Theta_2 r_2) - (b_{12}^0 r_2 - b_{22}^0 \Theta_2) (c_{21}^0 + c_{22}^0 \Theta_1 r_1).$$

We would like to emphasize that the relation (30) for the decrement $\eta$ is exact in the sense that it is not based on any assumptions of the magnitude of the deformations in the equilibrium configuration. This is very important since the effects of the stress-driven rearrangement instability are of second order with respect to the equilibrium stresses and deformations. Therefore, the conclusions based on the linear elasticity are questionable since the linear theory itself is based on the assumption that the second-order terms in deformations can be neglected. This does not mean, of course, that the results derived from the linear theory cannot be correct.

When the equilibrium deformations are much less than unity, equation (30) for the increment or decrement can be replaced with an approximate form:

$$\eta = \frac{K}{\rho^{02}} \frac{4\pi \tau^{02} k - \sigma k^2 + [(1 - \nu)/\mu] k^2}{1 + (1 - \nu)\sigma k/2\mu}.$$

This expression incorporates two instabilities. The first is the rearrangement instability due to the presence of electric charges. The driving force for this instability is the accumulated electrostatic energy. In the case of a liquid substance this instability was explored by Rayleigh (1900), Tonks (1935) and Frenkel' (1936) (Frenkel's approach is presented in the textbook by Landau and Lifshitz (1965)). In the case of a quasistatic rearrangement in an isotropic elastic solid the increment of this instability is given by the formula

$$\eta = \frac{K}{\rho^{02}} \frac{4\pi \tau^{02} k}{1 + (1 - \nu)\sigma k/2\mu}.$$
In the case of a rigid substance, at \( \mu = \infty \), we obtain the following formula in place of equation (35):

\[
\eta = \frac{K}{\rho^2} r \pi \tau^{02} k. \tag{36}
\]

Remarkably enough the two equations (35) and (36) have different short-wave length asymptotics: \( \eta \approx (K/\rho^{02})8\pi r^{02} \mu/(1 - \nu)\sigma \) in the former and \( \eta \approx (K/\rho^{02})4r\tau^{02} k \) in the latter.

The second instability is the now well known stress-driven rearrangement instability with the increment

\[
\eta = \frac{K}{\rho^{02}} \frac{1 - \nu}{\mu} \frac{\Gamma^2}{1 + (1 - \nu)\sigma k/2\mu} k. \tag{37}
\]

Nozières (1991) has given a simple intuitive explanation of it.

Both instabilities are opposed by the stabilizing influence of the surface energy which dominates at short wavelengths of the corrugation but cannot prevent the destabilization in the long-wavelength range.

\section{5. Modifications for Linear Elasticity}

We would like to emphasize again that equation (30) for the decrement \( \eta \) is exact in the sense that it is not based on any assumptions of smallness of the deformations in the equilibrium configuration. We find this circumstance to be of great importance since the effects of the various stress-driven rearrangement instabilities are of second order with respect to the stresses and deformations. Therefore, the conclusions based on the linear elasticity are questionable since the linear theory itself is based on the assumption that the second-order terms in deformations can be neglected. This does not imply, of course, that the results based on linear theory cannot be correct. In fact, some of them seem to be compatible with the results based on the exact nonlinear thermodynamics (see, in particular, Nozières (1991)). Linear elasticity is much more popular in the physics community and it is much simpler than nonlinear elasticity. We would like to emphasize that even in the framework of linear elasticity the master system of equations remains to be deeply nonlinear since we are still faced with a free (unknown) boundary problem. This is the most profound and the most interesting nonlinearity of the master system which is similar to all problems of equilibrium shape of crystals. In what follows we shall show some modifications of the exact master system (2)-(12) which are typical in linear elasticity but nonetheless yield the dispersion relation (30).

The electrostatic equations (2)-(5) do not change explicitly. There is, however, one important implicit change. In the exact nonlinear theory the boundary conditions (4) and (5) should be imposed on the deformed (actual) interface. In the linear theory these conditions are imposed on the undeformed interface. One has to distinguish clearly between the undeformed configuration and the configuration without rearrangement. The difference has been explained in detail by Grinfeld (1991) and in the references therein.

Under the approximations of linear elasticity, the elastic energy density \( \phi \) per unit volume is given by the quadratic form of the linear deformation tensor \( \varepsilon_{ij} = \frac{1}{2} (\nabla_i U_j + \nabla_j U_i) \equiv \nabla_i (U_j) \):

\[
\phi = \frac{1}{2} C^{ijkl} \varepsilon_{ij} \varepsilon_{kl} = \frac{1}{2} C^{ijkl} \nabla_i (U_j) \nabla_k (U_l), \tag{38}
\]
where $C^{ijkl}$ is the elastic modulus tensor (in particular, for an isotropic substance this tensor is given by the formula $C^{ijkl} = \lambda z^i z^j z^k z^l + \mu (z^i z^l + z^j z^k)$).

In the linear theory of elasticity the Murnaghan formula (9) is not valid and instead one has to use the following standard equation:

$$\sigma^{ij} = C^{ijkl} \epsilon_{kl} = C^{ijkl} \nabla \langle k U_l \rangle. \quad \text{(39)}$$

Consequently, in the linear version of the master system, one has to write, in place of equations (6) and (8), the following equations.

(a) Within the bulk,

$$\nabla \left( C^{ijkl} \nabla \langle k U_l \rangle \right) = \rho \nabla^i G. \quad \text{(40)}$$

(c) At the free interface $S_1$,

$$\frac{1}{2} C^{ijkl} \nabla \langle k U_l \rangle N_j = (2\pi \tau^2 + \sigma L B_\alpha^a) N^i. \quad \text{(41)}$$

Contrary to the exact nonlinear master system in its linear version, $\rho$ is the density in the undeformed configuration, $N^i$ and $B_\alpha^a$ are the unit normal and mean curvature of the undeformed interface, and $\tau$ is the charge density per unit area of the undeformed interface (we emphasize once again that the notions of 'undeformed' interface and of 'initial' interface are conceptually different; an undeformed interface can strongly deviate from the initial interface owing to rearrangement).

The expression for the tensorial chemical potential has to be substituted by

$$\chi^{ij} = \left( \frac{1}{2\rho} C^{mnkl} \nabla \langle m U_m \rangle \nabla \langle k U_l \rangle + G \right) z^{ij} - \frac{1}{\rho} C^{ijkl} \nabla \langle k U_l \rangle. \quad \text{(42)}$$

This relation contradicts linear elasticity more than any other. It contains terms of the first and second order with respect to small gradient displacements $\nabla \langle k U_l \rangle$.

The remaining kinetic equations (10) and (12) do not change their form:

$$\mu^{(N)} = \chi^{ij} N_j N_j, \quad \text{(43)}$$

$$J(\xi, t) = \rho (C - f^i N_i) = -K (\mu^{(N)}(\xi, t) - \langle \mu^{(N)} \rangle), \quad \text{(44)}$$

$$\langle A \rangle = \frac{1}{|S_i|} \int_S dS \ A, \quad \text{(45)}$$

but all the geometric parameters now characterize the undeformed interface.

§ 6. A NON-DEFORMABLE SUBSTANCE

In the case of a non-deformable substance the equations of 'mechanical' equilibrium have no physical meaning either in the bulk or at the interface and, instead of equation (12), the kinetic equation takes the following form:

$$C = \frac{J}{\rho}$$

$$= -K \left( G - \frac{1}{\rho} (2\pi \tau^2 + \sigma L B_\alpha^a) - \left( G - \frac{1}{\rho} (2\pi \tau^2 + \sigma L B_\alpha^a) \right) \right). \quad \text{(46)}$$
In the absence of electric and mass fields we arrive at the equation

\[ C = \frac{J}{\rho} = K \frac{\sigma}{\rho} (B_{\alpha} - \langle B_{\alpha} \rangle), \]  

which differs from the equation for ‘motion by mean curvature’ owing to the last term.

Linearizing the master system for a non-deformable crystal in the vicinity of the above solution and looking for the disturbances in the form we arrive at the following formula for the increment \( \eta \):

\[ \eta \frac{\rho^2}{K} = -g + 4\pi \tau^2 k - \sigma k^2. \]  

Obviously, the dispersion relation (48) for the increment of small disturbances growth in crystalline non-deformable substance leads us to the same stability criterion as the dispersion equation for the dynamic surface waves in a conducting liquid fluid established by Frenkel’ (1936) half a century ago (Landau and Lifshitz 1965).

\section*{Conclusion}

We proposed the master system (2)–(12) for the quasistatic evolution of shapes of such conductors owing to the rearrangement of their material particles. What drives this evolution is the fact that the total energy of the system is diminishing. The total energy of the system consists of the elastic, electrostatic and surface energies. The system allows one to investigate the stable equilibrium shapes of crystalline deformable conductors and the slow evolution of these conductors arising due to a loss of stability. The proposed master system is based on the assumption that the characteristic time scale of establishing equilibrium with respect to rearrangement of the particles of materials is much greater than the time scales of establishing equilibrium with respect to mechanical and electrostatic degrees of freedom. The proposed system is based on principles and notions of the exact nonlinear continuum mechanics in the Eulerian description.

We have formulated a new kinetic equation (12) of the mass rearrangement in the crystal. According to this equation, the mass flux at a certain point on the surface of the crystalline conductor is proportional to the local excess of the value of the generalized Bowen chemical potential tensor over its average value over the whole surface of the crystal. The suggested mechanism leads to an equation which is much easier to handle numerically than, say, the equation describing mass rearrangement via surface diffusion.

The exact master system leads to equation (30) for the increment or decrement \( \eta \) in the growth of small disturbances of an isotropic uniformly stressed elastic half-plane exposed to the action of a uniform electrostatic field. In the case of a small deviation from the unstressed state we express the coefficients of the dispersion relation in terms of the classical Lamé and Poisson moduli, namely equation (34).

We then show some modifications of the master system which yield the same dispersion equations without any appeal to the concepts of the exact theory. The great advantage of the simplified system is that it leads to the linear system of partial differential equations within the bulk of the conductor. There are many powerful packages for numerical solutions of such systems.

In the end we apply our system to the case of an incompressible substance and establish the dispersion equation (48) for the increment or decrement in the growth
of small disturbances. The suggested master system leads to the same stability criterion as established long ago for a liquid conductor in an electrostatic field by Frenkel' (1936).

ACKNOWLEDGEMENT

This memorial paper was written to reflect Michael Duesbery's concept of combined numerical modelling and visualization. Numerical computed-based simulation and visualization based on the master system (2)–(13) was planned to be a major part of the paper. This computer part is being developed right now and the results will be placed at www.geocities.com/pavel_grinfeld.

REFERENCES