

# Expected number of distinct part sizes in a random integer composition

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An asymptotics, as  $n \rightarrow \infty$ , for the expected number of distinct part sizes in a random composition of an integer  $n$  is obtained.

† Part of the research of the first author was carried out while he was visiting Department of Mathematics of Lehigh University in the spring semester of 1999. He would like to thank the Department for its hospitality

## 1. Introduction

In this note we obtain precise asymptotics, as  $n \rightarrow \infty$ , for the expected number of distinct part sizes in a random composition of an integer  $n$ . Let us recall that a multiset  $\lambda = \{\lambda_1, \dots, \lambda_k\}$  is a partition of an integer  $n$  if the  $\lambda_j$  are positive integers, called parts, such that  $\sum \lambda_j = n$ . The values of  $\lambda_j$ 's are called part sizes. Compositions are partitions in which the order of parts is significant. Thus, for example, the integer 3 admits three partitions,  $\{1, 1, 1\}$ ,  $\{2, 1\}$  and  $\{3\}$ , and four compositions, namely  $(1, 1, 1)$ ,  $(1, 2)$ ,  $(2, 1)$  and  $(3)$ . According to our terminology  $(1, 2)$  is a composition of 3 in two parts with sizes 1 and 2. In analogy with random partitions, by a random composition of an integer  $n$  we mean a composition of  $n$  that is chosen uniformly at random out of the set of all  $2^{n-1}$  compositions of an integer  $n$ . More formally, one considers the probability space consisting of the set  $C(n)$  of all compositions of  $n$  equipped with the uniform probability measure. In this setting, the number of distinct part sizes (or other characteristics) becomes a random variable whose probabilistic behavior is to be studied.

Investigation of random partitions from this probabilistic perspective originated with a paper by Erdős and Lehner [5] who studied the limiting distribution of the total number of parts in a random partition. Subsequently, Wilf [11] found an asymptotic formula for the expected number of distinct part sizes. Goh and Schmutz [7] obtained more precise information on the distribution of the number of distinct part sizes, namely they established the central limit theorem. Recently Corteel, Pittel, Savage and Wilf [3] obtained a refined version of Wilf's result concerning the expectation of the number of distinct part sizes in a random partition. Their result allows one to obtain as many terms for the asymptotic expansion of this expectation as one wishes. For example, on "o(1) level" this expectation is

$$\frac{\sqrt{6n}}{\pi} + \frac{3}{\pi^2} - \frac{1}{2} + o(1).$$

The aim of this note is to obtain an asymptotics for the same quantity in the case of random compositions. In order to state our result we need some notation. For an integer  $n$  consider the set  $C(n)$  of all compositions  $\kappa$  of  $n$  equipped with the uniform probability measure  $\mathbf{P}_n = \mathbf{P}$  (that is,  $\mathbf{P}(\kappa) = 2^{-n+1}$  for every  $\kappa \in C(n)$ .) For a composition  $\kappa = (\gamma_1, \dots, \gamma_k)$  the number of distinct part sizes,  $D_n(\kappa)$  is defined by the formula

$$D_n(\kappa) = 1 + \sum_{i=2}^k I_{\{\gamma_i \neq \gamma_j, j=1, \dots, i-1\}},$$

where  $I_A$  is the indicator function of the set  $A$ . We denote the integration with respect to  $\mathbf{P}$  on  $C(n)$  by  $\mathbf{E}$ . We have:

**Theorem.** *As  $n \rightarrow \infty$ ,*

$$\mathbf{E}D_n = \log_2 n + \frac{\gamma}{\ln 2} - \frac{3}{2} + g(\log_2 n) + o(1),$$

where  $\gamma$  is Euler's constant and  $g$  is a mean-zero function of period 1 satisfying  $|g| \leq 0.0000016$ .

Thus, the expected number of distinct part sizes in a composition of an integer  $n$  asymptotically behaves like  $\log_2 n$  plus a constant plus a small but periodic oscillation. This oscillatory behavior, which just a few years ago was considered surprising (to say the least) is by now a well documented and acknowledged feature of sequences of geometric random variables, see e.g. [2], [4], [8], [10].

We wish to observe that the asymptotics for the expected number of distinct part sizes is the same as the expected length of the longest run of heads in  $n$  tosses of a fair coin, see e.g. [2] or [8]. Since the size of the largest part is one plus the longest run of heads it follows that on average one expects to see parts of all but one sizes between 1 and the largest size (or runs of heads of all but one lengths between 1 and the longest run).

## 2. Outline of a proof

Quite often results like this are obtained through careful analysis of the the generating function. We will use a different approach. We will view random composition as (essentially) randomly stopped sequence of i.i.d. geometric random variables and we will express the number of distinct part sizes as a function of this sequence. This will allow for direct and straightforward estimates. The same approach was used successfully in [9] to handle a problem in which generating function approach was apparently futile. We believe that this technique will prove useful in many other problems concerning random compositions. Our proof in a natural way splits into two steps. In the first we will use the afore-mentioned representation and probabilistic estimates to extract the main contribution to  $\mathbf{E}D_n$ . Namely, we have

**Proposition 2.1.** *As  $n \rightarrow \infty$ ,*

$$\mathbf{E}D_n = \sum_{m=1}^{\infty} \left\{ 1 - \left(1 - \frac{1}{2^m}\right)^{\frac{n}{2}(1+o(1))} \right\} + o(1).$$

The second, purely analytical step is to analyse the asymptotic behavior of the infinite sum above. This goal could be accomplished by applying the so – called Rice method (see e.g. [6] for a very good description and examples). Since this method requires some tools from complex analysis we decided to take a different route. As a result our analysis is completely elementary (thus, making this paper fully accessible to advanced undergraduates, for example.) To facilitate our analysis we define

$$f(x) = \sum_{m=1}^{\infty} \left\{ 1 - \left(1 - \frac{1}{2^m}\right)^{2^x} \right\}.$$

With this definition we will show that  $f(x) - x$  tends to a limit as  $x$  tends to infinity along sequences of the form  $\{x_0 + k\}_{k \in \mathbb{Z}}$ , but does not possess an unrestricted limit as  $x \rightarrow \infty$ . More specifically, we have

**Proposition 2.2.** *For large positive  $k$*

$$f(x + k) = x + k + \gamma/\ln 2 - 1/2 + g(x) + o(2^{-x-k})$$

where  $\gamma$  is Euler's constant and

$$g(x) = -x - \gamma/\ln 2 + 1/2 - \sum_{m=-\infty}^0 \exp(-2^{-m+x}) + \sum_{m=1}^{\infty} (1 - \exp(-2^{-m+x}))$$

is a nonconstant, zero-mean function of period 1 satisfying  $|g(x)| \leq .0000016$ .

Clearly, Theorem follows by combining Propositions 2.1 and 2.2.

### 3. Proof of proposition 2.1

Central to our approach is the following proposition.

**Proposition 3.1.** *Let  $\Gamma_1, \Gamma_2, \dots$  be i.i.d. geometric random variables with parameter  $1/2$  (that is  $\mathbf{P}(\Gamma_1 = j) = 2^{-j}$ ,  $j = 1, 2, \dots$ ) and define*

$$\tau = \inf\{k \geq 1 : \Gamma_1 + \Gamma_2 + \dots + \Gamma_k \geq n\}.$$

*Then, the distribution of a randomly chosen composition in  $C(n)$  is given by*

$$(\Gamma_1, \Gamma_2, \dots, \Gamma_{\tau-1}, n - \sum_{j=1}^{\tau-1} \Gamma_j).$$

This proposition is nothing more than a reiteration of a known (see e.g. [1]) connection between compositions of integers and  $\{0, 1\}$ -valued sequences. Namely, a composition  $\kappa = (\gamma_1, \dots, \gamma_k)$  of an integer  $n$  into parts  $\gamma_1, \dots, \gamma_k$  is associated with a string of 0's and 1's of length  $n$  as follows: there is a 1 on the  $n$ th place and the numbers  $\gamma_1, \dots, \gamma_k$  are "waiting times" for the first, second,  $\dots$ , and  $k$ th appearance of 1. (For example, the composition  $(1, 2, 3, 1, 1)$  of 8 corresponds to the string 10100111 while  $(4, 2, 2)$  corresponds to 00010101.) Choosing a composition at random amounts to having the 0's and 1's on the first  $n - 1$  places occur according to a binomial  $\text{Bin}(n - 1, 1/2)$  law. We refer to [9] for more details. Let  $\tilde{\Gamma}_i(\kappa)$  denote parts of a randomly chosen composition  $\kappa$ , i.e.

$$\tilde{\Gamma}_i(\kappa) = \Gamma_i(\kappa), \quad \text{for } i < \tau(\kappa) \quad \text{and} \quad \tilde{\Gamma}_{\tau(\kappa)}(\kappa) = n - \sum_{i=1}^{\tau(\kappa)-1} \Gamma_i(\kappa).$$

Note that  $\tilde{\Gamma}_\tau \leq \Gamma_\tau$ . The expected value of  $D_n$  is computed as follows:

$$\begin{aligned} \mathbf{E}D_n &= 1 + \mathbf{E} \sum_{i=2}^{\tau} I_{\tilde{\Gamma}_i \neq \tilde{\Gamma}_j; j=1, \dots, i-1} \\ &= 1 + \mathbf{E} \sum_{i=2}^{\tau-1} I_{\Gamma_i \neq \Gamma_j; j=1, \dots, i-1} + \mathbf{P}(\tilde{\Gamma}_\tau \neq \Gamma_1, \dots, \Gamma_{\tau-1}). \end{aligned}$$

We will first show that the last probability is negligible. This is because  $\tau$  being a  $1 + \text{Bin}(n - 1, 1/2)$  random variable satisfies the bound

$$\mathbf{P}(|\tau - \mathbf{E}\tau| \geq t) \leq 2 \exp(-2t^2/(n - 1)),$$

so that

$$\mathbf{P}(|\tau - \mathbf{E}\tau| \geq \sqrt{(n-1)\log n}) \leq 2 \exp\{-2\frac{(n-1)\log n}{n-1}\} = O(1/n^2).$$

Therefore, letting  $t_n = \sqrt{(n-1)\log n}$  and then

$$n_0 \sim \mathbf{E}\tau - t_n = (n+1)/2 - \sqrt{(n-1)\log n},$$

and

$$n_1 \sim \mathbf{E}\tau + t_n = (n+1)/2 + \sqrt{(n-1)\log n},$$

we get

$$\begin{aligned} \mathbf{P}(\tilde{\Gamma}_\tau \neq \Gamma_1, \dots, \Gamma_{\tau-1}) &\leq \sum_{j=1}^n \mathbf{P}(\tilde{\Gamma}_\tau = j, \Gamma_1, \dots, \Gamma_{\tau-1} \neq j) \\ &\leq \sum_{j=1}^n \mathbf{P}(\tilde{\Gamma}_\tau \geq j, \Gamma_1, \dots, \Gamma_{\tau-1} \neq j) \\ &\leq \sum_{j=1}^n \mathbf{P}(\Gamma_\tau \geq j, \Gamma_1, \dots, \Gamma_{\tau-1} \neq j) \\ &\leq \sum_{j=1}^n \mathbf{P}(\Gamma_\tau \geq j, \Gamma_1, \dots, \Gamma_{\tau-1} \neq j, |\tau - \mathbf{E}\tau| \leq t_n) + n\mathbf{P}(|\tau - \mathbf{E}\tau| \geq t_n) \\ &\leq \sum_{j=1}^n \sum_{k=n_0}^{n_1} \mathbf{P}(\Gamma_k \geq j, \Gamma_1, \dots, \Gamma_{k-1} \neq j, \tau = k) + O(1/n) \\ &\leq \sum_{k=n_0}^{n_1} \sum_{j=1}^{\infty} \mathbf{P}(\Gamma_k \geq j, \Gamma_1, \dots, \Gamma_{k-1} \neq j) + O(1/n) \\ &= \sum_{k=n_0}^{n_1} \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} (1 - \frac{1}{2^j})^{k-1} + O(1/n) \\ &\leq \sum_{k=n_0}^{n_1} C \int_0^{\infty} \frac{1}{2^x} (1 - \frac{1}{2^{x+1}})^{k-1} dx + O(1/n) \\ &\leq \sum_{k=n_0}^{n_1} \frac{C}{k} + O(1/n) \leq C \frac{\sqrt{n \log n}}{n} \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ . As for the other term, we have

$$\begin{aligned} \mathbf{E} \sum_{i=2}^{\tau-1} I_{\Gamma_i \neq \Gamma_j, j < i} &\leq \mathbf{E} \left( \sum_{i=2}^{\tau-1} I_{\Gamma_i \neq \Gamma_j, j < i} \right) I_{|\tau - \mathbf{E}\tau| \leq t_n} \\ &\quad + \mathbf{E} \left( \sum_{i=2}^{\tau-1} I_{\Gamma_i \neq \Gamma_j, j < i} \right) I_{|\tau - \mathbf{E}\tau| > t_n} \\ &\leq \mathbf{E} \left( \sum_{i=2}^{\tau-1} I_{\Gamma_i \neq \Gamma_j, j < i} \right) I_{|\tau - \mathbf{E}\tau| \leq t_n} + (n-2) \mathbf{P}(|\tau - \mathbf{E}\tau| \geq t_n). \end{aligned}$$

The second term is bounded above by  $C/n$  and, of course, tends to 0 as  $n \rightarrow \infty$ . For the first one we have:

$$\begin{aligned} \mathbf{E}\left(\sum_{i=2}^{\tau-1} I_{\Gamma_i \neq \Gamma_j, j < i}\right) I_{|\tau - \mathbf{E}\tau| \leq t_n} &\leq \mathbf{E}\left(\sum_{i=2}^{n_1} I_{\Gamma_i \neq \Gamma_j, j < i}\right) I_{|\tau - \mathbf{E}\tau| \leq t_n} \\ &\leq \mathbf{E}\sum_{i=2}^{n_1} I_{\Gamma_i \neq \Gamma_j, j < i} = \sum_{i=2}^{n_1} \mathbf{P}(\Gamma_i \neq \Gamma_j, j < i) \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{E}\left(\sum_{i=2}^{\tau-1} I_{\Gamma_i \neq \Gamma_j, j < i}\right) I_{|\tau - \mathbf{E}\tau| \leq t_n} &\geq \mathbf{E}\left(\sum_{i=2}^{n_0} I_{\Gamma_i \neq \Gamma_j, j < i}\right) I_{|\tau - \mathbf{E}\tau| \leq t_n} \\ &= \mathbf{E}\sum_{i=2}^{n_0} I_{\Gamma_i \neq \Gamma_j, j < i} - \mathbf{E}\sum_{i=2}^{n_0} \mathbf{E}(\Gamma_i \neq \Gamma_j, j < i) I_{|\tau - \mathbf{E}\tau| > t_n} \\ &\geq \sum_{i=2}^{n_0} \mathbf{P}(\Gamma_i \neq \Gamma_j, j < i) - (n_0 - 1) \mathbf{P}(|\tau - \mathbf{E}\tau| \geq t_n) \\ &= \sum_{i=2}^{n_0} \mathbf{P}(\Gamma_i \neq \Gamma_j, j < i) - O(n^{-1}) \end{aligned}$$

We will now fix  $k$  and approximate  $\sum_{i=2}^k \mathbf{P}(\Gamma_i \neq \Gamma_j, j < i)$  as follows

$$\mathbf{P}(\Gamma_i \neq \Gamma_j, j < i) = \sum_{m=1}^{\infty} \mathbf{P}(\Gamma_i = m, \Gamma_j \neq m; j < i) = \sum_{m=1}^{\infty} \frac{1}{2^m} \left(1 - \frac{1}{2^m}\right)^{i-1}.$$

Hence, by summing up over  $i$  we get:

$$\begin{aligned} \sum_{i=2}^k \sum_{m=1}^{\infty} \frac{1}{2^m} \left(1 - \frac{1}{2^m}\right)^{i-1} &= \sum_{m=1}^{\infty} \frac{1}{2^m} \sum_{i=1}^{k-1} \left(1 - \frac{1}{2^m}\right)^i \\ &= \sum_{m=1}^{\infty} \frac{1}{2^m} \left(1 - \frac{1}{2^m}\right) \frac{1 - (1 - 2^{-m})^{k-1}}{1 - (1 - 2^{-m})} \\ &= \sum_{m=1}^{\infty} \left(1 - \frac{1}{2^m}\right) \left(1 - \left(1 - \frac{1}{2^m}\right)^{k-1}\right) \\ &= \sum_{m=1}^{\infty} \left\{ \left(1 - \frac{1}{2^m}\right) - \left(1 - \frac{1}{2^m}\right)^k \right\} \\ &= \sum_{m=1}^{\infty} \left\{ 1 - \left(1 - \frac{1}{2^m}\right)^k \right\} - \sum_{m=1}^{\infty} \frac{1}{2^m} \\ &= \sum_{m=1}^{\infty} \left\{ 1 - \left(1 - \frac{1}{2^m}\right)^k \right\} - 1. \end{aligned}$$

It follows that after ignoring terms of order  $o(1)$  we have

$$\sum_{m=1}^{\infty} \left\{ 1 - \left(1 - \frac{1}{2^m}\right)^{n_0} \right\} \leq \mathbf{E}D_n \leq \sum_{m=1}^{\infty} \left\{ 1 - \left(1 - \frac{1}{2^m}\right)^{n_1} \right\}.$$

since both  $n_0$  and  $n_1$  are of the form  $\frac{n}{2}(1 + o(1))$  Proposition 2.1 follows.

#### 4. Proof of Proposition 2.2

Recall that

$$f(y) = \sum_{m=1}^{\infty} \left\{ 1 - \left(1 - \frac{1}{2^m}\right)^{2^y} \right\}.$$

We first give a simple argument which gives the limiting behavior of  $f$  without, however, yielding an estimate for the rate of convergence. Let  $y = x + k$ , where  $0 \leq x < 1$ . We re-index the sum by  $m + k$  to obtain

$$f(k + x) - k - x = -x - \sum_{m=-k+1}^0 (1 - 2^{-m-k})^{2^{k+x}} + \sum_{m=1}^{\infty} (1 - (1 - 2^{-m-k})^{2^{k+x}}).$$

Permuting summation and limits as  $k \rightarrow \infty$  yields

$$f(x + k) - k - x = -x - \sum_{m=-\infty}^0 \exp(-2^{-m+x}) + \sum_{m=1}^{\infty} (1 - \exp(-2^{-m+x})) + o(1).$$

But this step is justified by dominated convergence using the following majorizing convergent series of positive terms independent of  $k$ :

$$\sum_{m=-k+1}^0 (1 - 2^{-m-k})^{2^{k+x}} \ll \sum_{m=-\infty}^0 \exp(-2^{-m+x})$$

and

$$\sum_{m=1}^{\infty} (1 - (1 - 2^{-m-k})^{2^{k+x}}) \ll \sum_{m=1}^{\infty} (1 - \exp(-2^{-m+x+1})).$$

These follow from the estimates  $\exp(-2ab) \leq (1 - b/\lambda)^{a\lambda} \leq \exp(-ab)$  if  $\lambda a > 0$  and  $b/\lambda \leq 1/2$  with  $\lambda = 2^k$ ,  $a = 2^x$  and  $b = 2^{-m}$ .

The series thus established as the limit of  $f(k + x) - (k + x)$  defines a function of period 1. Denoting its mean by  $c$  and its zero-mean part by  $g(x)$  we have

$$f(x + k) = x + k + c + g(x) + o(1) = x + k + c + g(x + k) + o(1).$$

To obtain the finer estimate stated in the proposition we use the higher order estimate  $\exp(-ab - ab^2/\lambda) \leq (1 - b/\lambda)^{a\lambda}$  if  $\lambda a > 0$  and  $b/\lambda \leq 1/2$ . We must bound

$$\begin{aligned} f(x + k) - (x + c + g(x)) &= \sum_{m=-\infty}^{-k} \exp(-2^{-m+x}) \\ &\quad + \sum_{m=-k+1}^{\infty} \left\{ \exp(-2^{-m+x}) - (1 - 2^{-m-k})^{2^{k+x}} \right\}. \end{aligned}$$

The first sum can be rewritten as

$$\sum_{m=0}^{\infty} \exp(-2^{m+k+x}) = \exp(-2^{k+x}) \sum_{m=0}^{\infty} \exp(-(2^m - 1)2^{k+x})$$

which is bounded by

$$\exp(-2^{k+x}) \sum_{m=0}^{\infty} \exp(-(2^m - 1))$$

and thus makes an exponentially small contribution to an error term of  $O(2^{-k-x})$ .

The second sum consists of positive terms and is bounded above by

$$\sum_{m=-\infty}^{\infty} \{\exp(-2^{-m+x}) - \exp(-2^{-m+x} - 2^{-2m+x-k})\}.$$

Then the inequality  $\exp(-a) - \exp(-a-b) \leq b \exp(-a)$  for positive  $a$  and  $b$  gives the bound

$$2^{-x-k} \sum_{m=-\infty}^{\infty} 2^{-2m+2x} \exp(-2^{-j+x}).$$

This bound has the form  $2^{-x-k}h(x)$  where  $h$  is a periodic function of  $x$  and is therefore bounded by a constant. This establishes the asserted rate of convergence.

It remains to calculate the mean of  $g$ . The mean of  $-x$  is  $-1/2$  and the mean of the residual series is

$$c_0 = - \sum_{m=-\infty}^0 \int_0^1 \exp(-2^{-m+x}) dx + \sum_{m=1}^{\infty} \int_0^1 (1 - \exp(-2^{-m+x})) dx.$$

On the  $m$ -th summand the change of variable  $u = 2^{-m+x}$  gives

$$c_0 \ln 2 = - \sum_{m=-\infty}^0 \int_{2^{-m}}^{2^{-m+1}} \exp(-u) \frac{du}{u} + \sum_{m=1}^{\infty} \int_{2^{-m}}^{2^{-m+1}} (1 - \exp(-u)) \frac{du}{u}$$

or

$$c_0 \ln 2 = - \int_1^{\infty} \exp(-u) \frac{du}{u} + \int_0^1 (1 - \exp(-u)) \frac{du}{u}.$$

Integrating each integral by parts yields a single integral

$$- \int_0^{\infty} \exp(-u) \ln u du$$

which is a well-known integral representing Euler's constant.

We remark that a little more similar reasoning shows that the periodic function

$$h(x) = \sum_{m=-\infty}^{\infty} 2^{-2m+2x} \exp(-2^{-m+x})$$

appearing in the rate estimate overestimates the error asymptotically by a factor of two and, in fact,

$$f(x+k) = x+k + \gamma/\ln 2 - 1/2 + g(x) + h(x)2^{-x-k-1} + O(2^{-2x-2k}).$$

The bound on  $g$  and its nonconstant character are easily checked numerically although calculations dealing with  $f$  rather than analytically derived asymptotic forms are rather sensitive. Alternately its complex Fourier coefficients are easily obtained by a simple variant of the calculation of  $c_0$  and have the form  $c_k = \Gamma(2\pi k i / \ln 2) / \ln 2$ . These are nonzero,

small and decrease geometrically in magnitude. For example  $2 | c_1 | = .00000157316$  accounts for the maximum contribution of the first harmonic while for the second harmonic  $2 | c_2 | < 10^{-12}$ .

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