Random Partitions With Restricted Part Sizes*

William M. Y. Goh,[1] Pawel Hitczenko[1,2]
[1] *Department of Mathematics, Drexel University, Philadelphia, Pennsylvania 19104; e-mail: wgoh@math.drexel.edu*
[2] *Department of Computer Science, Drexel University, Philadelphia, Pennsylvania 19104; e-mail: phitczenko@math.drexel.edu*

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ABSTRACT: For a subset $\mathcal{S}$ of positive integers let $\Omega(n, \mathcal{S})$ be the set of partitions of n into summands that are elements of $\mathcal{S}$. For every $\lambda \in \Omega(n, \mathcal{S})$, let $\mathbf{M}_n(\lambda)$ be the number of parts, with multiplicity, that λ has. Put a uniform probability distribution on $\Omega(n, \mathcal{S})$, and regard $\mathbf{M}_n$ as a random variable. In this paper the limiting density of the (suitably normalized) random variable $\mathbf{M}_n$ is determined for sets that are sufficiently regular. In particular, our results cover the case $\mathcal{S} = \{Q(k) : k \geq 1\}$, where $Q(x)$ is a fixed polynomial of degree $d \geq 2$. For specific choices of Q, the limiting density has appeared before in rather different contexts such as Kingman's coalescent, and processes associated with the maxima of Brownian bridge and Brownian meander processes. © 2007 Wiley Periodicals, Inc. Random Struct. Alg., 32, 440–462, 2008

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1. INTRODUCTION AND STATEMENT OF THE RESULT

In research on partitions, there have been great synergies between probabilistic, analytic, and combinatorial methods. The oldest literature on partition enumerations goes back to Euler and Sylvester. Hardy and Ramanujan [20], were the first to consider the asymptotic properties of the partition function and their work has a purely analytic flavor. Erdös and

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Correspondence to: Pawel Hitczenko
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Lehner [14] introduced a probabilistic viewpoint that was quite fruitful. Random partitions were developed by Erdös, Szalay, Turan and others, [15,32,34–37]. Szekeres in [38] studied the joint distribution of the number of parts and the maximal part size in integer partitions. Increasingly sophisticated probabilistic ideas have been introduced [3, 17], and these ideas have led to remarkably strong theorems about the joint distribution of part sizes of random integer partitions [28].

Some authors, e.g. [21, 22, 27, 30], have studied random partitions with summands restricted to proper subsets of the set of positive integers. Typically, the subset is described by a simple condition and in the most frequently encountered situation it is the range of a particular polynomial.

This actually goes back to the work of Hardy–Ramanujan who in their paper [20, Sec. 7.3] considered partitions into parts whose sizes are perfect squares, or more generally, perfect sth powers. Partitions into perfect squares turned out to be of interest in statistical mechanics as they have been used to model ideal gas. We refer to Vershik [39, Sec. 3, item 5.] for more details. Partitions whose parts are perfect cubes were considered by Richmond in [30] while Canfield, Corteel and Hitczenko [8] studied partitions whose part sizes are of the form $\binom{k+d}{d}$ where $k \geq 0$ and d is a fixed positive integer (the case $d = 1$ corresponds to ordinary partitions).

Investigation of partitions restricted in such way leads to generating functions of the form

$$\prod_{k\geq 1} \frac{1}{(1-z^k)^{b_k}}, \quad 0 < |z| < 1, \tag{1}$$

where (b_k) is a fixed sequence of non–negative numbers. Partitions whose parts are restricted to a subset $\mathcal{S} \subset \mathbf{N}$ correspond to $b_k := I_{k\in\mathcal{S}}$, $k \geq 1$, but other sequences (for example, $b_k = k^\alpha$, $\alpha \geq 0$) have been considered, particularly in statistical physics (see [39] and also [16, Sec. 6] for further discussion).

The primary focus was to develop the asymptotic expressions for the coefficients of the function (1). For example, Hardy and Ramanujan have done it for the case $b_k = I_{k\in\mathcal{S}}$, where $\mathcal{S} = \{j^s : j \geq 1\}$ is the set of perfect sth powers for some fixed $s \geq 1$, while Bringham [7] considered the set $\mathcal{S}$ of all prime numbers. The first order asymptotics when the set $\mathcal{S}$ is the range of a fixed polynomial was explicitly given in [8, Theorem 6]. A theorem of Meinardus [1, Chapter 6] provides a general tool for obtaining the asymptotics of the coefficients for many generating functions of the type (1).

In the present article we will be interested in the number of parts in random partition whose parts sizes are restricted to be in a proper subset of $\mathbf{Z}^+$ satisfying certain regularity assumptions. We obtain a general result concerning the limiting distribution of the (properly normalized) number of parts. The regularity assumptions we impose were considered by Ingham [23] and thus we will refer to such sets as Ingham sets. As was verified in [8, end of a proof of Theorem 6], any set that is the range of a polynomial is an Ingham set.

To make a precise statement let us introduce necessary notation. Let us recall that a partition λ of a positive integer n is a sequence of positive numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, $k \geq 1$, whose sum is n. Number k is called the number of parts and the λ_i's are called parts. Another convenient representation of a partition λ is to write $n = \sum aM_a(\lambda)$, where $M_a(\lambda) \geq 0$ is the multiplicity of part size a, i.e. the number of times that part a appears in λ. Let $\Omega(n, \mathcal{S})$ be the set of partitions of n into summands that are elements of the set $\mathcal{S} = \{s_j : j \in \mathbf{Z}^+\}$. Throughout this article we assume that the set $\mathcal{S}$ has gcd 1. Let $N(u)$ be the counting function associated with the set $\mathcal{S}$:

$$N(u) = \#\{h \in \mathcal{S} : h \le u\}.$$

Ingham [23] imposed the following condition

$$N(u) = Bu^{\beta} + R(u) \tag{2}$$

for some $B > 0$, $0 < \beta < 1$ and

$$\int_0^u \frac{R(v)}{v} dv = b_1 \log u + b_2 + o(1), u \to \infty. \tag{3}$$

For every $\lambda \in \Omega(n, \mathcal{S})$, let $\mathbf{M}_n(\lambda)$ be the number of parts that λ has. Put a uniform probability measure P_n on $\Omega(n, \mathcal{S})$, and regard $\mathbf{M}_n$ as a random variable. Note that $\mathbf{M}_n(\lambda) = \sum_s M_s(\lambda)$, where $M_s(\lambda)$ is the multiplicity of the part size s in the P_n-random partition λ. These random variables M_s are clearly not independent since they must satisfy the condition $\sum_{s\in\mathcal{S}} sM_s = n$. Fristedt [17] used a conditioning device that enables one to cope with this dependence. It quickly proved to be a powerful tool and has been used by several authors in the past decade, see e.g. [2, 3, 8, 10, 28, 29, 31]. Given a parameter $q \in (0, 1)$, let $\{G_s\}_{s\in\mathcal{S}}$ be independent geometric random variables with respective parameters $1 - q^s$, i.e. for all $a \in \mathcal{S}$, and for all non-negative integers k, we have $\mathsf{P}(G_s = k) = (1 - q^s)q^{sk}$. Here and in the sequel P is a generic probability measure on a probability space that is assumed to be rich enough to carry all the sequences of independent random variables that will be used. The joint distribution of the random variables $\{M_s\}_{s\in\mathcal{S}}$ (with respect to P_n) is exactly *equal* to the *conditional* distribution of the $\{G_s\}_{s\in\mathcal{S}}$, where the event conditioned on is that $\sum_{s\in\mathcal{S}} sG_s = n$. This is true for any choice of the parameter q. It will transpire that the most convenient choice is

$$q = q_n = \exp\left(-\frac{c_\beta}{n^{1/(1+\beta)}}\right), \quad n \ge 1, \tag{4}$$

with a specific value of the constant, namely

$$c_\beta = (B\beta\zeta(1+\beta)\Gamma(1+\beta))^{1/(1+\beta)}. \tag{5}$$

(A reason for that particular choice will become clear in Section 2.) We also choose the normalizing constants $\mu_n = n^{\frac{1}{\beta+1}}/c_\beta$.

Analogous methods have been used (with Poisson distributions in place of geometric distributions) in the context of random permutations [33]. As a matter of fact, it is quite common that the distribution of the components of random combinatorial structures are independent random variables conditioned on the sum of the sizes being fixed (see [2] for more information and references) and the idea of such representations appears already in [24, Chapters 4, 5].

Our aim is to prove the following:

Theorem 1. *Let $\mathcal{S} = \{s_1, s_2, \dots\}$ be an Ingham set. Assume that $\mathcal{S}$ contains four s_j's such that they are pairwise relatively prime. Then, for any positive real number x, we have*

$$\lim_{n\to\infty} \mathsf{P}_n\left(\frac{\mathbf{M}_n}{\mu_n} \le x\right) = \mathsf{P}(W_{\mathcal{S}} \le x), \tag{6}$$

where $W_{\mathcal{S}}$ is a random variable whose characteristic function is

$$\phi_{\mathcal{S}}(t) = \prod_{k\geq 1} \frac{1}{1 - it/s_k}. \tag{7}$$

Remarks

i. It is seen from (7) that $W_{\mathcal{S}}$ is distributed like the infinite sum of independent exponential random variables with parameters s_k, $k \geq 1$. Thus (see [11, Chapter 1, Sec. 1.4 (iv)]) the density of $W_{\mathcal{S}}$ is given by

$$f_{\mathcal{S}}(x) = \sum_{j=1}^{\infty} e^{-s_j x} s_j \left(\prod_{l\neq j} \frac{s_l}{s_l - s_j} \right). \tag{8}$$

ii. The extra assumption that $\mathcal{S}$ contains four pairwise relatively prime elements is technical (see the proof of Lemma 1 below). This is not a strict consequence of the assumption that gcd of $\mathcal{S}$ is 1 and it is not clear that it is a consequence of the Ingham's regularity condition (2). This extra assumption is satisfied in most of the applied situations. It would be interesting to know that if this is a consequence of the Ingham regularity condition (2). More generally, under Ingham regularity condition does $\mathcal{S}$ contain infinite many elements that are pairwise relatively prime?

The main step of our argument will be to show that the distribution of $\mathbf{M}_n$ is close to that of a sum of independent geometric random variables with suitably chosen parameters. This will be accomplished in the next section. Once this is known, one can approximate geometric variables by exponential ones which implies that the limiting distribution has a characteristic function given by (7).

If $\mathcal{S}$ is the range of a polynomial Q, then the expression (8) can be written more explicitly as an alternating series. We will develop such expression in Section 3. From that expression it will be seen that specific choices of Q lead to distributions that have already appeared in several, quite different, contexts. We will briefly mention a few such instances in Section 4.

2. PROOF OF (7)

2.1. Set-Up

We consider a doubly infinite array $\{G_{n,s_j} : j, n \geq 1\}$, where $G_{n,s}$ is a geometric random variable with parameter $1 - q_n^s$, q_n is defined in (5), and where for each $n \geq 1$, $\{G_{n,s_j} : j \geq 1\}$ are independent. It follows from [17] (see also [8]) that regardless of the value of q, the joint distribution of multiplicities of parts $(M_{s_1}(\lambda), M_{s_2}(\lambda), \ldots)$ in a randomly chosen partition of n is equal to that of $(G_{n,s_1}, G_{n,s_2}, \ldots)$ conditioned on the event $\{\sum_j s_j G_{n,s_j} = n\}$. That is,

$$\mathcal{L}(M_{s_1}, M_{s_2}, \ldots) = \mathcal{L}\left(G_{n,s_1}, G_{n,s_2}, \ldots \middle| \sum_{j\geq 1} s_j G_{n,s_j} = n \right),$$

where $\mathcal{L}(\cdot)$ denotes the probability distribution of a random vector in question. Hence, for any $x > 0$, we have

$$\mathsf{P}_n\left(\frac{\mathbf{M}_n}{\mu_n} \le x\right) = \mathsf{P}\left(\sum_{j=1}^{\infty} G_{n,s_j} \le \mu_n x \,\middle|\, \sum_{j=1}^{\infty} s_j G_{n,s_j} = n\right)$$
$$= \frac{\mathsf{P}\left(\sum_{j=1}^{\infty} G_{n,s_j} \le \mu_n x, \sum_{j=1}^{\infty} s_j G_{n,s_j} = n\right)}{\mathsf{P}\left(\sum_{j=1}^{\infty} s_j G_{n,s_j} = n\right)}. \tag{9}$$

The strategy is to argue that the events in the numerator of (9) are asymptotically independent. We will do it by arguing that, for a suitably chosen sequence (k_n), one can split each of the sums in two pieces ($j \le k_n$ and $j > k_n$) so that the dominant contribution to the value of $\sum G_{n,s_j}$ comes from indices $j \le k_n$ while the dominant contribution to $\sum s_j G_{n,s_j}$ comes from $j > k_n$. When these estimates are carried out, we will be left with two truncated sums over the disjoint sets of indices, plus error terms. The gain is that, unlike the original sums, the truncated sums are independent and thus can be handled with relative ease. It will be important, however, to control the approximation errors to ensure that they are negligible, even when divided by the denominator of (9); in particular, we will need a fairly precise information on the order of the magnitude of that denominator. To that end, we will establish a local limit theorem (at 0) for the normalized sum $\sum s_j G_{n,s_j}$ by refining an argument that was used by Fristedt in the case $\mathcal{S} = \mathbf{N}$ and repeated in [8] when $\mathcal{S}$ was the image of the polynomial $Q(x) = \binom{x+d}{d}$ with $d \ge 2$ fixed (see also [31] for a similar argument). First, in order to asymptotically maximize the denominator in (9) we choose q_n so that $\mathsf{E}(\sum s_j G_{n,s_j}) \sim n$. Since $G_{n,s}$'s are geometric, this means that we want

$$\nu_n := \mathsf{E}\sum_j s_j G_{n,s_j} = \sum_j s_j \frac{q^{s_j}}{1-q^{s_j}} \sim n.$$

The expression for ν_n as a function of q increases from 0 to infinity and thus is n for a unique value of q. We will employ the Riemann-Stieltjes integrals using the counting function $N(u)$ in estimating all sums involving s_j in the sequel. Thus

$$\begin{aligned}
\nu_n &= \sum_j s_j \frac{q^{s_j}}{1-q^{s_j}} = \int_0^\infty x \frac{q^x}{1-q^x} dN(x) \\
&= \frac{1}{\ln(1/q)} \int_0^\infty y \frac{e^{-y}}{1-e^{-y}} dN\left(\frac{y}{\ln(1/q)}\right) \\
&= \frac{1}{\ln(1/q)} \int_0^\infty N\left(\frac{y}{\ln(1/q)}\right) \frac{e^{-y}(e^{-y}-1+y)}{(1-e^{-y})^2} dy \\
&= \frac{1}{\ln(1/q)} \int_0^\infty \left(B\left(\frac{y}{\ln(1/q)}\right)^\beta + R\left(\frac{y}{\ln(1/q)}\right)\right) \frac{e^{-y}(e^{-y}-1+y)}{(1-e^{-y})^2} dy \\
&= \frac{C_\beta}{\ln^{1+\beta}(1/q)} + \frac{E(q)}{\ln(1/q)},
\end{aligned}$$

where

$$E(q) := \int_0^\infty R\left(\frac{y}{\ln(1/q)}\right) \frac{e^{-y}(e^{-y}-1+y)}{(1-e^{-y})^2} dy,$$

and where by [20, formula 3.411-7] the value of C_β is given by

$$C_\beta = B\int_0^\infty y^\beta \frac{e^{-y}(e^{-y}-1+y)}{(1-e^{-y})^2}dy = B\beta\int_0^\infty y^\beta \frac{e^{-y}}{1-e^{-y}}dy = B\beta\zeta(1+\beta)\Gamma(1+\beta).$$

To control the error we need to estimate $E(q)$ as $q\to 1$. Let

$$r(u) := \int_0^u \frac{R(v)}{v}dv.$$

So,

$$\int_0^u \frac{R(y/\ln(1/q))}{y/\ln(1/q)}dy = \ln(1/q)\int_0^{u/\ln(1/q)} \frac{R(y)}{y}dy = \ln(1/q) r\left(\frac{u}{\ln(1/q)}\right).$$

Use this and perform integration by parts to get

$$E(q) = -\int_0^\infty r\left(\frac{y}{\ln(1/q)}\right)\frac{d}{dy}\frac{ye^{-y}(e^{-y}-1+y)}{(1-e^{-y})^2}.$$

Referring to the condition of Ingham set as stated in (3) it is easy to get as $q\to 1$

$$E(q) = -\int_0^\infty \left(b_1\ln\left(\frac{y}{\ln(1/q)}\right)+b_2+o(1)\right)\frac{d}{dy}\frac{ye^{-y}(e^{-y}-1+y)}{(1-e^{-y})^2}.$$

Thus

$$E(q) = O(|\ln\ln(1/q)|).$$

Hence we have

$$\nu_n = \frac{C_\beta}{\ln^{1+\beta}(1/q)} + O\left(\frac{|\ln\ln(1/q)|}{\ln(1/q)}\right). \tag{10}$$

Setting $\frac{C_\beta}{\ln^{1+\beta}(1/q)} = n$ leads to the choice (4) where $c_\beta := C_\beta^{1/(1+\beta)}$ is given by (5). With that choice of q we can verify that

$$\nu_n = n + O(n^{1/(1+\beta)}\ln n).$$

From this we see that the Ingham's condition introduces a factor of $\ln n$ for the error estimate in the situation where the integrand is non-oscillatory. This phenomenon is persistent in many of the estimates in the sequel.

Next, we approximate the variance. Using the same change of variables and partial integration we see that

$$\begin{aligned}\sigma_n^2 &:= \mathrm{var}\left(\sum_j s_j G_{n,s_j}\right) = \sum_j s_j^2\frac{q^{s_j}}{(1-q^{s_j})^2}\\ &= \int_0^\infty x^2\frac{q^x}{(1-q^x)^2}dN(x)\\ &= \frac{1}{\ln^2(1/q)}\int_0^\infty y^2\frac{e^{-y}}{(1-e^{-y})^2}dN\left(\frac{y}{\ln(1/q)}\right)\\ &= -\frac{1}{\ln^2(1/q)}\int_0^\infty N\left(\frac{y}{\ln(1/q)}\right)d\left(y^2\frac{e^{-y}}{(1-e^{-y})^2}\right)\\ &= \frac{\kappa_\beta}{\ln^{2+\beta}(1/q)} + O\left(\frac{|\ln\ln(1/q)|}{\ln^2(1/q)}\right) = K_\beta n^{\frac{2+\beta}{1+\beta}} + O\left(n^{\frac{2}{1+\beta}}\ln n\right).\end{aligned} \tag{11}$$

The value of κ_β is

$$-B\int_0^\infty y^\beta d\left(y^2\frac{e^{-y}}{(1-e^{-y})^2}\right) = B\beta(\beta+1)\int_0^\infty y^\beta\frac{e^{-y}}{1-e^{-y}}dy = (\beta+1)c_\beta^{1+\beta},$$

and thus,

$$K_\beta = \frac{\kappa_\beta}{c_\beta^{2+\beta}} = \frac{\beta+1}{c_\beta} = \frac{\beta+1}{(B\beta\zeta(1+\beta)\Gamma(1+\beta))^{1/(1+\beta)}}. \tag{12}$$

2.2. Local Limit Theorem at 0

In this section we prove the following

Proposition 1. *If q is chosen according to* (4) *then*

$$\mathsf{P}\left(\sum_{j\ge1} s_jG_{n,s_j} = n\right) = \frac{1}{\sqrt{2\pi K_\beta}}\frac{1}{n^{(2+\beta)/(2(\beta+1))}}(1+o(1)), \tag{13}$$

where K_β is given by (12).

Proof. Following Fristedt we will first establish the central limit theorem for the normalized sums $\sum_j s_jG_{n,s_j}$ and then strengthen it to the local limit theorem by establishing additional bounds on the characteristic function. The simplest way to prove the CLT is to verify the Lyapunov condition (see [6, Theorem 27.3]). We will do it for $\delta = 1$, i.e. we will verify that

$$\frac{1}{\sigma_n^3}\sum_{j\ge1} s_j^3\mathsf{E}|G_{n,s_j} - \mathsf{E}G_{n,s_j}|^3 = o(1). \tag{14}$$

Since

$$\mathsf{E}|G_{n,s_j} - \mathsf{E}G_{n,s_j}|^3 \le 4\big(\mathsf{E}G_{n,s_j}^3 + (\mathsf{E}G_{n,s_j})^3\big) \le 8\mathsf{E}G_{n,s_j}^3,$$

it is enough to consider $\sum_{j\ge1} s_j^3\mathsf{E}G_{n,s_j}^3$. But

$$\mathsf{E}G_{n,s_j}^3 = \frac{q^{s_j}(1+4q^{s_j}+q^{2s_j})}{(1-q^{s_j})^3} \le 6\frac{q^{s_j}}{(1-q^{s_j})^3},$$

and by the same type of calculations as earlier we get

$$\begin{aligned}\sum_{j\ge1} s_j^3\frac{q^{s_j}}{(1-q^{s_j})^3} &= \frac{1}{\ln^3(1/q)}\int_0^\infty \frac{y^3e^{-y}}{(1-e^{-y})^3}dN\left(\frac{y}{\ln(1/q)}\right)\\ &= -\frac{1}{\ln^3(1/q)}\int_0^\infty N\left(\frac{y}{\ln(1/q)}\right)d\left(\frac{y^3e^{-y}}{(1-e^{-y})^3}\right)\\ &= O\left(\frac{1}{\ln^{3+\beta}(1/q)}\right) = O\left(\sigma_n^{\frac{2(3+\beta)}{2+\beta}}\right) = o(\sigma_n^3),\end{aligned}$$

where in the next to the last step we have used (11).

It remains to strengthen CLT to the local limit theorem. Let $\phi_n(t)$ be the characteristic function of the sum $\sum s_j G_{n,s_j}$. By the inversion formula

$$\mathsf{P}\left(\sum_{j\geq 1} s_j G_{n,s_j} = n\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itn}\phi_n(t)dt = \frac{1}{2\pi\sigma_n}\int_{-\pi\sigma_n}^{\pi\sigma_n} e^{-\frac{itn}{\sigma_n}}\phi_n(t/\sigma_n)dt,$$

and the goal is to show that last integral converges to $\sqrt{2\pi}$ as $n\to\infty$. We will write it as I_1+I_2, where, for some $0<\gamma<1$,

$$I_1 := \int_{-\gamma\sigma_n^{\beta/(2+\beta)}}^{\gamma\sigma_n^{\beta/(2+\beta)}} e^{-\frac{itn}{\sigma_n}}\phi_n(t/\sigma_n)dt,$$
$$I_2 := \int_{\gamma\sigma_n^{\beta/(2+\beta)}\leq|t|\leq\pi\sigma_n} e^{-\frac{itn}{\sigma_n}}\phi_n(t/\sigma_n)dt.$$

Since $\{G_{n,s_j}\}$ are independent, $\phi_n(t)$ is the product of the characteristic functions of its summands. Hence, we have

$$|\phi_n(t)| = \exp\left\{-\sum_{j\geq 1}\ln\left|\frac{1-q^{s_j}\exp(its_j)}{1-q^{s_j}}\right|\right\}.$$

Since

$$\left|\frac{1-q^{s_j}\exp(its_j)}{1-q^{s_j}}\right| = \left(1+\frac{2q^{s_j}(1-\cos(ts_j))}{(1-q^{s_j})^2}\right)^{1/2} \geq (1+2q^{s_j}(1-\cos(ts_j)))^{1/2}, \tag{15}$$

and $\ln(1+x)\geq x/5$ for $0\leq x\leq 4$, we get

$$|e^{-itn/\sigma_n}\phi_n(t/\sigma_n)| \leq \exp\left\{-c\sum_{j\geq 1} q^{s_j}(1-\cos(ts_j/\sigma_n))\right\}, \tag{16}$$

for some absolute constant $c\geq 1/10$. Consider t in the range of integration in I_1 and restrict the sum in (16) to those j for which

$$\delta\sigma_n^{2/(2+\beta)} \leq s_j \leq \sigma_n^{2/(2+\beta)}. \tag{17}$$

In that range of j's

$$\frac{|t|s_j}{\sigma_n} \leq \frac{|t|}{\sigma_n^{\beta/(2+\beta)}},$$

and

$$q^{s_j} = \exp(-s_j\ln(1/q)) \sim \exp\left(-s_j\frac{c_\beta\kappa_\beta^{1/(2+\beta)}}{\sigma_n^{2/(2+\beta)}}\right) \geq C > 0,$$

since by (11) $\ln(1/q)\sim c_\beta(\kappa_\beta/\sigma_n^2)^{1/(2+\beta)}$. Furthermore,

$$\frac{|t|s_j}{\sigma_n} \leq \frac{|t|}{\sigma_n^{\beta/(2+\beta)}} \leq \gamma < 1,$$

and thus

$$1-\cos(ts_j/\sigma_n) \geq \frac{t^2 s_j^2}{4\sigma_n^2}.$$

Hence,

$$2q^{s_j}(1-\cos(ts_j/\sigma_n)) \geq \frac{C}{2}\frac{t^2 s_j^2}{\sigma_n^2} \geq \frac{C\delta^2}{2}\frac{t^2}{\sigma_n^{2\beta/(2+\beta)}}.$$

Since there are $\Theta(\sigma_n^{2\beta/(2+\beta)})$ terms in the range specified by (17), we see that for $|t| \leq \gamma\sigma_n^{\beta/(2+\beta)}$ (16) is bounded by $\exp(-ct^2)$, for some absolute constant c. Since by the CLT

$$e^{-itn/\sigma_n}\phi_n(t/\sigma_n) \longrightarrow e^{-t^2/2}, \quad \text{as} \quad n \to \infty,$$

we conclude by the dominated convergence that

$$I_1 \longrightarrow \int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}.$$

It remains to show

Lemma 1. *$I_2 = o(1)$, as $n \to \infty$.*

Proof. We need to proceed with caution since here we handle an oscillatory sum. We claim that for $y \to \infty$, we have for some positive c

$$\frac{cy^\beta(1+o(1))}{\ln y} \leq N(y) \leq \frac{B2^\beta y^\beta(1+o(1))}{\beta \ln 2}. \tag{18}$$

From (2) we get

$$\int_0^y \frac{N(u)}{u}du = \int_0^y Bu^{\beta-1}du + \int_0^y \frac{R(u)}{u}du = \frac{B}{\beta}y^\beta + b_1 \ln y + b_2 + o(1)$$
$$= \frac{B}{\beta}y^\beta(1+o(1)).$$

Now

$$\int_0^y \frac{N(u)}{u}du = \int_{s_1}^y \frac{N(u)}{u}du \leq N(y)\int_{s_1}^y \frac{1}{u}du = N(y)(\ln y - \ln s_1).$$

Hence

$$\frac{By^\beta(1+o(1))}{\beta(\ln y - \ln s_1)} \leq N(y).$$

For the other direction consider

$$\int_0^y \frac{N(u)}{u}du \geq \int_{y/2}^y \frac{N(u)}{u}du \geq N\left(\frac{y}{2}\right)\int_{y/2}^y \frac{1}{u}du = N\left(\frac{y}{2}\right)\ln 2.$$

Hence

$$N\left(\frac{y}{2}\right) \leq \frac{1}{\ln 2}\int_0^y \frac{N(u)}{u}du.$$

This proves (18). Use (15) to get

$$|I_2| \le \int_{\gamma\sigma_n^{\beta/(2+\beta)} \le |t| \le \pi\sigma_n} \exp\left\{-\frac{1}{2}\sum_{j\ge1} \ln\left(1+\frac{2q^{s_j}\left(1-\cos\left(\frac{ts_j}{\sigma_n}\right)\right)}{(1-q^{s_j})^2}\right)\right\} dt$$
$$= J_1 + J_2 + J_3,$$

where

$$J_1 := \int_{\gamma\sigma_n^{\beta/(2+\beta)} \le |t| \le \sigma_n^{1/2}} \exp\{*\}dt, \quad J_2 := \int_{\sigma_n^{1/2} \le |t| \le \varepsilon_1\sigma_n} \exp\{*\}dt,$$

and

$$J_3 := \int_{\varepsilon_1\sigma_n \le |t| \le \pi\sigma_n} \exp\{*\}dt.$$

Here ε_1 is a small positive number. We start with estimating J_1. By a change of variable we have

$$J_1 := \sigma_n^{\beta/(2+\beta)} \int_{\gamma \le |t| \le \sigma_n^{(2-\beta)/2(2+\beta)}} \exp\left\{-\sum_{j\ge1} \ln\left(1+\frac{2q^{s_j}\left(1-\cos\left(\frac{ts_j}{\sigma_n^{2/(2+\beta)}}\right)\right)}{(1-q^{s_j})^2}\right)\right\} dt.$$

To get an upper estimate we keep the terms in the sum up to $s_j = o(\sigma_n^{1/2})$, say

$$s_j \le n^{\frac{1}{2(1+\beta)}}. \tag{19}$$

Since $|t| \le \sigma_n^{(2-\beta)/2(2+\beta)}$ we have $|ts_j| = o(\sigma_n^{2/(2+\beta)})$. For every term satisfying (19) we get $\frac{ts_j}{\sigma_n^{2/(2+\beta)}} = o(1)$, $s_j \ln(1/q) = o(1)$, and

$$\ln\left(1+\frac{2q^{s_j}\left(1-\cos\left(\frac{ts_j}{\sigma_n^{2/(2+\beta)}}\right)\right)}{(1-q^{s_j})^2}\right) = \ln\left(1+\frac{2(1+o(1))\left(\frac{1}{2}\left(\frac{ts_j}{\sigma_n^{2/(2+\beta)}}\right)^2(1+o(1))\right)}{(s_j\ln(1/q))^2(1+o(1))}\right)$$
$$= \ln\left(1+\frac{t^2(1+o(1))}{\sigma_n^{4/(2+\beta)}\ln^2(1/q)}\right) = (1+o(1))\ln(1+ct^2).$$

The integration variable t in J_1 is at least γ. Thus for all j with s_j satisfying (19) we have, for all large n,

$$\ln\left(1+\frac{2q^{s_j}\left(1-\cos\left(\frac{ts_j}{\sigma_n^{2/(2+\beta)}}\right)\right)}{(1-q^{s_j})^2}\right) > \frac{1}{2}\ln(1+c\gamma^2).$$

Hence, from (18)

$$J_1 \le \sigma_n^{\beta/(2+\beta)}\sigma_n^{(2-\beta)/2(2+\beta)} \exp\left\{-\frac{1}{4}\ln(1+c\gamma^2)N\left(n^{\frac{1}{2(1+\beta)}}\right)\right\}$$
$$\le \sigma_n^{1/2}\exp\left\{-c\frac{n^{\frac{\beta}{2(1+\beta)}}}{\ln n}\right\} = o(1).$$

Recall

$$J_2 = \int_{\sigma_n^{1/2} \le |t| \le \varepsilon_1 \sigma_n} \exp\left\{-\frac{1}{2}\sum_{j\ge 1} \ln\left(1 + \frac{2q^{s_j}\left(1-\cos\left(\frac{ts_j}{\sigma_n}\right)\right)}{(1-q^{s_j})^2}\right)\right\} dt$$

$$= \sigma_n \int_{\sigma_n^{-1/2} \le |t| \le \varepsilon_1} \exp\left\{-\frac{1}{2}\sum_{j\ge 1} \ln\left(1 + \frac{2q^{s_j}(1-\cos(ts_j))}{(1-q^{s_j})^2}\right)\right\} dt.$$

To make our arguments work, we choose ε_1 so small that $0 < \varepsilon_1 s_j \le \frac{1}{3}$ for $1 \le j \le 6$, and keep only the first 6 terms in the summation. Since the arguments of cosine are now in the interval $[0, \frac{1}{3}]$ where cosine decreases, we have

$$J_2 \le \varepsilon_1 \sigma_n \exp\left\{-\frac{1}{2}\sum_{1\le j\le 6} \ln\left(1 + \frac{2\left(1-\cos\left(\sigma_n^{-1/2}s_j\right)\right)}{(s_j \ln(1/q))^2(1+o(1))}\right)\right\}.$$

Use $\ln(1/q) \sim cn^{-1/(1+\beta)}$ and $\sigma_n \sim cn^{(1+\beta/2)/(1+\beta)}$ to simplify:

$$\ln\left(1 + \frac{2\left(1-\cos\left(\sigma_n^{-1/2}s_j\right)\right)}{(s_j \ln(1/q))^2(1+o(1))}\right) = \ln\left(1 + \frac{2\left(1-\cos\left(\sigma_n^{-1/2}s_j\right)\right)}{(s_j \ln(1/q))^2(1+o(1))}\right)$$

$$= \ln(1 + cn^{(1-\beta/2)/(1+\beta)}(1+o(1))) = \frac{1-\beta/2}{1+\beta}\ln n + o(1).$$

Hence,

$$J_2 \le \varepsilon_1 \sigma_n \exp\left\{-\frac{3(1-\beta/2)}{1+\beta}\ln n + o(1)\right\} = O\left(\sigma_n n^{\frac{-3(1-\beta/2)}{1+\beta}}\right) = O\left(n^{-\frac{2(1-\beta)}{1+\beta}}\right)$$

$$= o(1).$$

For J_3, we have

$$|J_3| \le 2\sigma_n \int_{\varepsilon_1 \le t \le \pi} \exp\left\{-\frac{1}{2}\sum_{j\ge 1} \ln\left(1 + \frac{2q^{s_j}(1-\cos(ts_j))}{(1-q^{s_j})^2}\right)\right\} dt.$$

We now use the assumption that $\mathcal{S}$ contains four s_j's which are pairwise relatively prime. Without loss of generality, we may assume them to be s_1, s_2, s_3, and s_4. We drop the remaining terms to get the estimate

$$|J_3| \le 2\sigma_n \int_{\varepsilon_1 \le t \le \pi} \exp\left\{-\frac{1}{2}\sum_{j=1}^{4} \ln\left(1 + \frac{2q^{s_j}(1-\cos(ts_j))}{(1-q^{s_j})^2}\right)\right\} dt.$$

Now let $\mathcal{A}$ be the union of the sets of roots of the four equations:

$$1 - \cos(ts_j) = 0, \quad 1 \le j \le 4, \tag{20}$$

that fall in the interval $[\varepsilon_1, \pi]$. The elements of $\mathcal{A}$ are of the form $\frac{2k\pi}{s_j}$, $1 \le j \le 4$. Since s_j's are pairwise relatively prime, different s_j gives rise to disjoint zero set of $1 - \cos(ts_j) = 0$ in $[\varepsilon_1, \pi]$. In other words, every element of $\mathcal{A}$ satisfies *exactly* one such equation in (20). We choose a fine partition $\mathcal{P}$ of $[\varepsilon_1, \pi]$ so that every subinterval $\mathcal{I}$ in $\mathcal{P}$ contains *at most* one point of $\mathcal{A}$. With $\mathcal{P}$ at our disposal, we decompose the interval of integration so that

$$|J_3| \le 2\sigma_n \sum_{\mathcal{I} \in \mathcal{P}} \int_{\mathcal{I}} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{4} \ln\left(1 + \frac{2q^{s_j}(1 - \cos(ts_j))}{(1 - q^{s_j})^2}\right)\right\} dt.$$

On each subinterval $\mathcal{I}$, either all four of $1 - \cos(ts_j)$ are monotone functions or three are monotone and the fourth one has a unique zero in $\mathcal{A}$. Hence by using the appropriate endpoints of $\mathcal{I}$ or an element of $\mathcal{A}$, we can get a lower estimate of the sum. Asymptotically we have

$$\sum_{j=1}^{4} \ln\left(1 + \frac{2q^{s_j}(1 - \cos(ts_j))}{(1 - q^{s_j})^2}\right) \ge \frac{3}{1 + \beta} \ln n + o(1).$$

As a consequence, for each subinterval $\mathcal{I}$

$$\int_{\mathcal{I}} \exp\left\{ -\frac{1}{2} \sum_{j=1}^{4} \ln\left(1 + \frac{2q^{s_j}(1 - \cos(ts_j))}{(1 - q^{s_j})^2}\right)\right\} dt = O\big(n^{-\frac{3}{2(1+\beta)}}\big).$$

Hence

$$|J_3| \le O\big(\sigma_n n^{-\frac{3}{2(1+\beta)}}\big) = O\big(n^{-\frac{1-\beta}{2(1+\beta)}}\big) = o(1).$$

Consequently $I_2 = o(1)$. This completes the proof of the lemma and the proof of (13). ∎

2.3. Conclusion of the Proof

We will now proceed to showing that the two events in the numerator of (9) are asymptotically independent. To that end consider $k = k_n$ whose value will be chosen later. As long as $k_n \to \infty$ and $s_k = o(n^{1/(\beta+1)})$, by the same calculations as before we have

$$\begin{aligned}
\mathsf{E} \sum_{j>k} G_{n,s_j} &= \int_{s_k \ln(1/q)}^{\infty} \frac{e^{-y}}{1 - e^{-y}} dN\left(\frac{y}{\ln(1/q)}\right) \\
&= \int_{s_k \ln(1/q)}^{\infty} \frac{e^{-y}}{(1 - e^{-y})^2} N\left(\frac{y}{\ln(1/q)}\right) dy + O\left(\frac{k}{1 - e^{-s_k \ln(1/q)}}\right) \\
&= \frac{B}{\ln^{\beta}(1/q)} (s_k \ln(1/q))^{\beta - 1} + O\left(\frac{k}{s_k \ln(1/q)}\right) \\
&= O\big(n^{1/(\beta+1)} s_k^{\beta-1}\big) = O\left(\frac{n^{1/(\beta+1)}}{k^{(1-\beta)/\beta}}\right).
\end{aligned}$$

In particular, we see that the expected value of the sum restricted to $j > k$ is of lower order than that of the full sum (which, by the same calculation is of order $n^{1/(\beta+1)}$). To confirm that the contribution of $\sum_{j>k} G_{n,s_j}$ is negligible we will establish the following concentration result:

Proposition 2. *Let (m_n) be any sequence such that $m_n \to \infty$ and $m_n = o(k_n)$ as $n \to \infty$. Set $\lambda_n = n^{1/(\beta+1)}/m_n^{(1-\beta)/\beta}$. Then*

$$\mathsf{P}\left(\sum_{j>k} G_{n,s_j} \geq \lambda_n\right) \leq e^{-cs_k^\beta}, \tag{21}$$

for some absolute constant c.

Proof of Proposition 2. Using independence of G_{n,s_j}, for any $t > 0$ we have

$$\mathsf{P}\left(\sum_{j>k} G_{n,s_j} \geq \lambda_n\right) \leq e^{-t\lambda_n} \prod_{j>k} \mathsf{E}e^{tG_{n,s_j}} = e^{-t\lambda_n} \prod_{j>k} \frac{1-q^{s_j}}{1-q^{s_j}e^t}$$
$$= e^{-t\lambda_n} \prod_{j>k} \frac{1}{1-\frac{q^{s_j}}{1-q^{s_j}}(e^t-1)}.$$

Suppose $0 < t < 1/2$ is chosen so that for $j \geq k$

$$\frac{q^{s_j}}{1-q^{s_j}}(e^t-1) \leq \frac{1}{2}. \tag{22}$$

Then, since $1/(1-x) \leq \exp(2x)$ whenever $0 \leq x \leq 1/2$, we have

$$\frac{1}{1-\frac{q^{s_j}}{1-q^{s_j}}(e^t-1)} \leq \exp\left(2\frac{q^{s_j}}{1-q^{s_j}}(e^t-1)\right) \leq \exp\left(4t\frac{q^{s_j}}{1-q^{s_j}}\right)$$
$$= \exp(4t\mathsf{E}G_{n,s_j}).$$

Hence the upper bound on the probability in (21) is

$$\exp\left(-t\left(\lambda_n - 4\mathsf{E}\sum_{j>k} G_{n,s_j}\right)\right).$$

By our choice of m_n and the definition of λ_n the term in the inner parentheses is of order λ_n. It remains to specify t. Since (s_j) is increasing, for $j \geq k$

$$\frac{q^{s_j}}{1-q^{s_j}} \leq \frac{q^{s_k}}{1-q^{s_k}},$$

so in order to satisfy (22) it will suffice to have

$$\frac{q^{s_k}}{1-q^{s_k}}(e^t-1) \leq \frac{1}{2},$$

for which it suffices that

$$e^t - 1 \le \frac{1}{2}(1 - q^{s_k}) = \frac{1}{2}\left(1 - \exp\left(-c\frac{s_k}{n^{1/(\beta+1)}}\right)\right).$$

But with $s_k = o(n^{1/(\beta+1)})$ that is easy to satisfy; t may be chosen to be a sufficiently small (but constant) multiple of $\frac{s_k}{n^{1/(\beta+1)}}$. If this multiple is, say, δ then the resulting bound on our probability is

$$\exp(-t\lambda_n) = \exp\left(-\delta\frac{s_k}{n^{1/(\beta+1)}} \cdot \frac{n^{1/(\beta+1)}}{m_n^{(1-\beta)/\beta}}\right) = \exp\left(-\frac{\delta s_k}{m_n^{(1-\beta)/\beta}}\right).$$

Since

$$m_n = o(k) = o(N(s_k)) = o\left(s_k^{\beta}\right),$$

it follows that $s_k/m_n^{(1-\beta)/\beta}$ is at least a multiple of s_k^{β} which proves (21). ∎

So far a good choice for the growth of k is, say, n^{α} with $0 < \alpha < \beta/(\beta+1)$ and it ensures that $\mathsf{P}(\sum_{j>k} G_{n,s_j} \ge \lambda_n)$ goes to zero even when divided by (13). Since $\lambda_n = o(\mathsf{E}\sum_{j\ge1} G_{n,s_j})$, it follows that the contribution of $\sum_{j>k} G_{n,s_j}$ to the full sum is asymptotically negligible.

To proceed note that for any two non-negative random variables X and W and any event B we have

$$\mathsf{P}(X + W \le x, B) \le \mathsf{P}(X \le x, B),$$

and also,

$$\begin{aligned}\mathsf{P}(X \le x - \varepsilon, B) &= \mathsf{P}(X \le x - \varepsilon, B, W \le \varepsilon) + \mathsf{P}(X \le x - \varepsilon, B, W > \varepsilon)\\ &\le \mathsf{P}(X + W \le x, B) + \mathsf{P}(W > \varepsilon).\end{aligned}$$

Using this with

$$X = \frac{\sum_{j\le k} G_{n,s_j}}{\mu_n}, \quad W = \frac{\sum_{j>k} G_{n,s_j}}{\mu_n}, \quad B = \left\{\sum_{j\ge1} s_j G_{n,s_j} = n\right\}$$

we get that

$$\mathsf{P}\left(\frac{1}{\mu_n}\sum_{j\le k} G_{n,s_j} \le x - \varepsilon, B\right) - O(e^{-ck_n}) \tag{23}$$

$$\le \mathsf{P}\left(\frac{1}{\mu_n}\sum_{j\ge1} G_{n,s_j} \le x, B\right) \le \mathsf{P}\left(\frac{1}{\mu_n}\sum_{j\le k} G_{n,s_j} \le x, B\right). \tag{24}$$

We now consider the last term in (24). Let $C_k = \{\sum_{j\le k} G_{n,s_j} \le x\mu_n\}$. We also set

$$\sum_{j\ge1} s_j G_{n,s_j} = \sum_{j\le k} s_j G_{n,s_j} + \sum_{j>k} s_j G_{n,s_j} := Y_1 + Y_2.$$

Then, for $\ell = \ell_n$ to be specified later we have

$$\begin{aligned}
\mathsf{P}(C_k \cap B) &\le \sum_{r=0}^{\ell} \mathsf{P}(C_k, Y_1 = r, Y_2 = n - r) + \mathsf{P}(Y_1 > \ell) \\
&= \sum_{r=0}^{\ell} \mathsf{P}(C_k, Y_1 = r) \cdot \mathsf{P}(Y_2 = n - r) + \mathsf{P}(Y_1 > \ell) \\
&\le \sup_{0 \le r \le \ell} \{\mathsf{P}(Y_2 = n - r)\} \mathsf{P}(C_k, Y_1 \le \ell) + \mathsf{P}(Y_1 > \ell) \\
&\le \sup_{0 \le r \le \ell} \{\mathsf{P}(Y_2 = n - r)\} \cdot \mathsf{P}(C_k) + O(\mathsf{P}(Y_1 > \ell)).
\end{aligned} \tag{25}$$

In the same fashion

$$\begin{aligned}
\mathsf{P}(C_k \cap B) &\ge \mathsf{P}(C_k \cap B, Y_1 \le \ell) = \sum_{r=0}^{\ell} \mathsf{P}(C_k, Y_1 = r, Y_2 = n - r) \\
&\ge \inf_{0 \le r \le \ell} \{\mathsf{P}(Y_2 = n - r)\} \mathsf{P}(C_k, Y_1 \le \ell) \\
&\ge \inf_{0 \le r \le \ell} \{\mathsf{P}(Y_2 = n - r)\} \mathsf{P}(C_k) - O(\mathsf{P}(Y_1 > \ell)).
\end{aligned} \tag{26}$$

We will first bound the error term in (25) and (26).

Proposition 3. *There exist absolute constants c_1, c_2 such that if $\ell \ge c_1 k n^{1/(\beta+1)}$ then*

$$\mathsf{P}(Y_1 \ge \ell) \le \exp(-c_2 k).$$

Proof of Proposition 3. Just as before we have

$$\mathsf{P}(Y_1 > \ell) = \mathsf{P}\left(\sum_{j \le k} s_j G_{n,s_j} > \ell\right) \le e^{-t\ell} \prod_{j \le k} \frac{1}{1 - \frac{q^{s_j}}{1-q^{s_j}}(e^{ts_j} - 1)},$$

and if we managed to pick $t > 0$ so that for $j \le k$

$$\frac{q^{s_j}}{1 - q^{s_j}}(e^{ts_j} - 1) \le \frac{1}{2}, \quad \text{and} \quad ts_j \le \frac{1}{2}, \tag{27}$$

then the argument used to justify (21), with (27) replacing (22) would show that the jth term in the product is bounded by

$$\exp\left(4ts_j \frac{q^{s_j}}{1 - q^{s_j}}\right) = \exp(4t\mathsf{E}s_j G_{n,s_j}),$$

and the probability in question by

$$\exp(-t(\ell - 4\mathsf{E}Y_1)) \le \exp(-t\mathsf{E}Y_1),$$

provided that $\ell \ge 5\mathsf{E}Y_1$.

Now, according to our choices thus far, $k = o(n^{\beta/(\beta+1)})$, so that $s_k = o(n^{1/(\beta+1)})$. It follows that,

$$\begin{aligned}
\mathsf{E}Y_1 &= \sum_{j=1}^{k} s_j \frac{q^{s_j}}{1-q^{s_j}} = \int_0^{s_k} \frac{tq^t}{1-q^t} dN(t) \\
&= \int_0^{s_k} N(t) \frac{q^t(q^t - 1 - t\ln q)}{(1-q^t)^2} dt + O(kn^{1/(\beta+1)}) \\
&= \frac{1}{\ln(1/q)} \int_0^{s_k \ln(1/q)} N\left(\frac{y}{\ln(1/q)}\right) \frac{e^{-y}(e^{-y} - 1 + y)}{(1-e^{-y})^2} dy + O(kn^{1/(\beta+1)}) \\
&= \Theta\big(s_k^{\beta+1}\big) + O(kn^{1/(1+\beta)}) = O(kn^{1/(1+\beta)}) = o(n).
\end{aligned}$$

In the last line we use the fact that $s_k \ln(1/q) = o(1)$ so that the main contribution to the integral comes from a small neighborhood of 0. It remains to choose t so that (27) is satisfied. First, $ts_k \leq 1/2$ would guarantee the second requirement in (27). The first is equivalent to

$$2(e^{ts_j} - 1) \leq \frac{1 - q^{s_j}}{q^{s_j}} = e^{s_j \ln(1/q)} - 1.$$

But, if $ts_k \leq 1/2$, then for $j \leq k$, $e^{ts_j} \leq 1 + 2ts_j$ and since $e^{s_j \ln(1/q)} \geq 1 + s_j \ln(1/q)$ it is enough to pick t so that $4t \leq \ln(1/q)$. Choosing t to be a small (but fixed) multiple of $\ln(1/q) = c_\beta / n^{1/(\beta+1)}$ we get that

$$\mathsf{P}(Y_1 > \ell) \leq \exp(-t\mathsf{E}Y_1) \leq \exp(-c_2 k),$$

provided $\ell \geq 5\mathsf{E}Y_1 = ck/\ln(1/q) := c_1 kn^{1/(1+\beta)}$. ∎

As a consequence, we see that if k is chosen to be a constant multiple of n^α where $\alpha < \beta/(\beta+1)$ then all the previous requirements on k are satisfied and ℓ can be chosen to be proportional to $n^{\alpha+1/(\beta+1)} = o(n^{(2+\beta)/(2(\beta+1))})$. Coming back to (25) and (26) let r_n, ρ_n be defined by

$$\sup_{0\leq r\leq \ell} \{\mathsf{P}(Y_2 = n - r)\} = \mathsf{P}(Y_2 = n - r_n),$$
$$\inf_{0\leq r\leq \ell} \{\mathsf{P}(Y_2 = n - r)\} = \mathsf{P}(Y_2 = n - \rho_n).$$

Repeating the same calculation as for $\sigma_n^2 = \mathsf{var}(\sum_{j\geq 1} s_j G_{n,s_j})$ we see that

$$\mathsf{var}(Y_1) = \sum_{j\leq k} s_j^2 \frac{q^{s_j}}{(1-q^{s_j})^2} \sim ck \cdot n^{\frac{2}{\beta+1}} = o\big(\sigma_n^2\big),$$

given our choice of k. Hence $\sigma_{n,2}^2 := \mathsf{var}(Y_2) \sim \mathsf{var}(\sum_{j\geq 1} s_j G_{n,s_j})$ and since $\sum_{j>k} s_j G_{n,s_j}$ satisfy exactly the same local limit theorem as the full sums do we obtain

$$\lim \sigma_{n,2} \mathsf{P}(Y_2 = n - r_n) = \lim \sigma_{n,2} \mathsf{P}\left(\frac{Y_2 - n}{\sigma_{n,2}} = \frac{-r_n}{\sigma_{n,2}}\right) = \frac{1}{\sqrt{2\pi}},$$

where the last equality is true because $r_n/\sigma_{n,2} \to 0$ as $n \to \infty$. This implies in particular, that

$$\lim \frac{\mathsf{P}(Y_2 = n - r_n)}{\mathsf{P}\left(\sum_{j\geq 1} s_j G_{n,s_j} = n\right)} = 1,$$

and the same holds for ρ_n's.

Combining all of this we see that (23) and (24) give

$$\lim \mathsf{P}\left(\frac{\sum_{j\le k} G_{n,s_j}}{\mu_n} \le x-\varepsilon\right) \le \lim \mathsf{P}_n\left(\frac{\mathbf{M}_n}{\mu_n} \le x\right) \le \lim \mathsf{P}\left(\frac{\sum_{j\le k} G_{n,s_j}}{\mu_n} \le x\right).$$

Since, as we will see in a moment, the limiting distribution of $\sum_{j\le k} G_{n,s_j}/\mu_n$ is continuous and ε is arbitrary we conclude that

$$\lim \mathsf{P}_n\left(\frac{\mathbf{M}_n}{\mu_n} \le x\right) = \lim \mathsf{P}\left(\frac{\sum_{j\le k} G_{n,s_j}}{\mu_n} \le x\right).$$

To identify the limiting distribution of the normalized sum on the right note that since G_{n,s_j} is geometric with parameter $1-q^{s_j}$, the jth summand has a characteristic function

$$\mathsf{E}e^{itG_{n,s_j}/\mu_n} = \frac{1-q^{s_j}}{1-e^{it/\mu_n}q^{s_j}} = \frac{1-e^{-s_j\ln(1/q)}}{1-e^{-s_j\ln(1/q)+it/\mu_n}}.$$

Since $s_j \le s_k = o(n^{1/(\beta+1)})$, $q = \exp(-c_\beta/n^{1/(\beta+1)})$, and $\mu_n = 1/\ln(1/q)$ using basic approximations we further have

$$\begin{aligned}\mathsf{E}e^{itG_{n,s_j}/\mu_n} &= \frac{s_j\ln(1/q) + O\big(s_j^2\ln^2(1/q)\big)}{s_j\ln(1/q) - it/\mu_n + O\big(s_j^2\ln^2(1/q)\big)}\\ &= \frac{1}{1-\frac{it}{s_j}}(1+O(s_j\ln(1/q))).\end{aligned}$$

Hence, by independence of the summands, for k in our range, we get

$$\begin{aligned}\phi_n(t) := \mathsf{E}e^{\frac{it}{\mu_n}\sum_{j\le k} G_{n,s_j}} &= \prod_{j\le k}\left(\frac{1}{1-\frac{it}{s_j}}(1+O(s_j\ln(1/q)))\right)\\ &= \left(\prod_{j\le k}\frac{1}{1-\frac{it}{s_j}}\right)Q_n,\end{aligned}$$

where

$$Q_n := \prod_{j\le k}(1+O(s_j\ln(1/q))).$$

For k of order n^α, with $\alpha < \beta/2(\beta+1)$,

$$ks_k\ln(1/q) = O(n^{1/2}n^{-1/(1+\beta)}) = o(1).$$

Clearly,

$$\sum_{j\le k} s_j = O(ks_k).$$

So,

$$\begin{aligned}Q_n &= \exp\left(\sum_{j\le k}\ln(1+O(s_j\ln(1/q)))\right) = \exp\left(O\left(\ln(1/q)\sum_{j\le k}s_j\right)\right)\\ &= \exp(O(ks_k\ln(1/q))) = \exp(o(1)) = 1+o(1).\end{aligned}$$

Finally, since for $k \to \infty$

$$\prod_{j>k} \frac{1}{1-\frac{it}{s_j}} = \exp\left(-\sum_{j>k} \ln\left(1-\frac{it}{s_j}\right)\right) = \exp(o(1)) \longrightarrow 1,$$

we conclude that $\phi_n(t)$ converge pointwise to $\phi_{\mathcal{S}}(t)$ given by (7).

3. THE CASE OF A POLYNOMIAL

As we mentioned before, the formula (8) may be further transformed when $\mathcal{S}$ is the range of a polynomial. We will show the following:

Proposition 4. *Let $Q(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $d \geq 2$. Let $r_1, r_2, \ldots, r_d$ be the complex roots of $Q(x)$ and for $j = 1, 2 \ldots, d-1$ let $\alpha_1(j), \ldots, \alpha_{d-1}(j)$ be those complex roots of $Q(x) - Q(j)$ that are not equal to j. If $\mathcal{S}$ is the range of Q, i.e. $s_j = Q(j)$, for $j \geq 1$ then the expression* (8) *for the density can be written as*

$$f_{\mathcal{S}}(x) = \sum_{j=1}^{\infty} (-1)^{j-1} e^{-Q(j)x} \frac{Q'(j)}{(j-1)!} \frac{\prod_{m=1}^{d-1} \Gamma(1-\alpha_m(j))}{\prod_{m=1}^{d} \Gamma(1-r_m)}, \qquad x > 0. \tag{28}$$

Proof of Proposition 4. To prove (28) it is enough evaluate the product in (8). Specifically, we will show

$$\prod_{\ell \neq j} \frac{s_\ell}{s_\ell - s_j} = \prod_{\ell \neq j} \frac{Q(\ell)}{Q(\ell) - Q(j)} = \frac{Q'(j)(-1)^{j+1}}{Q(j)(j-1)!} \frac{\prod_{t=1}^{d-1} \Gamma(1-\alpha_t(j))}{\prod_{t=1}^{d} \Gamma(1-r_t)}. \tag{29}$$

To this end we write

$$\begin{aligned}\prod_{\ell \neq j} \frac{Q(\ell)}{Q(\ell) - Q(j)} &= \lim_{s \to j} \left(\prod_{\ell \neq j} \frac{Q(\ell)}{Q(\ell) - Q(s)}\right) \\ &= \lim_{s \to j} \left(\frac{Q(j) - Q(s)}{Q(j)} \prod_{\ell \geq 1} \frac{Q(\ell)}{Q(\ell) - Q(s)}\right).\end{aligned} \tag{30}$$

We factor both $Q(\ell)$ and $Q(\ell) - Q(s)$ as a product of linear terms

$$Q(\ell) = a_d \prod_{m=1}^{d} (\ell - r_m),$$

$$Q(\ell) - Q(s) = a_d \prod_{m=1}^{d} (\ell - \alpha_m(s)).$$

We now use the following formula [40, Chapter XII, Sec. 12.13]: if $a_1 + \cdots + a_r = b_1 + \cdots + b_r$ then

$$\prod_{n=1}^{\infty} \frac{(n-a_1) \cdot \cdots \cdot (n-a_r)}{(n-b_1) \cdot \cdots \cdot (n-b_r)} = \prod_{m=1}^{r} \frac{\Gamma(1-b_m)}{\Gamma(1-a_m)}.$$

Applying this to the product in (30) we obtain

$$\prod_{\ell\neq j}\frac{Q(\ell)}{Q(\ell)-Q(j)}=\prod_{m=1}^{d}\frac{\Gamma(1-\alpha_m(s))}{\Gamma(1-r_m)}.$$

We know that exactly one of $\alpha_m(s)$'s is s and we assume without loss of generality that $\alpha_d(s)=s$. Since

$$\frac{1}{Q(j)\Gamma(1-r_d)}\prod_{m=1}^{d-1}\frac{\Gamma(1-\alpha_m(s))}{\Gamma(1-r_m)} \tag{31}$$

is continuous at $s=j$ we only need to be concerned with

$$\begin{aligned}\lim_{s\to j}((Q(j)-Q(s))\Gamma(1-\alpha_d(s))) &= \lim_{s\to j}\left(\frac{Q(j)-Q(s)}{j-s}(j-s)\Gamma(1-s)\right)\\ &= \lim_{s\to j}\left(\frac{Q(j)-Q(s)}{j-s}(j-s)(-s)\Gamma(-s)\right)\\ &= jQ'(j)\lim_{s\to j}((s-j)\Gamma(-s)).\end{aligned}$$

Since the residue of $\Gamma(z)$ at $-j$ is $(-1)^j/j!$ this last limit is $(-1)^{j-1}/j!$ which combined with (31) and (30) proves (29). ∎

4. FURTHER REMARKS

In this section we briefly discuss a few cases that are of special interest.

i. One such case, $Q(z)=\frac{z(z+1)}{2}$ arises naturally in the context of iterated functions and the coalescent [18], [25]. There the characteristic function is

$$\phi(t)=\prod_{m=2}^{\infty}\frac{\binom{m}{2}}{\binom{m}{2}-it}=\prod_{m=1}^{\infty}\frac{1}{1-it/\binom{m+1}{2}}. \tag{32}$$

In (29) we have $r_1=0, r_2=-1, \alpha_1(k)=-k-1$, and consequently

$$\prod_{\ell\neq k}\frac{Q(\ell)}{Q(\ell)-Q(k)}=\frac{\frac{2k+1}{2}}{\frac{k(k+1)}{2}}\frac{\Gamma(1+1+k)}{\Gamma(1-0)\Gamma(1+1)}\frac{(-1)^{k-1}}{(k-1)!}=(-1)^{k-1}(2k+1).$$

Hence inversion of (32) yields the probability density function

$$f(x)=\sum_{k=2}^{\infty}e^{-\binom{k}{2}x}\binom{k}{2}(-1)^k(2k-1).$$

This latter density is well-known in certain circles, it represents the total coalescent time in the least common ancestor process, and is generally attributed to Kingman [25]. See the unpublished manuscript [18] for a derivation that is related to the arguments in this paper.

ii. Similarly, for the special case $Q(x) = x^3$, we consider the number of parts of random partitions of n into parts that are cubes. For this particular class of partitions, Richmond [30] provided asymptotic estimates for the moments. Carleman's conditions are satisfied, therefore the limit distribution is uniquely determined. However, Richmond did not go beyond the computation of moments, and we are not aware of any previous work in which the limiting density is calculated. In fact, the density has an interesting form: for $x > 0$,

$$f(x) = 3\sum_{k=1}^{\infty} e^{-k^3x}\frac{(-1)^{k+1}k^3c_k}{k!},$$

where $c_k = \Gamma(1 - ke^{2\pi i/3})\Gamma(1 - ke^{-2\pi i/3}) = |\Gamma(1 - ke^{2\pi i/3})|^2$.

iii. The next case corresponds to $Q(x) = \binom{x+d}{d}$, for some fixed positive integer d. (Since $d = 1$ does not impose any restrictions we will assume $d \geq 2$. Also, $d = 2$ was a special case discussed in (i).) Such partitions are in bijection with partitions with dth differences non-negative. Some of their properties (although the limiting distribution of the number of parts was not one of them) were studied in [8]. We have $r_m = -m$, $m = 1, \ldots, d$ and thus

$$\prod_{m=1}^{d}\Gamma(1 - r_m) = \prod_{m=1}^{d} m!.$$

Further,

$$Q'(x) = \frac{1}{d!}\sum_{j=1}^{d}\prod_{\substack{1\leq \ell \leq d\\ \ell\neq j}}(x+\ell) = Q(x)\sum_{j=1}^{d}\frac{1}{x+j},$$

so that

$$Q'(k) = Q(k)(H_{k+d} - H_k),$$

where H_n is the nth harmonic number. Although there does not seem to be a simple way of handling the roots of $Q(x) - Q(k)$ in the general case, the case $d = 3$ can be managed (as can be any other polynomial of degree 3 since it leads to a quadratic equation after factoring $(x - k)$) and gives

$$f(x) = \frac{1}{2!\cdot 3!}\sum_{k=1}^{\infty}(-1)^{k-1}e^{-\binom{k+3}{3}x}\binom{k+3}{3}\frac{H_{k+3} - H_k}{(k-1)!}f_k,$$

where $f_k = |\Gamma(4 + \frac{k}{2} + \frac{i}{2}\sqrt{3k^2 + 12k + 8})|^2$.

If $d = 4$ then $\binom{x+4}{4} - \binom{k+4}{4}$ has a real root $-k - 5$ (in addition to k, of course) and the limiting density is

$$f(x) = \frac{1}{2!\cdot 3!\cdot 4!}\sum_{k=1}^{\infty}(-1)^{k-1}e^{-\binom{k+4}{4}x}\binom{k+4}{4}\frac{(H_{k+4} - H_k)(k+5)!}{(k-1)!}g_k,$$

where $g_k = |\Gamma(\frac{7}{2} + \frac{i}{2}\sqrt{4k^2 + 20k + 15})|^2$.

iv. Finally, we would like to conclude by observing that the choice $Q(x) = x^2$ corresponds to yet another interesting situation that arises in quite a different context. In view of (28) and (29) the probability density function corresponding to this choice is

$$f(x) = 2\sum_{k=1}^{\infty}(-1)^{k+1}k^2e^{-k^2x}, \quad x > 0.$$

Up to a scaling this is the density of the maximum of the Brownian bridge process or the Brownian meandering process (see [9, Section 3] and also [12, 13] for more details and information). Further interesting connections along with many more references to the literature are discussed in a relatively recent survey paper [4].

Distribution function corresponding to the last density is given by

$$F(x) = 1 - 2\sum_{k=1}^{\infty}(-1)^{k+1}e^{-k^2x} = \sum_{k=-\infty}^{\infty}(-1)^k e^{-k^2x}.$$

Changing variables, $x \to 2x^2$ and differentiating gives a density

$$4\sum_{k=-\infty}^{\infty}(-1)^{k-1}k^2xe^{-2k^2x^2},$$

which is the density of the Kolmogorov-Smirnov statistic used to measure the discrepancy between the true and empirical distribution functions. We refer the reader to [26] for the translation of the original work of Kolmogorov and to [5, Chapter 2, Sec. 13] for a detailed exposition.

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