

RENORMING DIVERGENT PERPETUITIES

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ABSTRACT. We consider a sequence of random variables (R_n) defined by the recurrence $R_n = Q_n + M_n R_{n-1}$, $n \geq 1$, where R_0 is arbitrary and (Q_n, M_n) , $n \geq 1$, are i.i.d. copies of a two dimensional random vector (Q, M) , and (Q_n, M_n) is independent of R_{n-1} . It is well known that if $E \ln |M| < 0$ and $E \ln |Q| < \infty$, then the sequence (R_n) converges in distribution to a random variable R given by $R \stackrel{d}{=} \sum_{k=1}^{\infty} Q_k \prod_{j=1}^{k-1} M_j$, and usually referred to as perpetuity. This note is the first step in exploring a situation in which the sequence (R_n) itself does not converge. We assume that $E \ln |M|$ exists but that it is non-negative and we ask if in this situation the sequence (R_n) , after suitable normalization, converges in distribution to a non-degenerate limit.

1. INTRODUCTION

We consider the following iterative scheme

$$(1.1) \quad R_n = Q_n + M_n R_{n-1}, \quad n \geq 1$$

where R_0 is arbitrary and (Q_n, M_n) , $n \geq 1$, are i.i.d. copies of a two dimensional random vector (Q, M) , and (Q_n, M_n) is independent of R_{n-1} . Writing out the above recurrence we see that

$$R_n = Q_n + M_n Q_{n-1} + M_n M_{n-1} Q_{n-2} + \cdots + M_n \cdots M_2 Q_1 + M_n \cdots M_1 R_0.$$

Note that although the above is not a partial sum, after renumbering (Q_n, M_n) in the opposite direction, we can write

$$(1.2) \quad R_n \stackrel{d}{=} \prod_{j=1}^n M_j R_0 + \sum_{k=1}^n Q_k \prod_{j=1}^{k-1} M_j,$$

where “ $\stackrel{d}{=}$ ” denotes the equality in distribution and we adopt the convention that the product or sum over the empty range is 1 or 0, respectively. Much of the impetus for studying such equations stems from numerous applications of schemes like (1.1) in mathematics and other disciplines of science. We refrain here from giving a long list of fields in which equation (1.1) appeared referring instead to some of the references we give (most notably

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to [6, 20] for the status up to the early – nineties of the past century) for more detailed information. Examples of more recent applications closer to statistics are given in [2], an application related to neuronal modeling may be found in [13], and for an application in the analysis of algorithms see [11] and references therein.

Essentially all of up-to-date research focused on the situation when the sequence (R_n) converges in distribution and on analyzing properties of its limit. It has been known for a long time that if

$$E \ln |M| < 0 \quad \text{and} \quad E \ln |Q| < \infty,$$

then the sequence of partial sums in (1.2) converges almost surely and the product in the first term asymptotically vanishes. Thus, (R_n) converges in distribution to a random variable R given by

$$R \stackrel{d}{=} \sum_{k=1}^{\infty} Q_k \prod_{j=1}^{k-1} M_j.$$

This R , referred to as perpetuity, is of central interest. In particular, its tail behavior has been thoroughly investigated (see e.g. [7, 8, 9, 14, 15, 20] and references therein for more information).

This note is the first step in exploring a situation in which the sequence (R_n) does not converge. While our terminology may be a bit controversial, we will refer in such situations to the whole sequence (R_n) as a divergent perpetuity. We will assume throughout that $E \ln |M|$ exists but that

$$(1.3) \quad \mu := E \ln |M| \geq 0.$$

Under the above assumption we ask if (R_n) can be renormed to converge in distribution to a non-degenerate limit. As is the case when (R_n) converges, the role of R_0 and Q seem to be of much lesser importance than that of M . In fact, since R_0 plays no significant role from now on we assume that $R_0 = 0$. Furthermore, assuming that M is non-random would lead to (R_n) being a sequence of sums of independent random variables. Since this situation has been extensively studied, we will exclude it from our considerations by assuming from now on that M is non-constant. On the other hand we freely impose moment conditions on Q when necessary, or sometimes even assume that it is of a special form. For example, if $Q = 1$, then (1.2) is just a partial sum of successive partial products of i.i.d. copies of M . While there is very little known technical connection, it is perhaps worth mentioning that an analogous problem of investigating the asymptotic properties of the consecutive products of partial sums

$$\prod_{k=1}^n \sum_{j=1}^k M_j \quad \text{versus} \quad \sum_{k=1}^n \prod_{j=1}^k M_j,$$

has been intensely investigated in the past several years, see for example [19, 18, 21].

As $x \rightarrow \ln x$ is concave the assumption (1.3) implies that necessarily $E|M| \geq 1$. Thus there are at least four situations to consider, namely

- (i) $E \ln |M| = 0$ and $E|M| = 1$,
- (ii) $E \ln |M| > 0$, $E|M| > 1$, $|M|$ non-random,
- (iii) $E \ln |M| > 0$, $E|M| > 1$, $|M|$ random,
- (iv) $E \ln |M| = 0$ and $E|M| > 1$.

These cases are discussed in detail below and we will show that in each of these situations (R_n) can be renormed (each time differently) so that it converges in distribution to a non-degenerate limit.

2. THE CASE $E \ln |M| = 0$ AND $E|M| = 1$

Under this assumption we in fact have that $|M| \equiv 1$, i.e., M takes on values ± 1 . Consequently, because of a non-degeneracy assumption on M , its expected value must satisfy: $-1 < EM < 1$. We have

Theorem 1. *Suppose that $EQ^2 < \infty$. Then, as $n \rightarrow \infty$,*

$$(2.4) \quad \frac{R_n}{\sqrt{n}} \xrightarrow{d} \beta \mathcal{N},$$

where $\beta^2 = EQ^2 + 2\frac{EQ}{1-EM}E(QM)$ and \mathcal{N} is the standard normal random variable.

Remark: Since $-1 < EM < 1$, by straightforward calculation we see that $ER_n = O(1)$ and $\text{var}(R_n) = \beta^2 n + O(1)$. Thus, (2.4) is equivalent to

$$\frac{R_n - ER_n}{\sqrt{\text{var}(R_n)}} \xrightarrow{d} \mathcal{N}.$$

Proof. Let $\alpha := EM$ and $q := EQ$ so that

$$\beta^2 = EQ^2 + 2\frac{q}{1-\alpha}EQM = \text{var}\left(Q + \frac{q}{1-\alpha}M\right).$$

To prove (2.4) we write

$$(2.5) \quad R_n = \sum_{k=1}^n Q_k \prod_{j=1}^{k-1} M_j = \sum_{k=1}^n (Q_k - q) \prod_{j=1}^{k-1} M_j + q \sum_{k=1}^n \prod_{j=1}^{k-1} M_j.$$

Furthermore, as

$$\sum_{k=1}^n (M_k - \alpha) \prod_{j=1}^{k-1} M_j = -\alpha + \prod_{j=1}^n M_j + (1-\alpha) \sum_{k=2}^n \prod_{j=1}^{k-1} M_j = -1 + \prod_{j=1}^n M_j + (1-\alpha) \sum_{k=1}^n \prod_{j=1}^{k-1} M_j,$$

the second term on the right-hand side of (2.5) is

$$\frac{q}{1-\alpha} \sum_{k=1}^n (M_k - \alpha) \prod_{j=1}^{k-1} M_j - \frac{q}{1-\alpha} \prod_{j=1}^n M_j + \frac{q}{1-\alpha}.$$

Set

$$d_k := (Q_k - q + \frac{q}{1-\alpha}(M_k - \alpha)) \prod_{j=1}^{k-1} M_j, \quad k = 1, \dots, n.$$

Then, by independence of (Q_j, M_j) , $j \geq 1$, (d_k) is a martingale difference sequence with respect to (\mathcal{F}_n) where $\mathcal{F}_k = \sigma(M_1, Q_1, \dots, M_k, Q_k)$, $k \geq 1$. Let E_k denote the conditional expectation given \mathcal{F}_k . Since $M_j^2 = 1$ we have

$$E_{k-1} d_k^2 = E \left(Q - q + \frac{q}{1-\alpha}(M - \alpha) \right)^2 = \beta^2,$$

so that, trivially,

$$\frac{\sum_{k=1}^n E_{k-1} d_k^2}{n} \xrightarrow{P} \beta^2,$$

where “ \xrightarrow{P} ” denotes convergence in probability. Moreover, since the M_k 's are uniformly bounded, for a given $\epsilon > 0$ and n sufficiently large

$$E_{k-1} d_k^2 I_{\{|d_k| > \epsilon \sqrt{n}\}} \leq 4EQ^2 I_{\{|Q| > \epsilon \sqrt{n}/2\}}.$$

Since $EQ^2 < \infty$, the last quantity converges to zero as $n \rightarrow \infty$ by the dominated convergence theorem. This verifies the conditional version of Lindeberg's condition:

$$\forall \epsilon > 0 \quad \frac{\sum_{k=1}^n E_{k-1} d_k^2 I_{\{|d_k| > \epsilon \sqrt{n}\}}}{n} \xrightarrow{P} 0.$$

It follows by the martingale version of the CLT (see e.g. [4, Theorem 35.12]) that

$$\frac{\sum_{k=1}^n d_k}{\sqrt{n}} \xrightarrow{d} \beta \mathcal{N}.$$

Now, by the above manipulations we have

$$\frac{R_n}{\sqrt{n}} = \frac{\sum_{k=1}^n d_k}{\sqrt{n}} + \frac{q}{(1-\alpha)\sqrt{n}} - \frac{q}{(1-\alpha)\sqrt{n}} \prod_{j=1}^n M_j.$$

Since each of the last two terms goes to 0 (deterministically and in probability, respectively), Theorem 1 follows. \square

3. THE CASE $E \ln |M| > 0$, $E|M| > 1$, AND NON-RANDOM $|M|$

In this section we consider the following situation; for a fixed $\rho > 1$ let

$$M \stackrel{d}{=} \rho\epsilon, \quad \text{where} \quad P(\epsilon = 1) = p, \quad P(\epsilon = -1) = 1 - p = q.$$

Theorem 2. *Let M be as above. Assume that $Q \stackrel{d}{=} \epsilon$ and that it is independent of M .*

(i) *Symmetric case: Let $p = 1/2$. Then*

$$\frac{R_n}{\rho^{n-1}} \xrightarrow{d} \sum_{k=1}^{\infty} \lambda^{k-1} \epsilon_k,$$

where $\lambda = \rho^{-1} < 1$ and (ϵ_k) is the sequence of i.i.d. copies of ϵ .

(ii) *Asymmetric case: Suppose $p \neq 1/2$. Then we have*

$$(3.6) \quad \frac{R_n}{\rho^{n-1}} \xrightarrow{d} rX,$$

where r is a Rademacher variable (i.e., $P(r = 1) = P(r = -1) = 1/2$), $X \stackrel{d}{=} \sum_{k=0}^{\infty} \lambda^k \prod_{j=1}^k \epsilon_j$, and r and X are independent.

Remark: The sums of the form $\sum_{k=1}^{\infty} \lambda^{k-1} r_k$ considered in part (i) are well-known objects, usually referred to as ‘‘symmetric Bernoulli convolutions’’. Their properties have been extensively studied since mid 1930s. In particular, it is known that the limiting distribution is uniform on $[-2, 2]$ if $\lambda = 1/2$, it is singular if $0 < \lambda < 1/2$, and is absolutely continuous for almost all (but not all) $\lambda \in (1/2, 1)$. A good description of the current state of knowledge can be found, for example, in [17].

Proof. We have

$$(3.7) \quad R_n = \epsilon_n + \rho\epsilon_n\epsilon_{n-1} + \cdots + \rho^{n-1} \prod_{j=1}^n \epsilon_j \stackrel{d}{=} \sum_{k=1}^n \rho^{k-1} \prod_{j=1}^k \epsilon_j.$$

If $p = 1/2$ then the sequences

$$(3.8) \quad (\epsilon_1, \epsilon_1\epsilon_2, \epsilon_1\epsilon_2\epsilon_3, \dots) \quad \text{and} \quad (\epsilon_1, \epsilon_2, \epsilon_3, \dots)$$

are identically distributed. Thus, if we normalize R_n by ρ^{n-1} we get

$$\frac{R_n}{\rho^{n-1}} \stackrel{d}{=} \sum_{k=1}^n \rho^{k-n} \epsilon_k \stackrel{d}{=} \sum_{k=1}^n \lambda^{k-1} \epsilon_k.$$

Since $0 < \lambda < 1$ the series of partial sums converges almost surely and thus R_n/ρ^{n-1} converges in distribution to the given limit.

If $p \neq 1/2$ then the distributional equality in (3.8) is no longer valid. However, we can write the right-hand side of (3.7) as

$$R_n \stackrel{d}{=} \sum_{k=1}^n \rho^{k-1} \prod_{j=1}^k \epsilon_j = \rho^{n-1} \prod_{j=1}^n \epsilon_j \left(\sum_{k=1}^n \frac{\lambda^{k-1}}{\prod_{j=1}^{k-1} \epsilon_j} \right).$$

Since $\epsilon_k = 1/\epsilon_k$ we get

$$(3.9) \quad \frac{R_n}{\rho^{n-1}} \stackrel{d}{=} T_n \prod_{j=1}^n \epsilon_j,$$

where

$$T_\ell = \sum_{k=1}^{\ell} \lambda^{k-1} \prod_{j=1}^{k-1} \epsilon_j.$$

Fix arbitrary m . Then, for any $n > m$, write the term on the right-hand side of (3.9) as

$$T_m \prod_{j=1}^m \epsilon_j + \prod_{j=1}^m \epsilon_j \left(\sum_{k=m+1}^n \lambda^{k-1} \prod_{j=1}^{k-1} \epsilon_j \right).$$

The second summand is bounded in absolute value by $\lambda^m/(1-\lambda)$ and thus it can be made arbitrarily small by choosing m sufficiently large. Consequently, we consider only the first part of the above expression. Note that T_m depends on $(\epsilon_1, \dots, \epsilon_m)$ only. Therefore upon conditioning on $(\epsilon_1, \dots, \epsilon_m)$ we obtain

$$E \left(e^{itT_m \prod_{j=1}^m \epsilon_j} \right) = E \left[\phi_{n-m} \left(tT_m \prod_{j=1}^m \epsilon_j \right) \right],$$

where ϕ_k is the characteristic function of the product $\prod_{j=1}^k \epsilon_j$. Since $\prod_{j=1}^n \epsilon_j = 1$ if and only if the number of j 's such that $\epsilon_j = -1$ is even, we have

$$P\left(\prod_{j=1}^n \epsilon_j = 1\right) - P\left(\prod_{j=1}^n \epsilon_j = -1\right) = \sum_{\substack{k=0 \\ k\text{-even}}}^n \binom{n}{k} q^k p^{n-k} - \sum_{\substack{k=0 \\ k\text{-odd}}}^n \binom{n}{k} q^k p^{n-k} = (p-q)^n \longrightarrow 0,$$

as $n \rightarrow \infty$. Thus the product $\prod_{j=1}^n \epsilon_j$ converges in distribution to r (recall that its characteristic function is $\cos(t)$). Therefore for arbitrary fixed m

$$E \left(e^{itT_m \prod_{j=1}^m \epsilon_j} \right) \xrightarrow{n \rightarrow \infty} E \cos \left(tT_m \prod_{j=1}^m \epsilon_j \right) = E \cos(tT_m).$$

As $m \rightarrow \infty$ the sum T_m converges almost surely to X . By the Lebesgue dominated convergence it follows therefore that

$$E \cos(tT_m) \xrightarrow{m \rightarrow \infty} E \cos(tX),$$

the latter being the characteristic function of rX , with r and X independent. This proves (3.6). \square

4. THE CASE $E \ln |M| > 0$, $E|M| > 1$, AND RANDOM $|M|$

In this section we assume that $\mu = E \ln |M| > 0$ and that $|M|$ is random, which forces $E|M| > 1$. Set $v^2 := \text{var}(\ln |M|)$. Then we have

Theorem 3. *Let (R_n) be given by (1.2) with the pair (Q, M) satisfying $\mu > 0$, $0 < v^2 < \infty$, and $E \ln |Q| < \infty$. Then, as $n \rightarrow \infty$*

$$\frac{R_n^{v\sqrt{n}}}{\exp\left(\frac{2\mu\sqrt{n}}{v}\right)} \xrightarrow{d} e^{2\mathcal{N}}$$

where \mathcal{N} is the standard normal random variable.

Proof. Write R_n as

$$(4.10) \quad R_n \stackrel{d}{=} \sum_{k=1}^n Q_k \prod_{j=1}^{k-1} M_j = \left(\prod_{j=1}^n M_j \right) \sum_{k=1}^n Q_k \prod_{j=k}^n \frac{1}{M_j}.$$

Consider the first factor. By the classical CLT

$$\frac{(\prod_{j=1}^n M_j)^{\frac{2}{v\sqrt{n}}}}{\exp\left(\frac{2\mu}{v}\sqrt{n}\right)} = \left(\frac{\prod_{j=1}^n M_j^2}{e^{2\mu n}} \right)^{\frac{1}{v\sqrt{n}}} = \exp \left\{ 2 \frac{\sum_{j=1}^n \ln |M_j| - n\mu}{v\sqrt{n}} \right\} \xrightarrow{d} e^{2\mathcal{N}}.$$

To finish the proof by Slutsky's theorem it suffices to show the second factor on the right hand side of (4.10) converges to 1 in distribution (recall that in the case of convergence to constant, convergence in probability and in distribution are equivalent). To this end note that

$$(4.11) \quad \sum_{k=1}^n Q_k \prod_{j=k}^n \frac{1}{M_j} = \sum_{k=1}^n \frac{Q_k}{M_k} \prod_{j=k+1}^n \frac{1}{M_j} \stackrel{d}{=} \sum_{k=1}^n \frac{Q_k}{M_k} \prod_{j=1}^{k-1} \frac{1}{M_j} =: S_n.$$

The expression on the right-hand side is a perpetuity generated by $(Q/M, 1/M)$. Since we are working under the condition $E \ln |M| > 0$, we have $E \ln |1/M| = -E \ln |M| < 0$. Furthermore, by our assumption on Q , $E \ln |Q/M| < \infty$ and thus (S_n) converges in distribution to, say S , (see [20, Theorem 1.6(b)]). Therefore, $S_n^{2/v} \xrightarrow{d} S^{2/v}$. Moreover, $P(1/M = 0) = 0$. Hence, by Theorem 1.3 of [1] (see also [10] and [3, Lemma 2.1]) it follows that the distribution of $S^{2/v}$ is continuous. In particular, it does not have atom at zero, which is all that is important for our purposes. Denote by ν_n and ν the distributions of

$S_n^{2/v}$ and $S_n^{2/v}$, respectively. We want to show that $S_n^{\frac{2}{v\sqrt{n}}} \xrightarrow{d} 1$. Consider first an arbitrary $x \in (0, 1)$, fix an arbitrary m and take any $n > m$. Then

$$P\left(S_n^{\frac{2}{v\sqrt{n}}} \leq x\right) = \nu_n([0, x^{\sqrt{n}}]) \leq \nu_n([0, x^{\sqrt{m}}]) \xrightarrow{n \rightarrow \infty} \nu([0, x^{\sqrt{m}}])$$

Taking now $m \rightarrow \infty$ we conclude that $P\left(S_n^{\frac{2}{v\sqrt{n}}} \leq x\right) \rightarrow 0$ for $x \in (0, 1)$.

Now we take an arbitrary $x \geq 1$. Then again we fix some m . For $n > m$ we obtain

$$P\left(S_n^{\frac{2}{v\sqrt{n}}} \leq x\right) = \nu_n([0, x^{\sqrt{n}}]) \geq \nu_n([0, x^{\sqrt{m}}]) \xrightarrow{n \rightarrow \infty} \nu([0, x^{\sqrt{m}}]) \xrightarrow{m \rightarrow \infty} 1$$

so that $P\left(S_n^{\frac{2}{v\sqrt{n}}} \leq x\right) \rightarrow 1$ for $x \geq 1$ and it follows that $S_n^{\frac{2}{v\sqrt{n}}} \xrightarrow{d} 1$ as desired. \square

Remark: Note that if it is additionally assumed that $M \geq 0$ a.s. then

$$\frac{R_n^{\frac{1}{v\sqrt{n}}}}{\exp\left(\frac{\mu\sqrt{n}}{v}\right)} \xrightarrow{d} e^{\mathcal{N}}.$$

In the general case, i.e. when negative values of M are allowed with positive probability one can still drop the coefficient 2, but then one has to take absolute value, which results in

$$\frac{\left(\sum_{k=1}^n \prod_{j=1}^k |M_j|\right)^{\frac{1}{v\sqrt{n}}}}{\exp\left(\frac{\mu\sqrt{n}}{v}\right)} \xrightarrow{d} e^{\mathcal{N}}.$$

5. THE CASE $E \ln |M| = 0$ AND $E|M| > 1$

To see the difference between current situation and the preceding one, note that because $E \ln |M| = 0$ the perpetuity defined by (4.11) in the course of the proof of Proposition 3 is not guaranteed to converge. Specifically, consider the following example: let X be non-degenerate, integrable, symmetric random variable and set $M = e^X$. Then,

$$E \ln |M| = EX = 0 \quad \text{and} \quad E|M| > e^{EX} = 1,$$

where the strict inequality follows from non-degeneracy of X . By the symmetry of X , M and $1/M$ have *exactly* the same distribution. Hence, if we take $Q = 1$ in (1.2) and (4.11) we find that $R_n \stackrel{d}{=} S_n$ for $n \geq 1$. Therefore, (R_n) and (S_n) converge or diverge in distribution simultaneously. If M is non-negative this difficulty can be handled by factoring the largest product in (1.2) rather than the last one, however, the limiting distribution will no longer be lognormal. It seems very likely that the case of general M is similar, however it remains open.

Before stating our result in the nonnegative case, recall that a function $h : \mathbf{R} \rightarrow \mathbf{R}$ is regularly varying at infinity with index ρ if for every $x > 0$

$$\lim_{t \rightarrow \infty} \frac{h(xt)}{h(t)} = x^\rho.$$

Recall also that any such function can be written as $h(t) = t^\rho \ell(t)$, where $\ell(t)$ is slowly varying at infinity, that is for every $x > 0$

$$\lim_{t \rightarrow \infty} \frac{\ell(xt)}{\ell(t)} = 1.$$

It follows from that definition in particular that for any $\delta > 0$, $t^\delta \ell(t) \rightarrow \infty$ as $t \rightarrow \infty$ (see [5] for these and much more on regularly varying functions). The following holds true:

Theorem 4. *Let (Q, M) be given by (e^Y, e^X) where $EX = 0$ and $v^2 := \text{var}(X)$ satisfies $0 < v^2 < \infty$ (so that $E \ln M = 0$, $EM > 1$, and $v^2 = \text{var}(\ln M) < \infty$).*

(i) *If $EY^2 < \infty$ then, as $n \rightarrow \infty$*

$$(5.12) \quad R_n^{\frac{1}{v\sqrt{n}}} \xrightarrow{d} e^{|\mathcal{N}|},$$

where $|\mathcal{N}|$, the absolute value of the standard normal random variable, has distribution given by

$$P(|\mathcal{N}| \leq x) = 2\Phi(x) - 1 = \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2} dt.$$

(ii) *Let $h(t) := P(Y > t)$ be a tail function of a random variable Y and define a sequence (γ_n) by*

$$\gamma_n := \inf \left\{ t : h(t) \leq \frac{1}{n} \right\}, \quad n \geq 1.$$

If $h(t)$ is regularly varying at infinity with index α for some $-2 < \alpha < 0$ then, as $n \rightarrow \infty$

$$(5.13) \quad R_n^{1/\gamma_n} \xrightarrow{d} e^{V_\alpha},$$

where V_α has Fréchet distribution Φ_α given by

$$P(V_\alpha \leq x) = \Phi_\alpha(x) = \begin{cases} 0, & x \leq 0; \\ \exp(-x^\alpha), & x > 0. \end{cases}$$

(iii) *Assume that $h(t)$ is regularly varying at infinity with index $\alpha = -2$, i.e. that $h(t) = t^{-2}\ell(t)$, where $\ell(t)$ is slowly varying at infinity. If $\lim_{t \rightarrow \infty} \ell(t) = \infty$ then (5.13) holds, while if $\lim_{t \rightarrow \infty} \ell(t) = 0$ then (5.12) holds.*

Proof. Set

$$W_n := \max_{1 \leq k \leq n} Q_k \prod_{j=1}^{k-1} M_j,$$

and write

$$(5.14) \quad R_n^{1/\sqrt{nv^2}} = W_n^{1/\sqrt{nv^2}} \left(\sum_{k=1}^n \frac{Q_k \prod_{j=1}^{k-1} M_j}{W_n} \right)^{1/\sqrt{nv^2}}.$$

Since $x \rightarrow \ln x$ is increasing

$$W_n^{1/\sqrt{nv^2}} = \exp \left\{ \frac{1}{v\sqrt{n}} \ln \left(\max_{1 \leq k \leq n} Q_k \prod_{j=1}^{k-1} M_j \right) \right\} = \exp \left\{ \frac{\max_{1 \leq k \leq n} \left\{ \ln Q_k + \sum_{j=1}^{k-1} \ln M_j \right\}}{v\sqrt{n}} \right\}.$$

We let $Y_k = \ln Q_k$, $X_k = \ln M_k$, $S_k = \sum_{j=1}^k X_j$, and for any sequence of random variables (Z_k) we will write $Z_m^* = \max_{1 \leq k \leq m} Z_k$.

Subadditivity of maxima implies that for any numerical sequences $(u_k), (w_k)$

$$(5.15) \quad \max_{1 \leq k \leq n} \{u_k\} - \max_{1 \leq k \leq n} \{|w_k|\} \leq \max_{1 \leq k \leq n} \{u_k + w_k\} \leq \max_{1 \leq k \leq n} \{u_k\} + \max_{1 \leq k \leq n} \{w_k\}.$$

To prove the first part of Theorem 4 we note that

$$\frac{|Y_n|^*}{\sqrt{n}} \xrightarrow{P} 0.$$

Indeed, for $\epsilon > 0$ we have

$$P \left(\frac{|Y_n|^*}{\sqrt{n}} > \epsilon \right) = 1 - (1 - P(|Y| > \epsilon\sqrt{n}))^n \leq nP(Y^2 > \epsilon^2 n) \rightarrow 0, \quad n \rightarrow \infty,$$

where the last assertion follows from $EY^2 < \infty$. Using this and (5.15) with $u_k = S_{k-1}$ and $w_k = Y_k$ we obtain that

$$\frac{\max_{1 \leq k \leq n} \{Y_k + S_{k-1}\}}{v\sqrt{n}} = \frac{S_{n-1}^*}{v\sqrt{n}} + o_P(1),$$

where $o_P(1)$ denotes a quantity that goes to zero in probability. Furthermore, our assumptions on M_j 's imply that $(\sum_{j=1}^k X_j)$ is a sequence of partial sums of random walk whose increments X_j have mean zero and a finite variance v^2 . Thus, Erdős–Kac Theorem for the maxima of random walks (see e.g. [12, Theorem 12.2]) implies that

$$W_n^{1/\sqrt{nv^2}} \xrightarrow{d} e^{|\mathcal{N}|}.$$

Just as in the proof of Proposition 3, to complete the proof we need to show that the second factor in (5.14) converges to 1 in distribution. But that is clear since on one hand,

$$\left(\sum_{k=1}^n \frac{Q_k \prod_{j=1}^{k-1} M_j}{W_n} \right)^{1/\sqrt{nv^2}} \leq n^{1/\sqrt{nv^2}} = \exp \left(\frac{\ln n}{v\sqrt{n}} \right) \rightarrow 1,$$

and, on the other hand, we clearly have

$$\sum_{k=1}^n \frac{Q_k \prod_{j=1}^{k-1} M_j}{W_n} \geq 1.$$

This implies that

$$\sum_{k=1}^n \frac{Q_k \prod_{j=1}^{k-1} M_j}{W_n} \xrightarrow{P} 1,$$

and proves the first part of Theorem 4.

The argument for the second part is parallel to the one just given with the following adjustment: in the first part, the assumption $EY^2 < \infty$ ensures that the maximum of random walk with increments X_j , $j < n$, dominates the maximum of $\{Y_k, k \leq n\}$. If this assumption is weakened this may no longer be true, and, in fact, the maximum of Y_k 's may dominate. In that case, we can just use basics of extreme value theory (Chapter 1 of [16] being more than enough) instead of Erdős–Kac theorem to complete the argument. This time, using (5.15) we write

$$Y_n^* - |S_{n-1}|^* \leq \max_{1 \leq k \leq n} Y_k + \sum_{j=1}^{k-1} X_j \leq Y_n^* + S_{n-1}^*.$$

By the characterization theorem in the extreme value theory (see e.g. [16, Theorem 1.6.2]), our assumption on Y is a necessary and sufficient condition for the existence of constants (a_n) , (b_n) for which

$$a_n(Y_n^* - b_n) \xrightarrow{d} V_\alpha,$$

where V_α has Fréchet distribution (also referred to as type II extreme value distribution) as described above. Furthermore, (see [16, Corollary 1.6.3]) we may take $a_n = 1/\gamma_n$ and $b_n = 0$, $n \geq 1$, and if we do we obtain that

$$\frac{Y_n^*}{\gamma_n} \xrightarrow{d} V_\alpha, \quad n \rightarrow \infty.$$

We now observe that $\gamma_n/\sqrt{n} \rightarrow \infty$. In fact, there exists $\beta > 1/2$ such that $\gamma_n \geq n^\beta$ for all sufficiently large n . Indeed, since h is decreasing it is enough to see that $h(n^\beta) > 1/n$. But as h is regularly varying, we have

$$h(n^\beta) = n^{\beta\alpha} \ell(n^\beta) = \frac{n^{1+\beta\alpha} \ell(n^\beta)}{n} = \frac{(n^\beta)^{\frac{1}{\beta} + \alpha} \ell(n^\beta)}{n} = \frac{(n^\beta)^\delta \ell(n^\beta)}{n}.$$

If we now take $1/2 < \beta < -1/\alpha$ (which is possible since $-2 < \alpha < 0$) then $\delta > 0$ and the numerator on the right-hand side goes to infinity with n , proving the claim that $h(n^\beta) > 1/n$ for large n . We now get

$$\frac{|S_{n-1}|^*}{\gamma_n} = \frac{|S_{n-1}|^* \sqrt{n}}{\sqrt{n} \gamma_n} \xrightarrow{P} 0,$$

and also, since $\gamma_n/\sqrt{n} \rightarrow \infty$,

$$1 \leq \left(\sum_{k=1}^n \frac{Q_k \prod_{j=1}^{k-1} M_j}{W_n} \right)^{1/\gamma_n} \leq n^{1/\gamma_n} \rightarrow 1,$$

which proves the second part.

Finally, the last part follows by essentially the same reasoning. Assume $\ell(t) \rightarrow \infty$ as $t \rightarrow \infty$. By the just given argument, to establish (5.13), it suffices to verify $\gamma_n/\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$. Assume this is not the case. Then there exist $C < \infty$ and an infinite subsequence (n_k) such that $\gamma_{n_k}/\sqrt{n_k} \leq C$ for all $k \geq 1$. By the definition of (γ_n) this means that $P(Y \geq C\sqrt{n_k}) \leq 1/n_k$, i.e. that $\ell(C\sqrt{n_k}) \leq C^2$. But that contradicts the assumption that $\ell(t) \rightarrow \infty$ as $t \rightarrow \infty$. If, on the other hand, $\ell(t) \rightarrow 0$ as $t \rightarrow \infty$, then $\gamma_n/\sqrt{n} \rightarrow 0$ (for otherwise there would exist $c > 0$ and a subsequence (n_k) such that $\gamma_{n_k}/\sqrt{n_k} \geq c$, $k \geq 1$ implying that $\ell(c\sqrt{n_k}) \geq c^2$ and contradicting $\ell(t) \rightarrow 0$ as $t \rightarrow \infty$). Now, it follows from the proof of the first part that $\gamma_n/\sqrt{n} \rightarrow 0$ is enough to conclude (5.12). The proof of part (iii) is completed. \square

Remark. The only case not covered by the above theorem is $\alpha = -2$ and $\ell(t) \sim \text{const}$. We suspect that in that case we have

$$\frac{\max_{1 \leq k \leq n} \{Y_k + S_{k-1}\}}{\sqrt{n}} \xrightarrow{d} V_\alpha + \sigma \mathcal{N},$$

where V_α is a Fréchet random variable with parameter $\alpha = -2$, σ is the variance of X , and V_α and \mathcal{N} are independent, but we have not managed to prove it.

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