

Random Partitions with non negative r^{th} differences

Rod Canfield^a, Sylvie Corteel^b, Pawel Hitczenko^c

^a *Dpt of Computer Science, UGA, Athens GA 30602, USA;* ^b *PRISM, UVSQ, 45 Av. des États Unis, 78035 Versailles, France and LaCIM, UQAM, C.P. 8888, Montréal, QC H3C 3P8, Canada;* ^c *Dpt of Maths and Computer Science, Drexel, Philadelphia, PA 19104, USA*

E-mail: erc@cs.uga.edu, corteel@math.uqam.ca, phitczen@mcs.drexel.edu

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Let $P_r(n)$ be the set of partitions of n with non negative r^{th} differences. Let λ be a partition of an integer n chosen uniformly at random among the set $P_r(n)$. Let $d(\lambda)$ be a positive r^{th} difference chosen uniformly at random in λ . The aim of this work is to show that for every $m \geq 1$, the probability that $d(\lambda) \geq m$ approaches the constant $m^{-1/r}$ as $n \rightarrow \infty$. This work is a generalization of a result on integer partitions [7] and was motivated by a recent identity by Andrews, Paule and Riese's Omega package [3]. To prove this result we use bijective, asymptotic/analytic and probabilistic combinatorics.

1. INTRODUCTION

A partition λ of n is a sequence of integers

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \text{ with } \lambda_1 \geq \lambda_2 \dots \geq \lambda_k \geq 1 \text{ and } \sum_{j=1}^k \lambda_j = n.$$

The same partition λ can also be written in its frequential notation, that is :

$$\lambda = n^{m_n} (n-1)^{m_{n-1}} \dots 1^{m_1} \text{ with } m_j = |\{i \mid \lambda_i = j\}|, 1 \leq j \leq n.$$

The number m_j is called the *multiplicity* of the part j in λ . We will use both representations. The *part sizes* of a partition λ are the indices j such that m_j is positive. The number of part sizes can therefore be defined as : $|\{j \mid m_j > 0\}|$. The *conjugate* of a partition λ is the partition $\lambda' = (\lambda'_1, \dots)$ where $\lambda'_i = |\{j \mid \lambda_j \geq i\}|$, $1 \leq i \leq \lambda_1$. We now define the r^{th} differences. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n and $\Delta^r(\lambda) = (\Delta_1^r(\lambda), \dots, \Delta_k^r(\lambda))$ be its r^{th} differences. The r^{th} differences can

be computed by the following recurrence :

$$\Delta_i^r(\lambda) = \begin{cases} \lambda_i & \text{if } i = k \text{ or } r = 0 \\ \Delta_i^{r-1}(\lambda) - \Delta_{i+1}^{r-1}(\lambda) & \text{otherwise} \end{cases}$$

In what follows we will write $\Delta_i^r(\lambda) : \Delta_i^r$ for short. Let P_r be the set of partitions with non negative r^{th} differences, that is to say, $(\lambda_1, \lambda_2, \dots, \lambda_k) \in P_r$ if and only if $\Delta_i^r \geq 0$ for $1 < i < k$. Let $P_r(n)$ be the set of partitions $\lambda \in P_r$ with $|\lambda| = n$ and let $p_r(n) = |P_r(n)|$.

This work was motivated by two previous results. The first result is an identity on partitions with non negative r^{th} differences. It was discovered by Andrews, Paule and Riese's Omega package. Let $F_r(n)$ be the set of partitions of n into parts in the set $S_r = \left\{ \binom{i+r}{r}, i \geq 0 \right\}$.

THEOREM 1.1. [2, 3] *There is a one-to-one correspondence between the partitions in $P_r(n)$ and the partitions in $F_r(n)$.*

The second result is on ordinary integer partitions (partitions with non negative 1^{st} differences) :

THEOREM 1.2. [7] *Let m be a fixed positive integer. Let λ be a partition of an integer n chosen uniformly at random among the set of all partitions of n . Let $s(\lambda)$ be a part size chosen uniformly at random from the set of all part sizes that occur in λ . The probability that the multiplicity of $s(\lambda)$ is equal to m approaches the constant $1/(m(m+1))$ as $n \rightarrow \infty$.*

This result can easily be restated in the following way :

COROLLARY 1.1. *Let m be a fixed positive integer. Let λ be a partition of an integer n chosen uniformly at random among the set of all partitions of n . Let $s(\lambda)$ be a part size chosen uniformly at random from the set of all part sizes that occur in λ . The probability that the multiplicity of $s(\lambda)$ is greater than or equal to m approaches the constant $1/m$ as $n \rightarrow \infty$.*

Note that if $r = 1$ there exists a one-to-one correspondence between the number of positive 1^{st} differences in any given partition and the number of part sizes of its conjugate λ' . For example if $\lambda = (7, 7, 6, 3, 2, 2, 2)$ then the 1^{st} differences are $\Delta^1(\lambda) = (0, 1, 3, 1, 0, 0, 2)$. Hence λ has four positive 1^{st} differences. The conjugate λ' is $(7, 7, 4, 3, 3, 3, 2)$ and has four part sizes: 7,4,3,2. Moreover this correspondance can be refined. It is easy to see that the i^{th} 1^{st} difference (i.e. $\lambda_i - \lambda_{i+1}$) is equal to the multiplicity of the part i in λ' . If we use again our example we have $\Delta_7^1(\lambda) = 2$ which is the multiplicity of the part 7 in λ' . Therefore Corollary 1.1 is equivalent to :

COROLLARY 1.2. *Let m be a fixed positive integer. Let λ be a partition of an integer n chosen uniformly at random among the set of all partitions of n . Let $d(\lambda)$ be a 1st difference chosen uniformly at random from the set of all positive 1st differences that occur in λ . The probability that $d(\lambda)$ is greater than or equal to m approaches the constant $1/m$ as $n \rightarrow \infty$.*

Our aim is therefore to generalize the result of Corollary 1.2 by using the identity of Theorem 1.1.

Let us now state our generalization :

THEOREM 1.3. *Let m and r be fixed positive integers. Let λ be a partition of an integer n chosen uniformly at random among the set $P_r(n)$ of all partitions of n with non negative r^{th} differences. Let $d(\lambda)$ be a positive r^{th} difference chosen uniformly at random from the set of all positive r^{th} differences that occur in λ . The probability that $d(\lambda) \geq m$ approaches the constant $m^{-1/r}$ as $n \rightarrow \infty$.*

The purpose of this paper is to prove Theorem 1.3. We now present the organization of the paper. We will first give in Section 2 a simple bijection of the identity of Theorem 1.1 that gives several refinements of the identity. This bijection was advertised/announced by Zeilberger in his very own journal [6]. We then present in Section 3 some asymptotic results on the number of partitions in $P_r(n)$ and some asymptotics results on the average number of r^{th} differences greater or equal to m in the partitions in $P_r(n)$. Finally in Section 4 we use some probabilistic arguments which generalize the works on ordinary partitions [8, 7]. The association of the three parts gives us the proof of our result. We conclude the paper by presenting some future work in Section 5.

2. BIJECTIVE COMBINATORICS

In this section we are going to present a bijection f between the partitions in $P_r(n)$ and the partitions of n into parts $\binom{i+r}{r}$, $i \geq 0$. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition in $P_r(n)$ then its image by the bijection f in its frequential notation is :

$$f(\lambda) = \binom{k-1+r}{r}^{\Delta_k^r} \binom{k-2+r}{r}^{\Delta_{k-1}^r} \dots \binom{r}{r}^{\Delta_1^r}$$

where the Δ_i^r ($1 \leq i \leq k$) are the r^{th} differences of λ . It is clear that $f(\lambda)$ is a partition into parts $\binom{r+i}{r}$ with $i \geq 0$. Now let us prove that $f(\lambda)$ is a

partition of n . From the definition of $\Delta^r(\lambda)$, it is easy to see that

$$\lambda_i = \sum_{j=i}^k \binom{r+j-i-1}{r-1} \Delta_j^r.$$

As $\binom{r+i-1}{r} = \sum_{j=0}^{i-1} \binom{r+j-1}{r-1}$, we have $|f(\lambda)| = n$. We can reconstruct λ from $\Delta^r(\lambda)$. The reverse mapping f^{-1} is then easy to define. Let μ be a partition of n into parts $\binom{r+i}{r}$ with $\mu_1 = \binom{k+r-1}{r}$. Let $\mu^{(i)}$ be the multiplicity of the part $\binom{i-1+r}{r}$ in μ , $1 \leq i \leq k$. Then

$$f^{-1}(\mu) = \left(\sum_{j=1}^k \binom{r+k-j-1}{r-1} \mu^{(j)}, \sum_{j=2}^k \binom{r+k-j-1}{r-1} \mu^{(j)}, \dots, \binom{r-1}{r-1} \mu^{(k)} \right).$$

It is then easy to see that f is a bijection. Note that if $r = 1$ then $f(\lambda) = \lambda'$, the conjugate of λ .

As we said in the introduction, this bijection gives some refinements of the identity. Let us now present these refinements :

THEOREM 2.1. *There is a one-to-one correspondance between the partitions in $P_r(n)$ with k parts and j positive r^{th} differences and the partitions of n into parts $\binom{i+r}{r}$, $i \geq 0$, whose largest part is $\binom{k-1+r}{r}$ and with j parts sizes.*

Proof. Straightforward with the bijection. \square

We even get more than the previous theorem. The bijection is indeed a bijection between a partition in $P_r(n)$ whose r^{th} differences are $(\Delta_1^r, \Delta_2^r, \dots, \Delta_k^r)$ and a partition in $E_r(n)$ where each part of size $\binom{i+r}{r}$ appears Δ_i^r times, $1 \leq i \leq k$.

Let us now illustrate these refinements with generating functions and a recurrence. Let $p_r(n, k, m)$ be the number of partitions in $P_r(n)$ with k parts and $\sum_{i=1}^k \Delta_i^r = m$. Then

$$\sum_{m, n, k \geq 0} p_r(n, k, m) q^n y^k x^m = 1 + \sum_{k \geq 1} \frac{y^k x q^{\binom{k-1+r}{r}}}{(1 - x q^{\binom{r}{r}})(1 - x q^{\binom{r+1}{r}}) \dots (1 - x q^{\binom{r+k-1}{r}})}$$

Let $p_{r \leq}(n, k, m)$ be the number of partitions in $P_r(n)$ with at most k parts and $\sum_{i=1}^k \Delta_i^r = m$, then

$$\sum_{m, n \geq 0} p_{r \leq}(n, k, m) q^n x^m = \prod_{i=0}^k \left(1 - x q^{\binom{i+r}{r}}\right)^{-1}.$$

Let $p_{r \leq}(n, k)$ be the number of partitions in $P_r(n)$ with at most k parts. We can get an easy recurrence to compute this number :

$$p_{r \leq}(n, k) = \begin{cases} 0 & \text{if } n < 0 \text{ or } k = 0 \text{ and } n > 0 \\ 1 & \text{if } n = 0 \text{ and } k = 0 \\ p_{r \leq}(n - \binom{r+k-1}{r}, k) + p_{r \leq}(n, k-1) & \text{otherwise.} \end{cases}$$

To conclude this section let us state what remains to prove Theorem 1.3 :

THEOREM 2.2. *Let m and r be fixed positive integers. Let λ be a partition of an integer n chosen uniformly at random among the set $F_r(n)$. Let $s(\lambda)$ be a part size chosen uniformly at random from the set of all part sizes that occur in λ . The probability that the multiplicity of $s(\lambda) \geq m$ approaches the constant $m^{-1/r}$ as $n \rightarrow \infty$.*

That will be the purpose of the next two sections.

3. ASYMPTOTIC COMBINATORICS

In this section we develop an asymptotic formula for $p_S(n)$, the number of partitions of the integer n whose parts lie in a set S which is the image of a given polynomial. As a special case, we have for our $p_r(n)$:

THEOREM 3.1. *As $n \rightarrow \infty$, we have:*

$$p_r(n) \sim cn^{-(r+1)/2} \exp\{C(1+r)n^{\frac{1}{1+r}}\} \quad (1)$$

where the constants C and c are given by

$$C = \{r^{1/r} r^{-1} \zeta(1+r^{-1}) \Gamma(1+r^{-1})\}^{\frac{r}{r+1}} \quad (2)$$

and

$$c = \left(\frac{C}{2\pi}\right)^{\frac{r+1}{2}} (r!)^{-r/2} (1+r^{-1})^{-1/2} \prod_{j=0}^{r-1} j!. \quad (3)$$

To prove Theorem 3.1 we can take $P(x) = \binom{x+r-1}{r}$ in Theorem 3.2 below. It is seen that $d = r$, $A = 1/r!$, $B = 1/(2(r-2)!)$, and $\rho_j = j-1$. We now state and prove the more general result.

THEOREM 3.2. *Let $P(x) = Ax^d + Bx^{d-1} + \dots$ be a polynomial of degree d which is positive and increasing for $x \geq 1$, and which assumes integral*

values for integral x . Let S be the set $\{P(1), P(2), \dots\}$, and assume that the set S has gcd 1. Let ρ_j be the negatives of the roots of $P(x)$, so that

$$P(x) = A \prod_{j=1}^d (x + \rho_j)$$

(in particular, $B/A = \sum \rho_j$). Then,

$$p_S(n) = c n^{-\kappa - \frac{1}{2}} \exp\{(1+d)Cn^{\frac{1}{d+1}} + o(1)\},$$

where

$$\begin{aligned} \kappa &= \frac{Ad + B}{A(d+1)} \\ C &= \{A^{-1/d} d^{-1} \zeta(1+d^{-1}) \Gamma(1+d^{-1})\}^{\frac{d}{d+1}} \end{aligned}$$

and

$$c = \prod_{j=1}^d \Gamma(1 + \rho_j) C^{1+B/A} A^{\frac{1}{2}+B/A} (1+d^{-1})^{-1/2} (2\pi)^{-(d+1)/2} \quad (4)$$

Proof. Let $N(u)$ be the counting function associated with the set S :

$$N(u) = \#\{h \in S : h \leq u\}.$$

Under the assumptions that

$$N(u) = Eu^\beta + R(u)$$

with $E, \beta > 0$ and

$$\int_0^u \frac{R(v)}{v} dv = b_1 \log u + b_2 + o(1), \quad u \rightarrow \infty$$

Ingham [10] has shown that for $n \rightarrow \infty$

$$\frac{\hat{P}(n) - \hat{P}(n-h)}{h} \sim \left(\frac{1-\alpha}{2\pi}\right)^{1/2} e^{b_2} M^{-(b_1-1/2)\alpha} n^{(b_1-1/2)(1-\alpha)-1/2} e^{\alpha^{-1}(Mn)^\alpha},$$

for each fixed $h \in S$, where $\alpha = \beta/(1 + \beta)$,

$$M = \{E\beta\Gamma(1 + \beta)\zeta(1 + \beta)\}^{1/\beta},$$

and

$$\hat{P}(n) = \sum_{j \leq n} p_S(j).$$

Observe that the right side of Ingham's formula is independent of h ; we shall denote it $G(n)$. Observe further that for any fixed k we have $G(n - k) \sim G(n)$; hence, for any fixed integer k and fixed $h \in S$,

$$p(n - k) + p(n - k - 1) + \cdots + p(n - k - h + 1) \sim hG(n). \quad (5)$$

Because the gcd of the set S is by assumption 1, there must be two finite, disjoint subsets $S', S'' \subseteq S$, and for each $h \in S' \cup S''$ a positive integer r_h such that

$$\sum_{h \in S'} r_h h = 1 + \sum_{h \in S''} r_h h.$$

Let $H = \sum_{h \in S''} r_h h$. If we divide the integers in the interval $[n - H, n - 1]$ (there are H such integers) into $\sum_{h \in S''} r_h$ disjoint subintervals, there being r_h subintervals of size h , for $h \in S''$, and we apply (5) to each of these, we obtain

$$\sum_{j=1}^H p_S(n - j) \sim \left(\sum_{h \in S''} r_h h \right) G(n), \quad n \rightarrow \infty.$$

Similarly, we may divide the integers in the interval $[n - H, n]$ (there are $H + 1$ such integers) into $\sum_{h \in S'} r_h$ disjoint subintervals, there being r_h subintervals of size h , for $h \in S'$, and find

$$\sum_{j=0}^H p_S(n - j) \sim \left(\sum_{h \in S'} r_h h \right) G(n), \quad n \rightarrow \infty.$$

Subtracting,

$$p_S(n) \sim G(n).$$

It may have been an oversight on Ingham's part not to notice that his formula gave an asymptotic for $p_S(n)$ in the typical situation that S has gcd 1. This was also overlooked by some later writers who reference Ingham's Tauberian theorem, but noticed, and explained somewhat differently than we have, in [4].

It remains only for us to determine the constants E , β , b_1 , and b_2 for the case that $S = \{P(1), P(2), \dots\}$. Our theorem then follows by using Ingham's $G(n)$. Given that the polynomial P is positive and increasing, we have

$$N(P(x)) = [x],$$

the brackets on the right denoting greatest integer. Hence, letting x_* denote the unique real solution of $P(x_*) = u$,

$$\begin{aligned} \int_0^u \frac{N(v)}{v} dv &= \int_{\lambda_1}^u \frac{N(v)}{v} dv \\ &= \int_1^{x_*} \frac{[x]P'(x)}{P(x)} dx \\ &= \int_1^{x_*} \frac{xP'(x)}{P(x)} dx - \frac{1}{2} \int_1^{x_*} \frac{P'(x)}{P(x)} dx - \int_1^{x_*} \frac{(x - [x] - 1/2)P'(x)}{P(x)} dx \\ &= I_1 - \frac{1}{2} \log P(x_*) + \frac{1}{2} \log P(1) - I_2. \end{aligned}$$

For the integral I_1 we integrate $x/(x + \rho_j)$, and sum on j to find

$$I_1 = d(x_* - 1) - \sum_{j=1}^d \rho_j \log(x_* + \rho_j) + \sum_{j=1}^d \rho_j \log(1 + \rho_j).$$

Using the estimates, valid for $u \rightarrow \infty$,

$$dx_* = d(u/A)^{1/d} - \sum_{j=1}^d \rho_j + o(1)$$

$$\log(x_* + \rho_j) = \log(x_*) + o(1) = \frac{1}{d} \log(u/A) + o(1),$$

we then find

$$I_1 = d(u/A)^{1/d} - \sum_{j=1}^d \rho_j - d - \frac{1}{d} \log(u/A) \sum_{j=1}^d \rho_j + \sum_{j=1}^d \rho_j \log(1 + \rho_j) + o(1).$$

According to the Euler-Maclaurin summation formula,

$$- \int_1^N \frac{x - [x] - 1/2}{x + \rho} dx = \int_1^N \log(x + \rho) dx - \sum_{k=1}^N \log(k + \rho) + \frac{1}{2} (\log(1 + \rho) + \log(N + \rho))$$

We can sum $\log(k + \rho)$ over k with the Gamma function,

$$\sum_{k=1}^N \log(k + \rho) = \log \Gamma(N + 1 + \rho) - \log \Gamma(1 + \rho),$$

and Stirling's formula,

$$\log \Gamma(N + 1 + \rho) = (N + \rho) \log(N) - N + \frac{1}{2} \log(2\pi N) + o(1).$$

Using this and the definite integral

$$\int_1^N \log(x + \rho) dx = (N + \rho) \log(N + \rho) - N + 1 - (1 + \rho) \log(1 + \rho),$$

we conclude, for $N \rightarrow \infty$,

$$- \int_1^N \frac{x - [x] - 1/2}{x + \rho} dx = 1 + \rho - \frac{1}{2} \log(2\pi) + \log \Gamma(1 + \rho) - \left(\frac{1}{2} + \rho\right) \log(1 + \rho) + o(1).$$

Summing over the negative roots ρ_j of $P(x)$, we have

$$-I_2 = d + \sum \rho_j - \frac{d}{2} \log(2\pi) + \sum \log \Gamma(1 + \rho_j) - \frac{1}{2} \log \frac{P(1)}{A} - \sum \rho_j \log(1 + \rho_j) + o(1).$$

Hence, replacing $P(x_*)$ by u and substituting for $I_1 - I_2$,

$$\int_0^u \frac{N(v)}{v} dv = d(u/A)^{1/d} - \frac{B}{Ad} \log(u/A) - \frac{1}{2} \log(u) - \frac{d}{2} \log(2\pi) + \sum \log \Gamma(1 + \rho_j) + o(1).$$

From this we can read off, directly,

$$\begin{aligned} E &= A^{-1/d} \\ \beta &= 1/d \\ b_1 &= -\frac{1}{2} - \frac{B}{Ad} \\ b_2 &= \left(\frac{1}{2} + \frac{B}{Ad}\right) \log(A) - \frac{d}{2} \log(2\pi) + \sum \log \Gamma(1 + \rho_j). \end{aligned}$$

The proof is then completed, as explained earlier, by substituting these four values into Ingham's formula. \square

Next, we use this result on $p_r(n)$ to compute the asymptotic behavior of the average number of positive r^{th} differences of the partitions in $P_r(n)$. This number $\delta_r(n)$ can be defined as follows for any n : $\delta_r(n) = \frac{1}{p_r(n)} \sum_{\lambda \in P_r(n)} |\{i \mid \Delta_i^r(\lambda) > 0\}|$. Thanks to the bijection it is straightforward to compute this value for any n and indeed :

$$\delta_r(n) = \frac{1}{p_r(n)} \sum_{i \geq 0} p_r \left(n - \binom{r+i}{r} \right).$$

PROPOSITION 3.1. *For suitable constant A*

$$\delta_r(n) \sim A n^{\frac{1}{1+r}} \tag{6}$$

namely, A is given by

$$A = \Gamma(1 + r^{-1}) r^{1/r} C^{-1/r}$$

where the constant C is given in (2).

Proof. Since

$$(n - K)^{\frac{1}{r+1}} = n^{\frac{1}{r+1}} - \frac{1}{r+1} K n^{\frac{1}{r+1}-1} + o(1),$$

uniformly for $K = o(n^{1-\frac{1}{2(r+1)}})$, we may use our asymptotic formula for $p_r(n)$ to conclude

$$\frac{p_r(n - K)}{p_r(n)} = \exp\{-C K n^{\frac{1}{r+1}-1}\},$$

as $n \rightarrow \infty$, uniformly for K restricted as above. For K we shall take $\binom{r+j-1}{r}$, $j \geq 1$; those terms for which K is out of range contribute negligibly to the sum; there results:

$$\delta_r(n) = \sum_{j \geq 1} \exp\{-C_n \binom{r+j-1}{r}\} + o(1),$$

where $C_n = C n^{\frac{1}{r+1}-1}$. We use Mellin's formula

$$e^{-y} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} y^{-s} \Gamma(s) ds,$$

which is valid for $\text{Re}(y) > 0$ and $a > 0$, with y replaced by $C_n \binom{r+j-1}{r}$. This gives

$$\delta_r(n) \sim \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} C_n^{-s} D(s) \Gamma(s) ds,$$

where $D(s)$ is the Dirichlet series

$$D(s) = \sum_{j \geq 1} \binom{r+j-1}{r}^{-s}.$$

By using

$$\{(r+j-1)(r+j-2)\cdots j\}^{-s} - j^{-rs} = (-s) \int_{j^r}^{(r+j-1)^r} u^{-s-1} du,$$

and some crude estimates, we can prove that $D(s)$ is equal to $(r!)^s \zeta(rs)$ plus a function analytic at least in the domain $\text{Re}(s) > 0$. Thus, $\delta_r(n)$ equals asymptotically the residue of $C_n^{-s} (r!)^s \zeta(rs) \Gamma(s)$ at $s = 1/r$. Since

$$\zeta(rs) = \frac{1}{rs-1} + \dots,$$

the latter residue is $C_n^{-1/r} (r!)^{1/r} r^{-1} \Gamma(r^{-1})$, and we arrive at the stated formula. \square

Let us now compute the asymptotic behavior of the average number of r^{th} differences greater or equal to m in the partitions in $P_r(n)$. This number $\delta_{r,m}(n)$ can be defined as follows for any n : $\frac{1}{p_r(n)} \sum_{\lambda \in P_r(n)} |\{i \mid \Delta_i^r(\lambda) \geq m\}|$. Thanks to the bijection it is straightforward to compute this value for any n and $m \geq 1$:

$$\delta_{r,m}(n) = \frac{1}{p_r(n)} \sum_{i > 0} p_r \left(n - m \binom{r+i}{r} \right).$$

PROPOSITION 3.2.

$$\delta_{r,m}(n) \sim m^{-1/r} A n^{\frac{1}{1+r}}$$

Proof. Similar to the proof of Proposition 1. \square

A straightforward consequence of Propositions 3.1 and 3.2 is then :

PROPOSITION 3.3.

$$\lim_{n \rightarrow \infty} \frac{\delta_{r,m}(n)}{\delta_r(n)} = \frac{1}{m^{1/r}}$$

4. PROBABILISTIC COMBINATORICS

In this section we will complete the proof of Theorem 1.3. In order to avoid unnecessary distractions, throughout this section we let K denote generic constant whose value may change from one use to the next. These various values may depend on r , but are completely irrelevant from the point of view of the asymptotic results.

The probability in question is the average value of the ratio

$$\mathbb{E} \frac{D_{n,r,m}}{D_{n,r}}, \quad (7)$$

where $D_{n,r,m} = \sum_k I(\Delta_k^r \geq m)$ and $D_{n,r} = D_{n,r,1}$. To compute this average we find it convenient to consider $F_r(n)$, the image of $P_r(n)$ under the bijection f . The explicit form of that bijection tells us that Δ_k^r 's are multiplicities of parts in the partitions of n whose part sizes are in the set $S_r = \{\binom{i+r}{r}, i = 0, 1, \dots\}$. Thus (7) asks for the asymptotic value of the probability that a randomly chosen part size in a random partition of n into parts in the set S_r has multiplicity at least m . To compute this probability we will use arguments which are the generalization of the work on ordinary partitions [7, 8]. Let \mathbf{Q} be the uniform probability measure on the set $P(n)$ of all partitions of n and \mathbf{Pr} the uniform probability measure on $F_r(n)$. Since $F_r(n) \subset P(n)$ the measure \mathbf{Pr} is a restriction of \mathbf{Q} to $F_r(n)$ or, in other words it is the conditional measure on *all* partitions given that a partition is in $F_r(n)$. That is, for any subset A of $F_r(n)$

$$\mathbf{Pr}(\lambda_{(r)} \in A) = \mathbf{Q}(\lambda \in A | \lambda \in F_r(n)) = \frac{\mathbf{Q}(\lambda \in A \cap \lambda \in F_r(n))}{\mathbf{Q}(\lambda \in F_r(n))},$$

where $\lambda_{(r)}$ signifies that a partition λ is considered as an element of $F_r(n)$. We are going to use the result of Fristedt [8], who proved that if $\{\Gamma_k : k \geq 1\}$ are independent geometric random variables with the parameters $1 - q^k$, respectively, defined on a probability space with the measure \mathbf{P} , then, regardless of the value of q , the joint distribution of multiplicities of parts (m_1, m_2, \dots) in a randomly chosen partition of n is equal to that of

$(\Gamma_1, \Gamma_2, \dots)$ conditioned on the event $\{\sum_j j\Gamma_j = n\}$. Hence,

$$\begin{aligned} \mathbf{Q}(\lambda \in F_r(n)) &= \mathbf{P}((\Gamma_j) \in F_r(n) \mid \sum_j j\Gamma_j = n) \\ &= \frac{\mathbf{P}(\{\Gamma_j = 0, j \neq \binom{\ell+r}{r}\} \cap \{\sum_j j\Gamma_j = n\})}{\mathbf{P}(\sum_j j\Gamma_j = n)} \\ &= \frac{\mathbf{P}(\{\Gamma_j = 0, j \neq \binom{\ell+r}{r}\} \cap \{\sum_\ell \binom{\ell+r}{r} \Gamma_{\binom{\ell+r}{r}} = n\})}{\mathbf{P}(\sum_j j\Gamma_j = n)} \\ &= \frac{\mathbf{P}(\Gamma_j = 0, j \neq \binom{\ell+r}{r}) \mathbf{P}(\sum_\ell \binom{\ell+r}{r} \Gamma_{\binom{\ell+r}{r}} = n)}{\mathbf{P}(\sum_j j\Gamma_j = n)}, \end{aligned}$$

where the last identity follows from the fact that the events $\{\Gamma_j = 0, j \neq \binom{\ell+r}{r}, \ell \geq 0\}$ and $\{\sum_\ell \binom{\ell+r}{r} \Gamma_{\binom{\ell+r}{r}} = n\}$ are independent. The same computation yields

$$\begin{aligned} \mathbf{Q}(\{\lambda \in A\} \cap \{\lambda \in F_r(n)\}) \\ &= \frac{\mathbf{P}(\Gamma_j = 0, j \neq \binom{\ell+r}{r}) \mathbf{P}(\{(\Gamma_{\binom{\ell+r}{r}}) \in A\} \cap \{\sum_\ell \binom{\ell+r}{r} \Gamma_{\binom{\ell+r}{r}} = n\})}{\mathbf{P}(\sum_j j\Gamma_j = n)}, \end{aligned}$$

and thus

$$\Pr(\lambda_{(r)} \in A) = \frac{\mathbf{P}(\{(\Gamma_{\binom{\ell+r}{r}}) \in A\} \cap \{\sum_\ell \binom{\ell+r}{r} \Gamma_{\binom{\ell+r}{r}} = n\})}{\mathbf{P}(\sum_\ell \binom{\ell+r}{r} \Gamma_{\binom{\ell+r}{r}} = n)}$$

Most of the effort is to establish a lower bound on the denominator. Following Fristedt [8] we will show that for a particular choice of the parameter $q = q_n$ there exists a constant κ_r such that

$$\mathbf{P}\left(\sum_\ell \binom{\ell+r}{r} \Gamma_{\binom{\ell+r}{r}} = n\right) \geq \kappa_r n^{-\frac{2r+1}{2(r+1)}}, \quad (8)$$

for large n . To this end we will choose the value of q which makes the expected value of the sum

$$X_n = \sum_{\ell \geq 0} \binom{\ell+r}{r} \Gamma_{\binom{\ell+r}{r}},$$

asymptotic to n and then we establish a local central limit theorem (CLT) for the normalized random variables X_n . This will show that the probability in question is of order 1 over the standard deviation of X_n . Since Γ 's

are geometric we have

$$\mathbb{E}X_n = \sum_{\ell=0}^{\infty} \binom{\ell+r}{r} \frac{q^{\binom{\ell+r}{r}}}{1 - q^{\binom{\ell+r}{r}}}, \quad (9)$$

and since they, in addition, are independent

$$\text{var}(X_n) = \sum_{\ell=0}^{\infty} \binom{\ell+r}{r}^2 \frac{q^{\binom{\ell+r}{r}}}{\left(1 - q^{\binom{\ell+r}{r}}\right)^2}. \quad (10)$$

Before proceeding any further, let us note that in subsequent computations, it will be frequently convenient to replace $\binom{\ell+r}{r}$ by $\ell^r/r!$ in the infinite sums like (9) or (10). This can be done without difficulty because, with our choice of q , these sums will grow to infinity at the faster rate than the individual terms. Hence, ignoring the first few terms will not affect the asymptotic behavior of the sum and for larger ℓ 's the approximation $\binom{\ell+r}{r} \sim \ell^r/r!$ is valid. For example, with regard to (9), since the function $uq^u/(1-q^u)$ is decreasing for $u > 0$

$$\sum_{\ell=0}^{\infty} \binom{\ell+r}{r} \frac{q^{\binom{\ell+r}{r}}}{1 - q^{\binom{\ell+r}{r}}} \leq \sum_{\ell=1}^{\infty} \frac{\ell^r}{r!} \frac{q^{\ell^r/r!}}{1 - q^{\ell^r/r!}}.$$

On the other hand

$$\begin{aligned} \sum_{\ell=0}^{\infty} \binom{\ell+r}{r} \frac{q^{\binom{\ell+r}{r}}}{1 - q^{\binom{\ell+r}{r}}} &\geq \sum_{\ell=0}^{\infty} \frac{(\ell+r)^r}{r!} \frac{q^{(\ell+r)^r/r!}}{1 - q^{(\ell+r)^r/r!}} \\ &= \sum_{\ell=r}^{\infty} \frac{\ell^r}{r!} \frac{q^{\ell^r/r!}}{1 - q^{\ell^r/r!}} \geq \sum_{\ell=1}^{\infty} \frac{\ell^r}{r!} \frac{q^{\ell^r/r!}}{1 - q^{\ell^r/r!}} - (r-1) \max_{1 \leq \ell < r} \left\{ \frac{\ell^r}{r!} \frac{q^{\ell^r/r!}}{1 - q^{\ell^r/r!}} \right\}. \end{aligned}$$

As we will see, the first term is of order $1/\ln^{(r+1)/r}(1/q)$ and $\max_{1 \leq \ell < r} \left\{ \frac{\ell^r}{r!} \frac{q^{\ell^r/r!}}{1 - q^{\ell^r/r!}} \right\} \leq 1/\ln(1/q)$. Our choice of q will guarantee that the second term is of lower order than the first one, thus justifying our claim. Other instances can be treated in a virtually the same manner. In order to evaluate (9) we consider the sum

$$\sum_{\ell=1}^{\infty} \frac{\ell^r}{r!} \frac{q^{\ell^r/r!}}{1 - q^{\ell^r/r!}} = \sum_{\ell=1}^{\infty} g(\ell),$$

where

$$g(x) = \frac{x^r}{r!} \frac{q^{x^r/r!}}{1 - q^{x^r/r!}}.$$

The function g is decreasing on a positive half-line. Therefore,

$$\int_0^\infty g(x)dx \geq \sum_{\ell=1}^\infty g(\ell) \geq \int_1^\infty g(x)dx \geq \int_0^\infty g(x)dx - g(0+).$$

Hence, an error resulting from replacing the sum by the integral

$$\int_0^\infty g(x)dx = \int_0^\infty \frac{x^r}{r!} \frac{q^{x^r/r!}}{1 - q^{x^r/r!}} dx$$

is no more than $g(0+) = 1/\ln(1/q)$. Changing variables to $q^{x^r/r!} = e^{-u}$ we see that this integral is

$$\frac{(r!)^{1/r}}{r} \frac{1}{\ln^{(r+1)/r}(1/q)} \int_0^\infty u^{1/r} \frac{e^{-u}}{1 - e^{-u}} du = \frac{(r!)^{1/r}}{r} \frac{H_r}{\ln^{(r+1)/r}(1/q)},$$

where $H_r = \zeta(1 + r^{-1})\Gamma(1 + r^{-1})$ (see [9, formula 3.411-7]). Thus we can set

$$q = q_n = \exp\left(-\frac{C}{n^{r/(r+1)}}\right),$$

where C is given by (2). With that choice, the approximation error between the expectation and n is no more than $\ln(1/q) = O(n^{r/(r+1)})$. Asymptotic evaluation of the variance follows the same pattern (with $g(x)$ replaced by $(x^r/r!)^2 q^{x^r/r!}/(1 - q^{x^r/r!})^2$) and yields

$$\sigma^2 \sim \frac{(r!)^{1/r}}{r} \frac{1}{\ln^{(2r+1)/r}(1/q)} \int_0^\infty u^{(r+1)/r} \frac{e^{-u}}{(1 - e^{-u})^2} du = \frac{K}{\ln^{\frac{2r+1}{r}}(1/q)} \sim K n^{\frac{2r+1}{r+1}}$$

with an error of approximation bounded by $n^{2r/(r+1)}$. The next step is to establish the local CLT. Continuing to follow Fristedt, we will first use characteristic functions to establish the CLT. Let us suppress the dependence on n and put

$$X = \sum_{\ell=0}^\infty \binom{\ell+r}{r} \Gamma_{\binom{\ell+r}{r}}, \quad \mu = \mathbb{E}X, \quad \sigma^2 = \text{var}(X), \quad Y = \frac{X - \mu}{\sigma}.$$

Let $\phi(t) = \mathbb{E}e^{itY}$ be the characteristic function of Y . By independence

$$\begin{aligned} \log \phi(t) &= -it\frac{\mu}{\sigma} + \sum_{\ell \geq 0} \log \left(\frac{1 - q^{\binom{\ell+r}{r}}}{1 - (q \exp(it/\sigma))^{\binom{\ell+r}{r}}} \right) \\ &= -it\frac{\mu}{\sigma} - \sum_{\ell \geq 0} \log \left(1 + \frac{q^{\binom{\ell+r}{r}}(1 - \exp(i\binom{\ell+r}{r}t/\sigma))}{1 - q^{\binom{\ell+r}{r}}} \right). \end{aligned}$$

Approximating $\log(1+v)$ by $v - v^2/2$ we see that the above series is asymptotic to

$$\sum_{\ell \geq 0} \left\{ \frac{q^{\binom{\ell+r}{r}} (\exp(i \binom{\ell+r}{r} t/\sigma) - 1)}{1 - q^{\binom{\ell+r}{r}}} + \frac{q^{2\binom{\ell+r}{r}} (1 - \exp(i \binom{\ell+r}{r} t/\sigma))^2}{2 \left(1 - q^{\binom{\ell+r}{r}}\right)^2} \right\}, \quad (11)$$

provided that an error from approximation is negligible. But this error is no more than a constant multiple of

$$\begin{aligned} \sum_{\ell \geq 0} \frac{q^{3\binom{\ell+r}{r}}}{\left(1 - q^{\binom{\ell+r}{r}}\right)^3} \left| 1 - \exp\left(i \binom{\ell+r}{r} t/\sigma\right) \right|^3 &\leq \frac{K|t|^3}{\sigma^3} \sum_{\ell \geq 0} \binom{\ell+r}{r}^3 \frac{q^{3\binom{\ell+r}{r}}}{\left(1 - q^{\binom{\ell+r}{r}}\right)^3} \\ &\sim \frac{K|t|^3}{\sigma^3} \int_0^\infty \left(\frac{x}{r!}\right)^3 \frac{q^{3x^r/r!}}{\left(1 - q^{x^r/r!}\right)^3} dx = \Theta\left(n^{-\frac{1}{2(r+1)}}\right). \end{aligned}$$

Next, we use the approximations

$$1 - \exp\left(i \frac{t}{\sigma} \binom{\ell+r}{r}\right) \sim -\frac{i \binom{\ell+r}{r} t}{\sigma} + \frac{\binom{\ell+r}{r}^2 t^2}{2\sigma^2} \quad \text{and} \quad 1 - \exp\left(i \frac{t}{\sigma} \binom{\ell+r}{r}\right) \sim -\frac{i \binom{\ell+r}{r} t}{\sigma}$$

in the first and second expressions in (11), respectively. Since the errors are, respectively, of order

$$\frac{\binom{\ell+r}{r}^3 t^3}{\sigma^3} \quad \text{and} \quad \frac{\binom{\ell+r}{r} t^2}{\sigma^2},$$

the total errors from approximating these sums are, respectively,

$$\frac{t^3}{\sigma^3} \sum_{\ell \geq 0} \frac{q^{\binom{\ell+r}{r}}}{1 - q^{\binom{\ell+r}{r}}} \cdot \binom{\ell+r}{r}^3 = \Theta\left(\frac{1}{\sigma^3 \ln^{(3r+1)/r}(1/q)}\right) = \Theta\left(n^{-\frac{1}{2(r+1)}}\right),$$

and

$$K \frac{t^3}{\sigma^3} \int_0^\infty x^{3r} \frac{q^{2x^r/r!}}{\left(1 - q^{x^r/r!}\right)^2} dx + K \frac{t^4}{\sigma^4} \int_0^\infty x^{4r} \frac{q^{2x^r/r!}}{\left(1 - q^{x^r/r!}\right)^2} dx = \Theta\left(n^{-1/(2(r+1))}\right),$$

and are thus negligible. It follows that

$$\begin{aligned} \phi(t) &= -\frac{it\mu}{\sigma} + \sum_{\ell \geq 0} \frac{q^{\binom{\ell+r}{r}}}{1 - q^{\binom{\ell+r}{r}}} \cdot \frac{it \binom{\ell+r}{r}}{\sigma} - \sum_{\ell \geq 0} \frac{q^{\binom{\ell+r}{r}}}{1 - q^{\binom{\ell+r}{r}}} \cdot \frac{t^2 \binom{\ell+r}{r}^2}{2\sigma^2} \\ &\quad - \sum_{\ell \geq 0} \frac{q^{2\binom{\ell+r}{r}}}{\left(1 - q^{\binom{\ell+r}{r}}\right)^2} \cdot \frac{t^2 \binom{\ell+r}{r}^2}{2\sigma^2} + o(1). \end{aligned}$$

The sum of the first two terms is zero while the sum of the remaining two is

$$-\frac{t^2}{2\sigma^2} \sum_{\ell \geq 0} \left\{ \frac{q^{\binom{\ell+r}{r}}}{1-q^{\binom{\ell+r}{r}}} \cdot \binom{\ell+r}{r}^2 + \frac{q^{2\binom{\ell+r}{r}}}{(1-q^{\binom{\ell+r}{r}})^2} \cdot \binom{\ell+r}{r}^2 \right\}.$$

And again,

$$\begin{aligned} & \sum_{\ell \geq 0} \left\{ \frac{q^{\binom{\ell+r}{r}}}{1-q^{\binom{\ell+r}{r}}} \cdot \binom{\ell+r}{r}^2 + \frac{q^{2\binom{\ell+r}{r}}}{(1-q^{\binom{\ell+r}{r}})^2} \cdot \binom{\ell+r}{r}^2 \right\} \\ & \sim \frac{(r!)^{1/r}}{r} \frac{1}{\ln^{(2r+1)/r}(1/q)} \left\{ \int_0^\infty \frac{u^{(r+1)/r} e^{-u}}{1-e^{-u}} du + \int_0^\infty \frac{u^{(r+1)/r} e^{-2u}}{(1-e^{-u})^2} du \right\} \\ & = \frac{(r!)^{1/r}}{r} \frac{1}{\ln^{(2r+1)/r}(1/q)} \int_0^\infty \frac{u^{(r+1)/r} e^{-u} (1-e^{-u}) + u^{(r+1)/r} e^{-2u}}{(1-e^{-u})^2} du \\ & = \frac{(r!)^{1/r}}{r} \frac{1}{\ln^{(2r+1)/r}(1/q)} \int_0^\infty \frac{u^{(r+1)/r} e^{-u}}{(1-e^{-u})^2} du, \end{aligned}$$

which is the same expression as the one appearing in the computation for σ^2 . Thus, $\forall t \in \mathbf{R}$

$$\phi(t) \rightarrow \exp\left(-\frac{t^2}{2}\right)$$

which establishes the CLT.

It remains to strenghten it to the local CLT. To this end we will appeal to [5, Theorem 2.9] with $h_n = 1/\sigma = 1/\sigma_n$. We need to find an integrable function ϕ^* and a sequence (β_n) , $\beta_n \rightarrow \infty$, such that

$$\forall t \quad |\phi_n(t)| I(|t| \leq \beta_n) \leq \phi^*(t), \quad (12)$$

and

$$\sup_{\beta_n \leq |t| \leq \pi\sigma_n} |\phi_n(t)| = o(1/\sigma_n). \quad (13)$$

To check (12), supressing a subscript n again, and using the earlier computation we have

$$\log |\phi(t)| = - \sum_{\ell \geq 0} \log \left| \frac{1 - q^{\binom{\ell+r}{r}} \exp(it \binom{\ell+r}{r} / \sigma)}{1 - q^{\binom{\ell+r}{r}}} \right|$$

and since

$$\left| \frac{1 - q^{\binom{\ell+r}{r}} \exp(it \binom{\ell+r}{r} / \sigma)}{1 - q^{\binom{\ell+r}{r}}} \right| = \left(1 + \frac{2q^{\binom{\ell+r}{r}} (1 - \cos(t \binom{\ell+r}{r} / \sigma))}{(1 - q^{\binom{\ell+r}{r}})^2} \right)^{1/2}$$

we get

$$\begin{aligned} \log |\phi(t)| &= -\frac{1}{2} \sum_{\ell \geq 0} \log \left(1 + \frac{2q^{\binom{\ell+r}{r}} (1 - \cos(t \binom{\ell+r}{r} / \sigma))}{(1 - q^{\binom{\ell+r}{r}})^2} \right) \\ &\leq -\frac{1}{2} \sum_{\ell \geq 0} \log \left(1 + 2q^{\binom{\ell+r}{r}} (1 - \cos(t \binom{\ell+r}{r} / \sigma)) \right), \end{aligned}$$

by the monotonicity of \log . To find a suitable upper bound we increase the right hand side by restricting ℓ to the range

$$\gamma_r \sigma^{2r/(2r+1)} \leq \binom{\ell+r}{r} \leq \sigma^{2r/(2r+1)}, \quad 0 < \gamma_r < 1, \quad (14)$$

and replacing the remaining terms by 0's. In that range we have

$$\frac{|t| \binom{\ell+r}{r}}{\sigma} \leq |t| \sigma^{-1/(2r+1)},$$

and

$$\begin{aligned} q^{\binom{\ell+r}{r}} &= \exp \left(-\frac{C}{n^{r/(r+1)}} \cdot \binom{\ell+r}{r} \right) \geq \exp \left(-\frac{K}{n^{r/(r+1)}} \cdot \sigma^{\frac{2r}{2r+1}} \right) \\ &= \exp(-K) \geq \text{const.} \end{aligned}$$

Hence, with $\beta_n = \gamma \sigma^{1/(2r+1)}$, $|t| < \beta_n$ implies $|t| \sigma^{-1/(2r+1)} \leq \gamma$, so that

$$2q^{\binom{\ell+r}{r}} \left(1 - \cos \frac{t}{\sigma} \binom{\ell+r}{r} \right) \geq K \frac{t^2}{\sigma^2} \binom{\ell+r}{r}^2,$$

and thus

$$\begin{aligned} \log \left(1 + 2q^{\binom{\ell+r}{r}} \left(1 - \cos \frac{t}{\sigma} \binom{\ell+r}{r} \right) \right) &\geq K \frac{t^2}{\sigma^2} \binom{\ell+r}{r}^2 \\ &\geq K \frac{t^2}{\sigma^2} \sigma^{\frac{4r}{2r+1}} = K t^2 \sigma^{-\frac{2}{2r+1}}. \end{aligned}$$

Since there are at least $K\sigma^{2/(2r+1)}$ ℓ 's in the range (14) summing within that range gives

$$\log \left(1 + 2q^{\binom{\ell+r}{r}} \left(1 - \cos \frac{t}{\sigma} \binom{\ell+r}{r} \right) \right) \geq K\sigma^{\frac{2}{2r+1}} t^2 \sigma^{-\frac{2}{2r+1}} = Kt^2,$$

and it follows that in order to fulfill (12) we may take

$$\phi^*(t) = \exp(-Kt^2).$$

It remains to verify (13). Pick a t satisfying

$$\gamma\sigma^{1/(2r+1)} \leq |t| \leq \pi\sigma,$$

and consider the sum

$$\sum_{\ell \geq 0} \log \left(1 + 2q^{\binom{\ell+r}{r}} \left(1 - \cos \frac{t}{\sigma} \binom{\ell+r}{r} \right) \right)$$

Since $\log(1+x) \geq x/5$ for $0 \leq x \leq 4$, this sum is at least

$$\frac{1}{5} \sum_{\ell \geq 0} q^{\binom{\ell+r}{r}} \left(1 - \cos \frac{t}{\sigma} \binom{\ell+r}{r} \right).$$

Dealing with the series in the usual way we see that it is asymptotic to

$$\int_0^\infty q^{\frac{x^r}{r!}} \left(1 - \cos \left(\frac{t}{\sigma} \frac{x^r}{r!} \right) \right) dx,$$

which, using [9, Formula 3.944-6], we find to be asymptotic to

$$\frac{K}{\ln^{1/r}(1/q)} \left\{ 1 - \frac{\cos \left(\frac{1}{r} \arctan \left(\frac{t}{\sigma \ln(1/q)} \right) \right)}{\left(\frac{t^2}{\sigma^2 \ln^2(1/q)} + 1 \right)^{1/(2r)}} \right\}.$$

Since $t \geq \gamma\sigma^{1/(2r+1)}$, it follows that

$$\frac{t^2}{\sigma^2 \ln^2(1/q)} \geq \gamma^2,$$

and hence, using the relationship between σ and n we get

$$\log |\phi(t)| \leq -K \frac{1}{\ln^{1/r}(1/q)} \sim -Kn^{1/(r+1)} \sim -K\sigma^{2/(2r+1)},$$

which implies (13). We now appeal to Theorem 2.9 of [5] with

$$X_n = \sum_{\ell \geq 0} \binom{\ell+r}{r} \Gamma(\ell+r), \quad Y_n = \frac{X_n - \mu_n}{\sigma_n}, \quad h_n = \frac{1}{\sigma_n}, \quad \text{and} \quad y_n = \frac{n - \mu_n}{\sigma_n}.$$

Since $n - \mu_n = O(n^{r/(r+1)})$, we have $y_n \rightarrow 0$ and thus

$$\lim_{n \rightarrow \infty} n^{\frac{2r+1}{2(r+1)}} \mathbf{P}(X_n = n) = \lim_{n \rightarrow \infty} \frac{1}{h_n} \mathbf{P}(Y_n = y_n) = \frac{1}{\sqrt{2\pi}},$$

which clearly implies (8).

To complete the proof, let $I_j = \{\Gamma(\binom{j+r}{r}) \geq 1\}$. Then, denoting for simplicity a set and its indicator by the same symbol, we have

$$\begin{aligned} \Pr(|D_{n,r} - \mathbf{E}D_{n,r}| \geq t) &= \frac{\mathbf{P}(\{|\sum_j (I_j - \mathbf{E}I_j)| \geq t\} \cap \{\sum_{\ell} \binom{\ell+r}{r} \Gamma(\ell+r) = n\})}{\mathbf{P}(\sum_{\ell} \binom{\ell+r}{r} \Gamma(\ell+r) = n)} \\ &\leq K n^{\frac{2r+1}{2(r+1)}} \mathbf{P}(|\sum_j (I_j - \mathbf{E}I_j)| \geq t). \end{aligned}$$

Since the random variables $(I_j - \mathbf{E}I_j)$ are independent, mean-zero, and uniformly bounded by 1, the last probability can be controlled by virtue of Prokhorov's "arcsinh" inequality ([11, Theorem 5.2.2(ii)])

$$\mathbf{P}\left(\sum_j |I_j - \mathbf{E}I_j| \geq t\right) \leq 2 \exp\left\{-\frac{t}{2} \operatorname{arcsinh}\left(\frac{t}{2\operatorname{var}(\sum_j I_j)}\right)\right\}$$

Since $\operatorname{var}(I_j) = q^{\binom{j+r}{r}}(1 - q^{\binom{j+r}{r}})$, by independence we get

$$\begin{aligned} \operatorname{var}(\sum_j I_j) &= \sum_{j \geq 0} q^{\binom{j+r}{r}}(1 - q^{\binom{j+r}{r}}) \sim \frac{(r!)^{1/r}}{r} \frac{1}{\ln^{1/r}(1/q)} \int_0^\infty u^{-(r-1)/r} e^{-u} (1 - e^{-u}) du \\ &= \frac{(r!)^{1/r} (1 - 2^{-1/r}) \Gamma(1/r)}{r \ln(1/q)} \sim K n^{1/(r+1)}. \end{aligned}$$

Thus, selecting $t = t_n$ so that $t_n = o(n^{1/(r+1)})$ and $t_n^2/n^{1/(r+1)} \rightarrow \infty$ not too slow, say, $t_n = \Theta(n^{2/(3(r+1))})$, and using the fact that $\operatorname{arcsinh} x \geq Kx$ for x close to 0, we see that

$$\begin{aligned} \Pr(|D_{n,r} - \mathbf{E}D_{n,r}| \geq t_n) &\leq K n^{(2r+1)/(2(r+1))} \exp\left(-K \frac{t_n^2}{n^{1/(r+1)}}\right) \\ &= K n^{(2r+1)/(2(r+1))} \exp\left(-K n^{1/(3(r+1))}\right) \rightarrow 0. \end{aligned}$$

Hence, since $t_n = o(\mathbb{E}D_{n,r})$, integrating $D_{n,r,m}/D_{n,r}$ over the set $\{|D_{n,r} - \mathbb{E}D_{n,r}| \leq t_n\}$ and its complement yields

$$\mathbb{E} \frac{D_{n,r,m}}{D_{n,r}} = \frac{\mathbb{E}D_{n,r,m}}{\mathbb{E}D_{n,r}} + O\left(\max\left\{\frac{t_n}{\mathbb{E}D_{n,r}}, \mathbb{P}(|D_{n,r} - \mathbb{E}D_{n,r}| > t_n)\right\}\right) = \frac{1}{m^{1/r}} + o(1),$$

as desired.

5. FUTURE WORK

One interesting question would be to show how fast the probability converges. It appears that our proof does come with specific rates. The question is therefore to show if they are optimal or not. Let us note we also showed that the probability that a random part size of a random partition into parts $\binom{i+r}{r}$ of n has multiplicity at least m is $m^{-1/r}$ when $n \rightarrow \infty$. Our aim is now to identify the sets S such that the probability that a random part size of a random partition into parts in S of n has multiplicity at least m is a constant when $n \rightarrow \infty$.

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