Accelerated Calculus I — Group Work I

**Due Date:** Monday, October 3, in class

1. If $|x| < 1$, find the limit approached by the product

$$ f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16}) \cdots $$

as the number of factors is indefinitely increased. One way to do this is to form the product

$$ (1 - x)f(x) = (1 - x)(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16}) \cdots $$

combine the first two factors on the right, then multiply by the third factor. Next repeat this procedure again and again. Finally, solve for $f(x)$.

2. By factoring each term and then combining consecutive terms determine the value of the infinite product

$$ \left( 1 - \frac{1}{2^2} \right) \left( 1 - \frac{1}{3^2} \right) \left( 1 - \frac{1}{4^2} \right) \cdots \left( 1 - \frac{1}{n^2} \right) \cdots. $$

3. Let $A_n$ be the area and $p$ the perimeter of a regular polygon of $n$ sides. If $n$ increases and $p$ remains constant, find the limit

$$ \lim_{n \to \infty} A_n. $$

4. By expressing each term as a power of 2, show that the sequence

$$ \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots $$

converges, and find its limit.

5. Consider the unit circle (of radius 1). In this problem, we shall circumscribe and inscribe it with regular polygons with $n$ sides. Let $C_n$ denote the perimeter of the circumscribed polygon while $I_n$ is the perimeter of the inscribed polygon.

For convenience we introduce the further notations:

$$ A(a, b) = \frac{a + b}{2}, \quad \sqrt{ab} = G(a, b), \quad \frac{2ab}{a + b} = H(a, b). $$

(These are the arithmetic, geometric, and harmonic means).

(a) Find the quantities $C_4$, $I_4$, $C_6$, $I_6$. What are the corresponding estimates of the circumference of the unit circle?

(b) Show that $C_{2n} = H(C_n, I_n)$ and $I_{2n} = G(I_n, C_{2n})$.

(c) Archimedes gave the estimate of $\pi$ as $3 \frac{10}{71} < \pi < 3 \frac{1}{7}$ using circumscribing and inscribing polygons of 96 sides. Can you verify this estimate by using parts (a) and (b)?