1. Construct a bounded set of real numbers with exactly three limit points.

**Answer:** One possible answer is given by:

\[
\left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \cup \left\{ 1 + \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \cup \left\{ 2 + \frac{1}{n} : n \in \mathbb{Z}^+ \right\}
\]

with limit points 0, 1, and 2.

2. (a) Construct a compact set of real numbers whose limit points form a countable set. Note: a set will be compact if it is closed and bounded. (b) Construct an infinite set with no limit points.

**Answer:** (a) One possible answer is:

\[
\bigcup_{n=1}^{\infty} \left( \bigcup_{m=1}^{\infty} \left\{ \frac{1}{n} + \frac{1}{m} \right\} \right) \cup \{0\}
\]

whose limit points are \(\bigcup_{n=1}^{\infty} \{\frac{1}{n}\} \cup \{0\}\). Since the indicated set is bounded and contains all its limit points, it is compact.

(b) One possible example is given by the set of positive integers.

3. (a) Show that the intersection of two convex sets is convex. (b) Prove or disprove: the union of two convex sets is convex provided their intersection is non-empty.

**Answer:** (a) Let \(S\) and \(T\) be two convex subsets of \(\mathbb{R}^k\). Let \(p, q \in S \cap T\). Let \(t\) be a real number satisfying \(0 < t < 1\). Then \(tp + (1-t)q \in S\) since \(p\) does and \(S\) is convex; similarly \(tp + (1-t)q \in T\) since \(p\) does and \(T\) is convex. Hence, the intersection of \(S\) and \(T\) is also convex.

(b) Let \(S\) be the \(x\)-axis and \(T\) the \(y\)-axis in \(\mathbb{R}^2\). Then \(S\) and \(T\) are both convex with a non-empty intersection. But the line segment joining \((1,0)\) with \((0,1)\) does not lie in their union.

4. Let \(f\) be a real-valued function defined for every \(x \in [0,1]\). Suppose there is a positive number \(M\) having the property that for every choice of finite number of distinct points \(x_1, x_2, \ldots, x_n\) in the interval \([0,1]\), the sum \(|f(x_1) + f(x_2) + \ldots + f(x_n)| \leq M\). Let \(S\) be the set of those \(x\) in \([0,1]\) for which \(f(x) \neq 0\). Prove that \(S\) is countable.

**Answer:** Consider the sets

\[S_n = \left\{ x \in [0,1] : f(x) \geq \frac{1}{n} \right\}, \quad T_n = \left\{ x \in [0,1] : f(x) \leq -\frac{1}{n} \right\}\]

Then \(|S_n| \leq Mn\) and \(|T_n| \leq Mn\) as well. But the set \(S\) equals \(\bigcup_{n=1}^{\infty} S_n \cup \bigcup_{n=1}^{\infty} T_n\). Since the countable union of finite sets is countable, \(S\) itself is countable.

5. (a) Give an explicit one-to-one and onto map of \(J\), the positive integers, onto the odd positive integers;

(b) Give an explicit one-to-one and onto map of \(J\), the positive integers, onto the set of all integers;

(c) Let \(A\), \(B\), and \(C\) be sets. Show explicitly that if \(A \sim B\) and \(B \sim C\), then \(A \sim C\).

**Answer:** (a) One possible map is \(n \mapsto 2n - 1\).

(b) One possible map \(f\) is:

\[f(n) = \begin{cases} 
\frac{n}{2}, & \text{if } n \text{ is even;} \\
-\frac{(n-1)}{2}, & \text{if } n \text{ is odd.}
\end{cases}\]

(c) Since \(A \sim B\) and \(B \sim C\), we know that there exist two one-to-one maps \(f\) and \(g\) which map \(A\) onto \(B\) and \(B\) onto \(C\). Their composition \(g \circ f\) from \(A\) to \(C\) is both one-to-one and onto. Hence, \(A \sim C\).