Due: Wednesday, January 26

Note: We are be re-visiting questions about metric spaces throughout the term.

1. Let $X$ be an infinite set. For $p, q \in X$, define
   \[ d(p, q) = \begin{cases} 
   1, & \text{if } p \neq q, \\
   0, & \text{if } p = q. 
   \end{cases} \]
   Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

2. For $x, y \in \mathbb{R}$, define
   \[ \begin{align*}
   d_1(x, y) &= (x - y)^2, \\
   d_2(x, y) &= \sqrt{|x - y|}, \\
   d_3(x, y) &= |x^2 - y^2|, \\
   d_4(x, y) &= |x - 2y|, \\
   d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}.
   \end{align*} \]
   Determine for each of these whether it is a metric or not.

3. (a) Show that if $\{K_\alpha\}$ is a collection of closed subsets of $\mathbb{R}$ such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then the full intersection $\bigcap_\alpha K_\alpha$ may be empty.
   (b) Show that if $\{K_\alpha\}$ is a collection of bounded subsets of $\mathbb{R}$ such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then the full intersection $\bigcap_\alpha K_\alpha$ may be empty.
   (c) Find a metric space $X$ and a subset $K$ of $X$ which is closed and bounded but not compact.

4. Let $E'$ be the set of all limit points of a set $E$. Prove that $E'$ is closed. Prove that $E$ and $\overline{E}$ have the same limit points. Do $E$ and $E'$ always have the same limit points?

5. (a) If $A$ and $B$ are disjoint closed sets in a metric space $X$, prove that they are separated.
   (b) Prove the same for disjoint open sets.