Due: Wednesday, February 9

Note: We are be re-visiting questions about metric spaces, sequences, and series throughout the term.

1. Let $a_1 = \sqrt{2}$, and
   
   \[ a_{n+1} = \sqrt{2 + \sqrt{2}}, \quad n = 1, 2, \ldots, \]

   prove that $a_n$ converges and that $a_n < 2$, for $n = 1, 2, \ldots$. Hint: look at my lecture notes!

2. Prove that the convergence of $\sum a_n$ implies the convergence of

   \[ \sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}, \]

   provided $a_n \geq 0$. Hint: Use the Cauchy-Schwarz inequality.

3. Find the upper and lower limits of the sequence \( \{a_n\} \) defined by

   \[ a_1 = 0, \quad a_{2m} = \frac{a_{2m-1}}{2}, \quad a_{2m+1} = \frac{1}{2} + a_{2m}. \]

4. Find an example of two convergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ such that their product $\sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k b_{n-k})$ diverges.

5. (a) Prove that if the sum and the difference of two real sequences converge, then both the sequences must converge.

   (b) Prove that $\lim_{n \to \infty} a_n = 0$ if and only if $\lim_{n \to \infty} |a_n| = 0$. 