1. Let \( f(x) = e^x \) and \( x_0 = 0 \). Find the \( n \)-th taylor polynomial \( p_n(x) \) for \( f(x) \) about \( x_0 \). Find a value of \( n \) necessary for \( p_n(x) \) to approximate \( f(x) \) to within \( 10^{-6} \) on \([0, 0.5]\).

**Comment:** If \( f(x) = e^x \), then the Lagrange Form of the Remainder states:

\[
e^x - p_n(x) = \frac{f^{n+1}(c)x^{n+1}}{(n+1)!},\]

where \( c \) is between 0 and \( x \). Hence, the error is bounded by

\[
\left| \frac{(0.5)^{n+1}e^{0.5}}{(n+1)!} \right| \leq 10^{-6}.
\]

This inequality first holds for \( n = 7 \).

2. The error function defined by \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \) gives the probability that any one of series of trails will lie within \( x \) units of the mean, assuming the trails have a normal distribution with mean 0 and standard deviation \( \sqrt{2}/2 \).

(a) Integrate the maclurin series for \( e^{-x^2} \) to show that \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!} \).

(b) The error function can also be expressed in the form \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)} \). Verify that the two series agree for \( k = 1, 2, \) and 3.

(c) Use the series in part (a) to approximate \( \text{erf}(1) \) to within \( 10^{-7} \).

(d) Use the same number of terms as in part (c) to approximate \( \text{erf}(1) \) with the series in part (b).

(e) Explain why difficulties occur using the series in part (b) to approximate \( \text{erf}(x) \).

**Comment:** The maclurin series for \( e^{-x^2} \) is \( \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!} \). Hence \( \int_0^1 \left( \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{k!} \right) dt \). Hence the series expansion for the error function is \( \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)k!} \). Since this is an alternating series, to approximate its full sum by the \( \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k t^{2k+1}}{(2k+1)k!} \) has at most an absolute error bounded by \( \frac{2}{\sqrt{\pi}} \frac{t^{2(n+1)+1}}{(2(n+1)+1)(n+1)!} \) or \( \frac{2}{\sqrt{\pi}} \frac{t^{2n+3}}{(2n+3)(n+1)!} \). In particular, to approximate \( \text{erf}(1) \) to within \( 10^{-7} \) requires \( \frac{2}{\sqrt{\pi}} \frac{1}{(2n+3)(n+1)!} < 10^{-7} \). Hence \( n = 9 \). The approximation is .8427007792 while the exact value is .8427007929. The partial sum from part (b) with the same number of terms yields 0.8427008. Note: there is no useful bound in using the series from part (b).

3. Use three-digit rounding arithmetic to perform the following calculations. Compare the absolute and relative errors with the exact value determined to at least five digits. (a) \((121 - 0.327) - 119\), (b) \((121 - 119) - 0.327\).

**Comment:** (a) 2 with absolute error 0.327 and relative error .00179.

4. Suppose two points \((x_0, y_0)\) and \((x_1, y_1)\) are on a straight line with \( y_0 \neq y_1 \). Two formulas are available to find the \( x \)-intercept of the line: \( x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} \), \( x = x_0 - \frac{(x_1 - x_0)y_0}{y_1 - y_0} \).
(a) Show that both formulas are algebraically correct.

(b) Use the data \((x_0,y_0) = (1.31,3.24)\) and \((x_1,y_1) = (1.93,4.76)\) with three-digit rounding arithmetic to compute the \(x\)-intercept both ways. Which method is better and why?

Comment: The first formula gives \(-0.00658\) while the second \(-0.0100\). The second formula is better since it postpones subtracting nearly equal quantities until the last computation.

5. Let \(f(x) = (e^x - e^{-x})/x\). (a) Find \(\lim_{x\to 0} f(x)\). (b) Use three-digit rounding arithmetic to evaluate \(f(0.1)\). (c) Replace each exponential function with its third degree maclaurin polynomial and repeat part (b). (d) The actual value \(f(0.1) = 2.003335000\). Find the relative error for the values obtained in parts (a) and (b).

Comment: The limit is 2 since \((e^x - e^{-x})/x = [(e^x - 1) - (e^{-x} - 1)]/x\) as \(x\to 2\) tends to the derivative evaluated at \(x = 0\) of \(e^x - e^{-x}\) which equals 2. In three-digit rounding arithmetic, \((e^x - e^{-x})/x = (1.11 - .905)/.1 = 2.05\). Next we write \((e^x - e^{-x})/x\) as \([1+x+x^2/2+x^3/6] - (1-x+x^2/2-x^3/6)\)/\(x\) as \(2+x^2/3\) in terms of third-degree maclaurin polynomials. Then \(f(0.1)\) is approximated as \(2.00\). The relative error for value \(2.05\) is \(0.0233\) while for \(2.00\) is \(0.001665\).

6. Use three-digit chopping arithmetic to compute the sum \(\sum_{j=1}^{10} 1/j^2\) first as \(\frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{1000}\) and then by \(\frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{200}\). Which method is more accurate and why?

Comment: For three-digit rounding arithmetic both sums agree and equal 1.55. Using three-digit chopping arithmetic and adding the terms from largest to smallest, the sum is 1.53 while from smallest to largest 1.54.

7. Find the rate of convergence as \(h \to 0\) of (a) \(\lim_{h\to 0} (1 - \cos h)/h = 0\), (b) \(\lim_{h\to 0} (\sin h - h\cos h)/h = 0\).

Comment: (a) Write \((1 - \cos h)/h\) as \((1 - [1 - h^2/2 + h^4/24 - h^6/720 + O(h^8)])/h\) or \((h^2/2 - h^4/24 + h^6/720 + O(h^8))/h\) which reduces to \(h - h^3/24 + h^5/720 + O(h^7)\). Hence the rate of convergence is 1. (b) Write \(\sin(h)\) as \(h - h^3/6 + h^5/120 - h^7/5040 + O(h^9)\) and \(h\cos(h)\) as \(h(1 - h^2/2 + h^4/24 - h^6/720 + O(h^8))\) or \(h - h^3/2 + h^5/24 - h^7/720 + O(h^9)\). Hence, \((\sin h - h\cos h)/h\) becomes \(h^2/3 - h^4/30 + O(h^6)\). In particular, the rate of convergence is 2.

8. (a) Suppose \(0 < q < p\) and that \(\alpha_n = \alpha + O(n^{-q})\). Show that \(\alpha_n = \alpha + O(n^{-q})\). (b) Make a table listing \(1/n\), \(1/n^2\), \(1/n^3\), and \(1/n^4\) for \(n = 5, 10, 100,\) and 1000. Discuss the varying rates of convergence of these sequences as \(n\) becomes large.

Comment: The answer is in the back of the textbook!