1. In Chapter Two, two methods are proposed to restore quadratic convergence for Newton’s method if the root is not simple. In the first modified method to find the root of \( f(x) \), the new approximations have the form

\[
x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)}.
\]

The second method requires full knowledge of the multiplicity or order of the root \( x \). Suppose it has order \( m \). Then the the second generalization gives the approximations:

\[
x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}.
\]

Let \( f(x) = 1 + \ln x - x \). It has a root at \( x = 1 \).

(a) Apply the standard Newton method to \( f(x) \) with starting value \( p_0 = 2 \) with ten iterations. Verify the order of convergence is linear by making a list of the ratios \( |e_{n+1}/e_n| \) and observing it has a non-zero limit.

(b) Apply the first generalization of Newton’s method to \( f(x) \) with initial value \( p_0 = 2 \). Perform five iterations. Verify the order of convergence by making a list of the appropriate error ratios.

(c) Apply the second generalization of Newton’s method to \( f(x) \) with initial value \( p_0 = 2 \). Perform five iterations. Verify the order of convergence by making a list of the appropriate error ratios.

2. Consider the function \( f(x) = e^x \).

(a) Construct the Lagrange form of the interpolating polynomial \( p_2(x) \) for \( f \) passing through the points \( (-1, e^{-1}), (0, 1), \) and \( (1, e) \). NOTE: use must construct explicitly the three Lagrange polynomials \( L_0(x), L_1(x), \) and \( L_2(x) \) for the data to obtain full credit.

(b) Plot the polynomial \( p_2(x) \) obtained in part (a) on the same axes as \( f(x) = e^x \). Use the range \([-1, 1]\) for \( x \). Next, generate a plot of the difference between the polynomial interpolant and the exponential.

(c) Find the theoretical upper bound for \( |f(x) - p_2(x)| \) in Lagrange Form. Compare it with the actual error.

(d) Using the same points as in part (a), find the polynomial interpolant by using Newton’s divided differences.

(e) Recall that the zeros \( \{x_k\}_{k=1}^{n} \) of the Chebyshev polynomials \( T_n(x) \) give optimal points of interpolation for functions on \([-1, 1]\). Find the polynomial interpolant to \( e^x \) using the zeros of \( T_3(x) \). Next, generate a plot of the difference between this polynomial interpolant and the exponential. Compare the result with uniformly spaced nodes.
3. Let $\tilde{T}_n(x)$ denote the monic version of the $n$-th degree Chebyshev polynomial (that is, the multiple of $T_n(x)$ whose coefficient of $x^n$ is 1). Then we saw that $\max\{|\tilde{T}_n(x)| : x \in [-1, 1]\} = 1/2^{n-1}$. This can be used to “economize” polynomials by lowering their degree with minimum error. Recall the set-up.

Let $P_n(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ on $[-1, 1]$. Let $Q_{n-1}(x)$ be the “economized” polynomial of degree $n-1$ formed by:

$$Q_{n-1}(x) = P_n(x) - a_n\tilde{T}_n(x).$$

An upper bound on the error committed is:

$$|P_n(x) - Q_{n-1}(x)| \leq |a_n| \max\{|\tilde{T}_n(x)| : x \in [-1, 1]\} \leq |a_n| 2^{1-n}.$$

Consider the function $f(x) = e^x$ on $[-1, 1]$ and its fifth degree Maclurin polynomial $p(x) = 1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120$. We have the standard upper bound on the truncation error $|f(x) - p(x)| \leq 0.003775391428$ from the Lagrange form of the remainder. Use Chebyshev economization to find a fourth degree polynomial $q(x)$ so that $|f(x) - q(x)| \leq 0.005$ on $[-1, 1]$.

4. Consider the following data which gives experimental values for the emittance of tungsten as a function of temperature (in kelvin):

(300, 0.024), (400, 0.035), (500, 0.46), (600, 0.058), (700, 0.067), (800, 0.083),
(900, 0.097), (1000, 0.111), (1100, 0.125).

Construct the natural cubic spline interpolant $s(x)$ for this data. On the same plot, give the graph of the spline and mark the data points with diamonds. On another plot, give the graph of the spline together with the standard polynomial interpolant $p(x)$.