1. How many iterations of the Bisection Method is needed to find a solution accurate to within $10^{-2}$ for the root finding problem $x^3 + 4x^2 - 10 = 0$ on $[1, 2]$.

**Comments:** Use the theorem from Section 2.1: Suppose that $f$ is a continuous function on $[a, b]$ and $f(a) \cdot f(b) < 0$. The Bisection Method generates a sequence $x_n$ approximating a zero $x^*$ of $f$ with

$$|x_n - x^*| \leq \frac{b-a}{2^n}, \quad \text{when } n \geq 1.$$  

In this problem $a = 1$ and $b = 2$. Hence, we want to find $n$ so that $1/2^n < 10^{-2}$ or $100 < 2^n$. This first occurs when $n = 7$ so $2^7 = 128$.

2. Consider the question of computing $\sqrt{2}$ by Newton’s Method. Let $f(x) = x^2 - 2$ so its unique positive zero is $\sqrt{2}$. (a) Write out as explicitly as possible the formula given by Newton’s Method. Simplify it algebraically. (b) With initial value $x_0 = 1.0$, find the first three iterations $x_1, x_2, x_3$ given by Newton’s Method. (c) Find the relative errors for the approximations in part (b).

**Comments:** (a) Recall that $x_{n+1} = x_n - f(x_n)/f'(x_n)$, for $n = 1, 2, \ldots$. Now $f(x) = x^2 - 2$ so $f'(x) = 2x$. Hence, Newton’s Method first becomes: $x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - (x_n^2 - 2)/(2x_n)$ or $= (2x_n^2 - x_n^2 - 2)/(2x_n)$ or $(x_n^2 - 2)/(2x_n)$. Finally we write this as:

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{2}{x_n} \right).$$

(b) We use $x_0 = 1$ in the above formula. Then $x_1 = 1.500000000$, $x_2 = 1.416666667$, and $x_3 = 1.414215686$.

(c) Let $a = \sqrt{2}$. Then the relative errors are: $|x_0 - a|/|a| = 0.2928932186$, $|x_1 - a|/|a| = 0.06066017206$, $|x_2 - a|/|a| = 0.001734607181$, and finally $|x_3 - a|/|a| = 0.1501894804 \times 10^{-5}$.

3. Use the error estimate for fixed point iteration to determine the number of iterations to guarantee that the approximation is within $10^{-2}$ of the fixed point. Here $f(x) = 0.2(\sin x + \cos x)$.

![Graph of derivative](https://example.com/derivative_graph.png)

**Comments:** We use the corollary to the theorem in Section 2.2: Let $g$ be a continuous function on $[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Further, assume that $g'$ exists on $(a, b)$ and there is a constant $k$ with $0 < k < 1$ such that

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b).$$
Let \( x_0 \) be any number in \([a, b]\) and define \( x_n = g(x_{n-1}) \) for \( n \geq 1 \). Then \( x_n \) converges to the unique fixed point \( x^* \in [a, b] \) with error bounds:

\[
|x_n - x^*| \leq k^n \max\{x_0 - a, b - x_0\}.
\]

By the plot of the derivative, we see that \( |g'(x)| \leq k = 0.2 \) for \([a, b] = [0, 1]\). With the choice of \( x_0 = 0.5 \) which minimizes the quantity \( \max\{x_0 - a, b - x_0\} \), we have:

\[
|x_n - x^*| \leq 0.5 (0.2)^n \leq 10^{-2}.
\]

We find that \( n = 2 \) yields .020 while \( n = 2 \) is .0040.

4. Circle the correct choice. Let \( x_n = 1/n^2 \). Then the sequence converges to 0:
   (a) of order 1,     (b) of order 2,     (c) neither.

   **Comments:** \( x_n \) converges to 0 of order \( \alpha \) if

\[
\lim_{n \to \infty} \frac{|x_{n+1}|}{|x_n|^\alpha} = \lambda > 0.
\]

   We get \( \lambda = 1 \) if \( \alpha = 1 \). Verify!

5. Circle the correct choice. Let \( x_n = 0.5 \times 2^{-n} \). Then the sequence converges to 0:
   (a) of order 1,     (b) of order 2,     (c) neither.

   **Comments:** The order is 1.

6. Circle the correct choice. The fixed point of a function \( y = f(x) \) has the following geometric interpretation.
   (a) The fixed points of \( f \) are the points where \( f \) crosses the \( x \)-axis. (b) The fixed points of \( f \) are the points where \( f \) crosses the \( y \)-axis. (c) The fixed points of \( f \) are the points where \( f \) crosses the line \( y = x \)-axis. (d) None of the above.

   **Comments:** Fixed points are the intersection points of the graph of \( f \) with the straight line \( y = x \).