1. Consider the nodes \( x_0 = 2.0, x_1 = 2.5, x_2 = 4.0 \). The Lagrange polynomials for these nodes are:

\[
L_0(x) = (x - 6.5)x + 10 \\
L_1(x) = \frac{-(4x + 24)x - 32}{3}, L_2(x) = \frac{(x - 4.5)x + 5}{3}
\]

Let \( f \) be a function defined on \([2,4]\) such that \( f(2) = 0.5, f(2.5) = 0.4, f(3.0) = -0.6, f(3.5) = 0.8 \), and \( f(4.0) = 0.25 \). Find the polynomial interpolant to \( f \) at the nodes \( 2.0, 2.5, 4.0 \).

**Answer:** Form the interpolant in Lagrange polynomial form \( f(2.0)L_0(x) + f(2.5)L_1(x) + f(4.0)L_2(x) \) or \( 0.5L_0(x) + 0.4L_1(x) + 0.25L_2(x) \).

2. Find an upper bound for the absolute error on the interval \([x_0, x_5]\) for the polynomial interpolant \( p_5(x) \) for the function \( f(x) = \cos(2x) \) where the points of interpolation are \( x_0 = 1.0, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4, x_5 = 1.5 \). Justify your answer. It is not necessary to find the interpolant.

**Answer:** We use the error estimate \( |f(x) - p_n(x)| \leq M_n + \frac{h^{n+1}}{n(n+1)} \). In this case, \( n = 5 \) and \( h = 0.1 \). Since \( f(x) = \cos(x) \), we find \( M_5 = 1 \). Hence, the upper bound becomes: \((0.1)^6/20 = 0.5 \times 10^{-7}\).

3. Use divided differences to find the polynomial interpolant of degree four to the data given by the table

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.0</th>
<th>0.1</th>
<th>0.3</th>
<th>0.6</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-6.0000</td>
<td>-5.89483</td>
<td>-5.65014</td>
<td>-5.17788</td>
<td>-4.28172</td>
</tr>
</tbody>
</table>

**Answer:**

4. Let \( x_0 = 1.0, x_1 = 1.3, x_2 = 1.6, x_3 = 1.9, \) and \( x_4 = 2.2 \). Let \( f \) be a function whose second order differences are:

\[
f[x_0, x_1, x_2] = -0.1087, \\
f[x_1, x_2, x_3] = -0.0494, \\
f[x_2, x_3, x_4] = 0.0118
\]

Find the third order differences \( f[x_1, x_2, x_3, x_4] \) and \( f[x_0, x_1, x_2, x_3] \).

**Answer:** Use the definition! \( f[x_1, x_2, x_3, x_4] = (f[x_2, x_3, x_4] - f[x_1, x_2, x_3])/(x_4 - x_1) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.0</th>
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</table>

5. Given that \( p_4(x) \) is the interpolating polynomials for \( f \) at points \( x_0 = 1.0, x_1 = 1.3, x_2 = 1.6, x_3 = 1.9, \) and \( x_4 = 2.2 \) and

\[
p_4(x) = 0.7651 - 0.4837(x - 1.0) - 0.1087(x - 1.0)(x - 1.3) + 0.9658(x - 1.0)(x - 1.3)(x - 1.6) + 0.0018(x - 1.0)(x - 1.3)(x - 1.6)(x - 1.9)
\]

Find the following second and fourth order divided differences for \( f \): \( f[x_0, x_1, x_2] \) and \( f[x_0, x_1, x_2, x_3, x_4] \).

**Answer:** These differences can be read off the polynomial interpolant directly. Recall \( f[x_0, x_1, x_2] \) is the coefficient of \( (x - 1.0)(x - 1.3) \) while \( f[x_0, x_1, x_2, x_3, x_4] \) is the coefficient of \( (x - 1.0)(x - 1.3)(x - 1.6)(x - 1.9) \).

6. A natural cubic spline \( S \) on \([0,2]\) is defined as follows: on \([0,1]\), \( S(x) = S_0(x) = 2 + 3x + x^3 \) and on \([1,2]\), \( S(x) = S_1(x) = 6 + b(x - 1) + c(x - 1)^2 + d(x - 1)^3 \). Find the values of the parameters \( b, c, d \).

**Answer:** We use the consistency conditions and the vanishing of the second derivative at the endpoints to find linear equations for \( b, c, d \). We start with \( S_0(1) = S_1(1) \) which gives \( 6 = 6 \), then \( S_0'(1) = S_1'(1) \) which gives \( 6 = b \), and finally \( S_0''(1) = S_1''(1) \) which becomes \( 6 = 2c \) or \( c = 3 \). To determine \( d \) use \( S_0''(2) = 0 \) or \( 6d = 0 \) or \( d = -1 \).