1. We are given the following data about a polynomial \( P(x) \) of unknown degree: \( P(0) = 2 \), \( P(1) = -1 \), \( P(2) = 4 \). Determine the coefficient of \( x^3 \) in \( P(x) \) if all the third-order differences are 1.

2. A fourth-degree polynomial \( P(x) \) satisfies \( \Delta^4 P(0) = 24 \), \( \Delta^3 P(0) = 6 \), and \( \Delta^2 P(0) = 0 \), where \( \Delta P(x) = P(x + 1) - P(x) \). Compute \( \Delta^2 P(10) \). Hints: use the Newton forward-difference formula on page 127; further, neither the constant nor linear terms play any role in evaluating the second-order difference.

3. Find the constants \( x_0, x_1 \), and \( c_1 \) so that the quadrature formula \( \int_0^1 f(x) \, dx = \frac{1}{2} f(x_0) + c_1 f(x_1) \) has the highest possible degree of precision.

4. For a function \( f \), the Newton’s interpolatory divided-difference formula gives the interpolating polynomial: \( p_3(x) = 1 + 4x + 4(x - 0.25) + \frac{4}{3} x(x - 0.25)(x - 0.5) \), on the nodes \( x_0 = 0, x_1 = 0.25, x_2 = 0.5, \) and \( x_3 = 0.75 \). Find the value \( f(0.75) \).

5. Consider the nodes \( x_0 = 2.0, x_1 = 2.5, x_2 = 4 \). The Lagrange polynomials for these nodes are:
   \[
   L_0(x) = \frac{(x - 6.5)x + 10}{3}, \quad L_1(x) = \frac{(-4x + 24)x - 32}{3}, \quad L_2(x) = \frac{(x - 4.5)x + 5}{3}
   \]
   Let \( f \) be a function defined on \([2,4]\) such that \( f(2) = 0.5, f(2.5) = 0.4, f(3.0) = -0.6, \) \( f(3.5) = 0.8 \), and \( f(4.0) = 0.25 \). Then find the polynomial interpolant to \( f \) at the nodes 2.0, 2.5, 4.0.

6. Find an upper bound for the absolute error on the interval \([x_0, x_4]\) for the polynomial interpolant \( p_5(x) \) for the function \( f(x) = \cos(x) \) where the points of interpolation are \( x_0 = 1.0, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4 \), \( x_4 = 1.5 \). Justify your answer. It is NOT necessary to find the interpolant.

7. Find an upper bound for the absolute error on the interval \([x_0, x_4]\) for the Hermite polynomial interpolant \( H(x) \) for the function \( f(x) = \cos(x) \) where the points of interpolation are \( x_0 = 1.0 \) and \( x_1 = 1.2 \). Justify your answer. It is NOT necessary to find the interpolant.

8. Let \( x_0 = 1.0, x_1 = 1.3, x_2 = 1.6, x_3 = 1.9, \) and \( x_4 = 2.2 \). Let \( f \) be a function whose second order differences are:
   \[ f[x_0, x_1, x_2] = -0.1087, \quad f[x_1, x_2, x_3] = -0.0494, \quad f[x_2, x_3, x_4] = 0.0118 \]
   Find the third order differences \( f[x_1, x_2, x_3, x_4] \) and \( f[x_0, x_1, x_2, x_3] \).

9. Given that \( p_4(x) \) is the interpolating polynomials for \( f \) at points \( x_0 = 1.0, x_1 = 1.3, x_2 = 1.6, x_3 = 1.9, \) and \( x_4 = 2.2 \) and
   \[
   p_4(x) = 0.7651 - 0.4837(x - 1.0) - 0.1087(x - 1.0)(x - 1.3) + 0.9658(x - 1.0)(x - 1.3)(x - 1.6) + 0.0018(x - 1.0)(x - 1.3)(x - 1.6)(x - 1.9).
   \]
   Find the following second and fourth order divided differences for \( f \):
   \[
   f[x_0, x_1, x_2] \quad \text{and} \quad f[x_0, x_1, x_2, x_3, x_4].
   \]

10. A natural spline \( S \) on \([0,2]\) is defined as follows: on \([0,1]\), \( S(x) = S_0(x) = 2 + 3x + x^3 \);
    on \([1,2]\), \( S(x) = S_1(x) = 6 + b(x - 1) + c(x - 1)^2 + d(x - 1)^3 \). Find the values of the parameters \( b, c, \) and \( d \).

11. Determine the value of the step size \( h \) necessary to find an approximation to \( \int_0^2 \sin(3x) \, dx \) to within \( 10^{-2} \) using the composite Simpson Rule.
12. Find the values for the constants $c_0, c_1, c_2$ and $x_0$ so that the quadrature formula:

\[ \int_0^1 f(x) \, dx = a f(0) + b f(1/2) + c f(1) \]

has the highest possible degree of precision.

13. Consider the following numerical approximation for the first derivative of $f(x)$ at $x = a$:

\[ f'(a) \approx \frac{f(a - 2h) - 4f(a - h) + 3f(a)}{2h} \]

Find the order of the error of this approximation in the form of a dominant term and the order of the higher order terms, by using a Taylor series expansion. Your answer should have the form: const $\cdot h^r + O(h^s)$.

**Multiple Choice Questions**

1. Suppose that $L = L(h) + k_1 h + k_2 h^2 + \cdots$ is an approximation to the value $L$. If Richardson’s extrapolation is applied once we obtain the formula where $L(1/2) = L(h/2) + aL(h/2) - L(h)$, where
   (a) $a = 1$,
   (b) $a = 1/2$,
   (c) $a = 1/3$.

2. Suppose that $L = L(h) + k_1 h^2 + k_2 h^4 + \cdots$ is an approximation to the value $L$. If Richardson’s extrapolation is applied twice to $L(h)$ we obtain the formula where $L(2)(h) = L(1)(h/2) + aL(1)(h/2) - L(1)(h)$, where
   (a) $a = 1$,
   (b) $a = 1/3$,
   (c) $a = 1/15$.

3. Consider the interpolating polynomial $p_2(x)$ for $f(x)$ at $x = 1, 2, 3$ where $p_2(x) = 3 + 4(x - 1) + 1(x - 1)(x - 2)$. Then the value for $f(3)$ is:
   (a) 5,
   (b) 10,
   (c) 13.

4. Consider the two cubic polynomials: $S_0(x) = 1 + 3x + 2x^3$ and $S_1(x) = 6 + 9(x - 1) + 6(x - 1)^2 + 2(x - 1)^3$. Do they form a natural cubic spline $S(x)$ on the interval $[0, 2]$ where $S = S_0$ on $[0, 1]$ and $S = S_2$ on $[1, 2]$? TRUE OR FALSE.

5. Consider the data: $x_0 = 0, x_1 = 1, f_0 = 1, f_1 = 1, f_0' = 3, f_1' = -2$. Let $H(x)$ be the cubic polynomial satisfying: $H(x_0) = f_0, H(x_1) = f_1$ and $H'(x_0) = f_0', H'(x_1) = f_1'$. Write $H(x) = 1 + 3x - a * x^2 + x^3$, where the coefficient $a$ must be determined. Then $a$ has the value:
   (a) 4,
   (b) $-4$,
   (c) 2.

6. Let $S(x)$ be the natural cubic spline for $f(x) = e^x$ at the nodes $x_0 = 0, x_1 = 2$, and $x_2 = 2$. Let $S_1(x)$ be the cubic from the spline on $[0, 1]$. Then $S_1(x)$ is a cubic Hermite interpolating polynomial for $f(x)$ on $[0, 1]$. TRUE OR FALSE.