Review questions for TRUE or FALSE:

There is a homomorphism \( \phi \) from \( \mathbb{Z}_3 \) onto \( \mathbb{Z}_7 \).
There is a non-abelian group of order 30.
Every group of order 29 is abelian.
A non-abelian group can never have a quotient group which is abelian.
There is at least two non-abelian groups of order 24.
There is a non-abelian group of order 71.
Let \( G \) be a finite group. If \( G/Z(G) \) is cyclic, then \( G \) must be abelian.
Let \( G \) be a finite group. If \( G/Z(G) \) is abelian, then \( G \) too must be abelian.
The number of subgroups of a finite group \( G \) must divide the order of the group.
The order of the permutation \( (2, 3, 4) (5, 6, 7, 8) (9, 10, 11, 12) \) is 12.
The external direct product \( \mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_8 \) has an element of order 8.
If the order of a permutation is even, then the permutation itself is an even permutation.
Every group of order 4 has an element of order 4.
If \( G \) is a finite cyclic group of order \( m \) and \( d \) divides \( m \), then \( G \) has an element of order \( d \).

If \( G \) is any finite group of order \( m \) and \( d \) divides \( m \), then \( G \) has an element of order \( d \).
If \( G \) is any finite abelian group of order \( m \) and \( d \) divides \( m \), then \( G \) has an element of order \( d \).

If \( \phi \) is a homomorphism from the dihedral group \( D_8 \) into the symmetric group \( S(16) \),
then the kernel of \( \phi \) need NOT be a normal subgroup of \( D_8 \).
Every automorphism of a group is automatically a homomorphism.
If \( \phi \) is a homomorphism of a finite group back into itself with \( \text{Ker}(\phi) = \{e\} \), then \( \phi \) is an automorphism.
The quotient of a non-abelian group may be cyclic.
The quotient of any cyclic group is cyclic.
The number of normal subgroups of a finite group must divide the order of the group.
Any group of order 17 has no proper subgroups.
Any finite abelian group is isomorphic to a subgroup of some symmetric group.
Let \( G_1 \) and \( G_2 \) be two finite groups with a homomorphism \( \phi \) mapping \( G_1 \) onto \( G_2 \). Let
\( g \) be any element from \( G_1 \). Then the order of \( \phi(g) \) equals the order of \( g \); that is, \(|\phi(g)| = |g|\).
Let \( n \) be any positive number. Then there exists an abelian group \( G \) such that \(|G| = n\).
Let \( \phi \) be a homomorphism from \( G_1 \) onto \( G_2 \). Then \( G_2 \) is isomorphic to a quotient group of \( G_1 \).
Let \( \phi \) be a homomorphism from \( G_1 \) onto \( G_2 \). Then the order of \( G_2 \) must divide the order of \( G_1 \).
Every automorphism \( \alpha \) of a finite abelian group \( A \) is inner; that is, it has the form:
\( \alpha(x) = gxg^{-1} \) for some fixed group element \( g \in A \).
Every finite group has at least two subgroups.
The empty set is a subgroup of any finite group.
The empty set is automatically a normal subgroup of a group.
The set \{e\} is automatically a normal subgroup of a group.

Let \( G \) be a finite group of even order, say \( 2m \). Then \( G \) must have a quotient group of order 2.

Every subgroup of index 2 is normal.
Every subgroup of order 2 is normal.
The quotient \( G/Z(G) \) is always abelian.
The subgroup \( Z(G) \) is always abelian.

Let \( G \) be a finite group of even order, say \( 2m \). If \( G \) has a subgroup \( H \) of order \( m \), then \( H \) must be a normal subgroup.

The dihedral group \( D_4 \) of order 8 has exactly 4 subgroups of order 2.
The dihedral group \( D_4 \) of order 8 has exactly 5 subgroups of order 2.
The dihedral group \( D_3 \) has exactly three normal subgroups.
The dihedral group \( D_3 \) is NOT isomorphic to the symmetric group \( S(3) \).
The center of \( D_3 \) has two elements.
It is possible for the center of group to equal the empty set.
Every group has a quotient group of order one.
The group of rotations of a tetrahedron is isomorphic to the alternating group \( A_4 \).
The order of \( A_5 \) is 75.
If \( \alpha \) is an automorphism of a finite group, then its kernel must have order 1.
The quotient of an infinite group can never be finite.
It is possible that \( D_3 \) is isomorphic to a quotient of an abelian group.

It is possible that \( Z_2 \) is isomorphic to a quotient of a non-abelian group.
The external direct product of two abelian groups need NOT be abelian.
The external direct product of two cyclic groups need NOT be cyclic.
The external direct product of two cyclic groups need NOT be abelian.
The left coset \( (2,1) + \langle(1,2)\rangle \) in the quotient group \( (Z_4 \oplus Z_4)/\langle(1,2)\rangle \) has order 4.
The group \( (Z_4 \oplus Z_4)/\langle(1,2)\rangle \) is cyclic.
The quotient group \( (Z \oplus Z)/\langle(1,1)\rangle \) is finite.

**Sample Theory Questions:**

1. Let \( G \) be a group of order \( pq \), where \( p \) and \( q \) are prime numbers. PROVE that every proper subgroup of \( G \) is cyclic.

2. PROVE that any finite group with at least two elements but with no proper subgroups must be of prime order.

3. Let \( G \) be a finite group of order \( 2n \). If \( H \) is a subgroup of order \( n \), PROVE that \( H \) is a normal subgroup.

4. PROVE that the kernel of any homomorphism is a normal subgroup.

5. PROVE that the external direct product of two abelian groups is abelian.

6. Let \( H \) be a normal subgroup of \( G \) with index \( m = [G : H] \). PROVE that \( a^m \in H \)

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for all $a \in G$.

(7) Let $H_1$ and $H_2$ be two subgroups of $G$. Prove (i) their intersection $H_1 \cap H_2$ is a subgroup of $G$; (ii) if $H_1$ and $H_2$ are assumed to be normal subgroups, show that their intersection is a normal subgroup.

Sample Calculation Questions:

(1) Find the orders of the following quotient groups:

$$(\mathbb{Z}_2 \oplus \mathbb{Z}_4)/\langle(0, 1)\rangle, (\mathbb{Z}_2 \oplus \mathbb{Z}_4)/\langle(1, 2)\rangle,$$

$$(\mathbb{Z}_2 \oplus \mathbb{Z}_4)/\langle(2, 1)\rangle, (\mathbb{Z}_4 \oplus \mathbb{Z}_8)/\langle(1, 2)\rangle,$$

(2) How many non-isomorphic abelian groups are there of order $5^3 \cdot 7^2$? How many non-isomorphic abelian groups are there of order $13^3 \cdot 17^2$?

(3) Find the order of the element $(2, 3) + \langle(2, 2)\rangle$ in the group $\langle(\mathbb{Z}_2 \oplus \mathbb{Z}_8)/\langle(2, 2)\rangle\rangle$.

(4) (a) Find three different homomorphisms from $\mathbb{Z}$ into $\mathbb{Z}$. (b) Find four different automorphisms of $\mathbb{Z}_5$.

(5) Find the order of $(8, 4, 10)$ in the group $\mathbb{Z}_{12} \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{24}$.

(6) Find all the left cosets of (4) in the group $\mathbb{Z}_{16}$.

(7) Find all subgroups of order 3 in $\mathbb{Z}_9 \oplus \mathbb{Z}_3$.

(8) The group $\mathbb{Z}_2 \oplus D_3$ is isomorphic to one of the following: $\mathbb{Z}_{12}, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, A_4, D_6$. Determine which one by elimination. The answer is $D_6$.

(9) Determine all homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. There are four.

(10) Suppose that $\beta$ is the cycle and $\beta^2 = (1, 3, 5, 2, 4)$. Determine $\beta$. Use the fact that $\beta^5$ is the identity permutation.

(*) Review permutations calculations from the quiz.