1. Let $a$ and $b$ be positive integers. Show that $\sqrt{2}$ always lies between $a/b$ and $(a+2b)/(a+b)$. Further $(a+2b)/(a+b)$ is always closer to $\sqrt{2}$ that $a/b$.

**Comments:** There are two cases: either $a/b$ is smaller than $\sqrt{2}$ or larger. We focus on the first case that $a/b < \sqrt{2}$. Then we find

$$\frac{a + 2b}{a + b} < \sqrt{2} = \frac{(\sqrt{2} - 1)(b\sqrt{2} - a)}{a + b} \geq 0$$

since $\sqrt{2} > 1$ and $a < b\sqrt{2}$. Hence, $(a+2b)/(a+b) > \sqrt{2}$.

To show that $(a+2b)/(a+b)$ is closer than $a/b$, we consider the difference:

$$\left(\sqrt{2} - \frac{a}{b}\right) - \left(\frac{a + 2b}{a + b} - \sqrt{2}\right) = \frac{(a + 2b - b\sqrt{2})(b\sqrt{2} - a)}{b(a + b)} \geq 0$$

since $b\sqrt{2} - a > 0$ and $a + 2b - \sqrt{2} - b\sqrt{2} > 0$. The second case is handled similarly.

2. If $x$ is irrational, show that there are infinitely many rational numbers $h/k$ ($k > 0$) such that $|x - h/k| < 1/k^2$.

**Comments:** We are allowed to use the following result: For any real number $x$ and integer $N > 1$, there are integers $h$ and $k$ with $0 < k \leq N$ such that $|kx - h| < 1/N$. We argue by contradiction. So assume there are only finitely many rationals $h_1/k_1, \ldots, h_m/k_m$ with $0 < k_j \leq N$ ($1 \leq j \leq m$) and $|k_jx - h_j| < 1/N$. Since $x$ is irrational, $\delta = \min\{|x - h_j/k_j| : 1 \leq j \leq m\}$ is positive. Next, there exists a positive integer $N$ such that $N > 1/\delta$. Further by the quoted result, there are positive integers $h'$ and $k'$ with $0 < k' \leq N$ so $|k'x - h'| < 1/N$; that is, $|x - h'/k'| < 1/[Nk'] \leq 1/(k')^2$. But this contradicts the choice of $\delta$ since $|k'x - h'| < \delta$.

3. (a) $\sin n\theta = \sin^n \theta \left\{ \left(\frac{n}{1}\right) \cot^{n-1} \theta - \left(\frac{n}{3}\right) \cot^{n-3} \theta + \cot^{n-5} \theta + \cdots \right\}$. **Comments:** Since $(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$, we find by the binomial theorem and taking imaginary parts that:

$$\sin(n\theta) = \sum_{j=1}^{m} (-1)^{j+1} \left(\frac{n}{2j - 1}\right) \cos^{n-2j+1}(\theta) \sin^{2j-1}(\theta)$$

$$= \sin^n \theta \sum_{j=1}^{m} (-1)^{j+1} \left(\frac{n}{2j - 1}\right) \cot^{n-2j+1}(\theta)$$

where $m$ is the ceiling of $n/2$.

(b) Suppose $0 < \theta < \pi/2$. Then for any odd integer $2m + 1 \sin((2m + 1)\theta) = \sin^{2m+1} P_m(\cos^2 \theta)$ where $P_m(x)$ is the polynomial given as $P_m(x) = \sum_{j=1}^{m} (-1)^{j+1} \left(\frac{2m + 1}{2j - 1}\right) x^{m-j+1}$ by part (a). [Since $\cos^2 t$ is periodic with period $\pi/2$ and has singularities at both 0 and $\pi/2$, the restriction $0 < \theta < \pi/2$ is natural.] Now the roots of $P_m(x)$ are found using the zeros of $\sin((2m + 1)\theta)$; that is, $(2m + 1)\theta = \pi k$, where $k$ is any integer. In particular, $P_m(x)$ has $m$ distinct roots at $x_k = \cot^2(\pi k/[2m + 1])$, for $k = 1, 2, \ldots, m$.

(c) We recall the some relationships between the coefficients of a general monic polynomial $q(x)$ and its roots. Write $q(x)$ as $x^n - a_1 x^{n-1} + a_2 x^{n-2} + \cdots + (-1)^m a_0$ in factored form $(x - r_1)(x - r_2) \cdots (x - r_m)$.
Then the coefficient $a_1$ of $x^{m-1}$ equals $-(r_1 + r_2 + \cdots + r_m)$ while the coefficient $a_2$ of $x^{m-2}$ is $\sum \{r_i r_j : 1 \leq i < j \leq m \}$. Let $S_2$ denote the sum of the squares of the roots, so $S_2 = a_1^2 - 2a_2$. In our case, we find

$$\sum_{j=1}^{m} \cot^2 \left( \frac{\pi j}{2m+1} \right) = \left( \frac{2m+1}{3} \right) / \left( \frac{2m+1}{1} \right)$$

and

$$\sum_{j=1}^{m} \cot^4 \left( \frac{\pi j}{2m+1} \right) = \left[ m(2m-1) \right]^2 - \left( \frac{2m+1}{5} \right) / \left( \frac{2m+1}{1} \right) = \frac{m(2m-1)(4m^2 + 10m - 9)}{45}.$$

4. Show $\prod_{k=1}^{m} \sin(k\pi/m) = m 2^{1-m}$.

**Comments:** Since $z^{m-1} = \prod_{k=1}^{m} (z - \exp[2\pi ik/m])$, we can cancel out $z - 1$ to obtain $z^{m-1} + z^{m-2} + \cdots + 1 = \prod_{k=1}^{m-1} (z - \exp[2\pi ik/m])$. For $z = 1$, we find that $m = \prod_{k=1}^{m-1} (1 - \exp[2\pi ik/m])$. This product simplifies to $m 2^{1-m}$ as follows.

$$\prod_{k=1}^{m-1} (1 - \exp[2\pi ik/m]) = \prod_{k=1}^{m-1} [(1 - \cos(2\pi k/m)) - i \sin(2\pi k/m)]$$

$$= \prod_{k=1}^{m-1} [(2 \sin^2(\pi k/m)) - 2 i \sin(\pi k/m) \cos(\pi k/m)]$$

$$= 2^{n-1} \prod_{k=1}^{m-1} [\sin(\pi k/m) \cos(\pi k/m)]$$

$$= 2^{n-1} \prod_{k=1}^{m-1} [\sin(\pi k/m)(-i \exp(\pi i k/m))]$$.

But $\prod_{k=1}^{m-1} (-i \exp(\pi i k/m)) = (-i)^{m-1} \exp[\pi im(m-1)/2m] = (-i)^{m-1} \exp[i(m-1)/2] = (-i)^{m-1} i^{m-1} = 1$.

5. (a) The set of circles in the complex plane with rational centers and radii is countable. (b) Any collection of disjoint intervals of positive length is countable.

**Comments:** (a) A circle $C$ is uniquely determined by its center $(x_C, y_C)$ and radius $\rho_C$. Let $\mathcal{C}$ denote the set of all circles in the complex plane with rational centers and radii. Then we have a natural map $F : \mathcal{C} \to \mathbb{Q}^3$ given by $F(C) = (x_C, y_C, \rho_C)$. We know that the set of all rational number is countable so its cartesian product with itself three times is also countable. Furthermore, the image $F(\mathcal{C})$ is countable. Hence $\mathcal{C}$ itself is countable since $F$ is an injection.

(b) Let $I$ be any collection of disjoint intervals of positive length from the real line $\mathbb{R}$. Given $I \in \mathcal{I}$, choose a rational number $r_I$ from $I$. Since the intervals are mutually disjoint, there is a map $F : \mathcal{I} \to \mathbb{Q}$ given by $I \mapsto r_I$ which is an injection. Since $\mathbb{Q}$ is countable, so is $\mathcal{I}$.

6. Let $f : [0, 1] \to \mathbb{R}$. Assume there exists a positive constant $M$ so that for every choice $x_1, \ldots, x_n \in [0, 1]$, $|f(x_1) + \cdots + f(x_m)| \leq M$. Then the sets of points where $f$ is non-zero is countable.

**Comments:** Let $S = \{ x \in [0, 1] : f(x) \neq 0 \}$ and $S_n = \{ x \in [0, 1] : f(x) > 1/n \}$ if $n \in \mathbb{Z}^+$ while $S_n = \{ x \in [0, 1] : f(x) < 1/n \}$ if $n \in \mathbb{Z}^+$. Note that $S = \bigcup \{ S_n : n \in \mathbb{Z} \setminus \{0\} \}$. However each $S_n$ is a finite set. To see this, consider first the case $n > 0$. Then for any $x_1, \ldots, x_k \in S_n$ we have $k/n < f(x_1) + \cdots + f(x_k) \leq M$ so $k < Mn$. The case $n < 0$ is handled similarly.